Viability Kernels for Nonlinear Control Systems Using Bang Controls

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***Abstract*—This paper develops a method to construct viability kernels for multi-input, single-output nonlinear control systems affine in the control. The safe set is the superlevel set of a smooth function, and the control is constrained to take values in a compact polyhedron. The results make use of the Frankowska method and the notion of viable capture basins.**

***Index Terms*—viability kernels, capture basins, set invariance, control affine systems, nonsmooth analysis**

I. Introduction

The purpose of this paper is to develop a methodology to explicitly construct viability kernels for control affine systems. The central problem can be roughly described as enforcing a control system to evolve in a "safe set" of the state space starting from any initial condition inside the set, by proper assignment of the control input. When no control exists to satisfy this requirement, then the problem is to find a largest subset inside the safe set, called a viability kernel, and an associated controller, called a viability controller, so that the system remains inside the safe set, starting from any initial condition in the viability kernel, using a viability controller. The theory of viability kernels has been developed over the last two decades by J.-P. Aubin and his co-workers [1]. It has numerous applications in diverse disciplines such as ecology, mathematical biology, economics, and robotics. Several interesting results have recently appeared on numerical methods to compute viability kernels [3], [4], [5], [10], [12]. The present paper is among the first results on explicit construction of viability kernels.

We study the following situation: we have a multi-input, single-output nonlinear system affine in the control. The safe set is the sublevel set of a smooth function and geometrically is a manifold with boundary. We want to find the viability kernel associated with the safe set, and a viability controller, assuming that the control is constrained to take values in a compact, convex set. The proposed solution is motivated by features of viability problems arising in applications. First, we are interested in viability controllers which terminate in finite time. For example, in collision avoidance of two vehicles [13], it is desirable that the vehicles can return to a useful activity after performing collision avoidance. The termination requirement is addressed by introducing a target set and formulating the viability problem in terms of viable capture basins [20]. Second, we focus on systems which satisfy a relative degree-like condition. This condition makes it clear how to select the target set. Finally, we focus on bang controls. This restriction has the important consequence that a formula for the viable capture basin can be obtained. To test that the proposed viable capture basin is correct, one verifies tangential conditions based on the Frankowska method [14]. In certain cases, the entire procedure can be carried out analytically. An example on fisheries management is provided at the end.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of $\mathcal{K}$ is $\neg\mathcal{K} := \mathbb{R}^n \setminus \mathcal{K}$, the closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The *tangent cone* to $\mathcal{K}$ at a point $x \in \mathcal{K}$ is denoted by $T_{\mathcal{K}}(x)$, and the closure of the convex hull of $T_{\mathcal{K}}(x)$ is denoted $\overline{\text{co}}(T_{\mathcal{K}}(x))$. For $x \in \mathbb{R}^n$, $\Pi_{\mathcal{K}}(x)$ denotes the set of points in $\mathcal{K}$ that achieve the infimal distance to x. Let $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^n$. If $x \in \overline{\mathcal{K}_1 \cup \mathcal{K}_2}$, then $T_{\mathcal{K}_1 \cup \mathcal{K}_2}(x) = T_{\mathcal{K}_1}(x) \cup T_{\mathcal{K}_2}(x)$ [1]. If $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \to \mathbb{R}$, then $L_f h(x) = \frac{\partial h}{\partial x} f(x)$, $L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$, and we define recursively, $L_f^0 h(x) = h(x)$ and $L_f^k h(x) = \frac{\partial \left(L_f^{k-1} h\right)}{\partial x} f(x)$.

II. Problem Formulation

Consider the multi-input, single output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x)\,, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are Lipschitz and smooth functions, and $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth submersion, i.e. the gradient ∇h is non-vanishing everywhere in $\mathbb{R}^n$. The input space is a compact, convex polyhedron $U \subset \mathbb{R}^m$. A control $u : [0,\infty) \to U$ is a measurable function in t which takes values in U. Let $\phi_u(t, x_0)$ denote the unique solution of (1) starting at x_0 and using control u. The set of q vertices of U is denoted as $V = \{v^1, \ldots, v^q\}$. Also, let $I := \{1, \ldots, q\}$ be the set of indices. A *bang control* is a control that takes a single constant control value in V. A *bang-bang control* is a control that is piecewise constant and takes values in V. The domain of the state space that we want to render positively invariant by proper choice of control, called the *safe set*, is

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\}\,. \tag{2}$$

Assumption 1: There exists $1 < r \leq n$ such that for all $x \in \mathbb{R}^n$ and for all $k < r-1$, $L_g L_f^k h(x) = 0$.

Remark 2: The assumption says that each component of the row vector $L_g L_f^k h(x) = \frac{\partial (L_f^k h)}{\partial x} g(x)$ is zero for $k < r-1$; that is, no input appears before r differentiations of the output. One interpretation is that the system does not have relative degree less than two at any point. The condition arises from a structural property of the system. It is a reasonable assumption

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for the given problem in the sense that if $L_g h(x) \neq 0$ on some set, then the viability kernel is trivially computable on that set because the control can be used to maintain any value of h. Assumption 1 implies that the derivative of h along solutions of (1) is $\frac{dh(t)}{dt} = L_f h(\phi_u(t, x_0))$. Thus, we can define the set of states where h is strictly decreasing as

$$\mathcal{W} := \{ \, x \in \mathbb{R}^n \mid L_f h(x) < 0 \, \} \, .$$

A subset $\mathcal{K}$ is said to be a *viability domain* if for each $x_0 \in \mathcal{K}$, there exists a control $u(t)$ such that the unique solution $\phi_u(t, x_0)$ of (1) stays in $\mathcal{K}$ for all $t \geq 0$. If $\mathcal{K}$ is not a viability domain, then there exists a largest closed (possibly empty) viability domain $Viab(\mathcal{K})$ contained in $\mathcal{K}$, which is called the *viability kernel* of $\mathcal{K}$. A control u which renders $Viab(\mathcal{K})$ viable is called a *viability controller*. The notion of a viability kernel with target was introduced in [20]. A related notion is that of viable capture basin of a set.

Definition 3: Let $\mathcal{C} \subset \mathcal{K}$. The subset $Capt(\mathcal{K}, \mathcal{C})$, called the *viable capture basin*, is the set of all initial states $x_0 \in \mathcal{K}$ such that there exists a control $u(t)$ such that the unique solution $\phi_u(t, x_0)$ of (1) stays in $\mathcal{K}$ until reaching $\mathcal{C}$ in finite time.

We are interested in finding the viability kernel of the set $\mathcal{K} := \mathcal{S} \cap \overline{\mathcal{W}}$, where $\mathcal{S} \cap \overline{\mathcal{W}}$ is the closed set of states where the system is safe but in danger of reaching an unsafe state. We also impose the practical requirement that the system reach a target set $\mathcal{C} \subset \mathcal{K}$ from the set $\mathcal{K}$ in finite time. This formulation is meaningful if we can guarantee that the system can remain in $\mathcal{S}$ after arriving at $\mathcal{C}$. To do so, we define the closed sets

$$\begin{aligned} \mathcal{C}^+ &:= \left\{ x \in \mathbb{R}^n \mid h(x) \geq 0, L_f h(x) \geq 0, \ldots, L_f^{r-1} h(x) \geq 0 \, \right\} \\ \mathcal{C} &:= \mathcal{C}^+ \cap \mathcal{K} \, . \end{aligned} \tag{3}$$

Assumption 4: For all $x_0 \in \mathcal{C}$, there exists an open-loop control $u_p : \mathbb{R}^+ \to U$ such that $\frac{d^r}{dt^r} h(\phi_{u_p}(t, x_0)) \geq 0$, for all $t \geq 0$.

Remark 5: When Assumption 4 holds we say that $\mathcal{C}^+$ is the *viability core* of $\mathcal{S}$. Its importance is in providing concrete termination conditions for the viability problem, and it is inspired by applications in ecology, biology and robotics, where a viability core often arises. Without such a termination condition the computation of the viability kernel is significantly more complex.

Our viability problem is formally stated as follows.

Problem 1: Given a control affine system (1), the closed set $\mathcal{K} = \mathcal{S} \cap \overline{\mathcal{W}}$, and a target set $\mathcal{C} = \mathcal{C}^+ \cap \mathcal{K}$, find $u^\star$, a viability controller, and $\mathcal{S}^\star := Capt(\mathcal{K}, \mathcal{C})$, the viable capture basin.

III. Viable Capture Basin

In this section we present a construction of the viable capture basin for the set $\mathcal{K}$ with target $\mathcal{C}$. Our construction is centered on bang controls. This is motivated by the fact that, under reasonable conditions, there always exists a subset of $\mathcal{K}$ that can reach $\mathcal{C}$ in finite time via a bang control (for if $\mathcal{C}$ is not reachable by bang control then it is not reachable by bang-bang control). It is also motivated by applications where it is often known that bang controls are the correct controls for a particular domain, without having explicit knowledge of system trajectories.

Consider $x_0 \in \mathbb{R}^n$ and for each $i \in I$, define $\phi_i(t, x_0)$ to be the unique solution of the autonomous system $\dot{x} = f(x) + g(x) v^i$ with initial condition x_0. Define the *hitting time* $\overline{t}_i(x_0)$ to be the first time when $\phi_i(t, x_0)$ reaches $\mathcal{C}$ before possibly leaving $\mathcal{K}$. If $\phi_i(t, x_0)$ does not reach $\mathcal{C}$ or it leaves $\mathcal{K}$ before reaching $\mathcal{C}$, set $\overline{t}_i(x_0) = \infty$. For $x_0 \in \mathcal{C}$, set $\overline{t}_i(x_0) = 0$. Define the set $\mathcal{X}_i := \{ x_0 \in \mathbb{R}^n \mid \overline{t}_i(x_0) < \infty \}$. It can be shown that for each $i \in I$, $\overline{t}_i$ is lower semicontinuous on $\mathcal{X}_i$ [1].

Next, for $x_0 \in \mathbb{R}^n$, we define $\overline{h}_i(x_0)$ to be the value of h at $\overline{t}_i(x_0)$, i.e., $\overline{h}_i(x_0) := h(\phi_i(\overline{t}_i(x_0), x_0))$. If $\overline{t}_i(x_0) = \infty$, set $\overline{h}_i(x_0) := -\infty$. Notice that by construction $\overline{h}_i$ is constant when evaluated along the trajectory $\phi_i(t, x_0)$ over the interval $[0, \overline{t}_i(x_0)]$.

For $x \in \mathcal{K}$, define the set of indices

$$I^\star(x) = \mathrm{argmax}_{i \in I} \{ \, \overline{h}_i(x) \mid \overline{t}_i(x) < \infty \} \, . \tag{4}$$

Note the cardinality of this set may vary with x. Define the function $\mu^\star : \mathcal{K} \to V$ by $\mu^\star(x) := v^j$, where $j \in I^\star(x)$ is selected arbitrarily. Finally, for each initial condition $x_0 \in \mathcal{K}$ we define

$$u^\star(t, x_0) := \mu^\star(x_0) \, , \qquad\qquad t \in [0, \overline{t}(x_0)] \, , \tag{5}$$

where $\overline{t}(x_0) := \overline{t}_j(x_0)$ if $\mu^\star(x_0) = v^j$. Intuitively, this choice of controller maximizes the first local minimum value of h on an interval $[0, \overline{t}]$, by using only a single control value in V. The controller $u^\star$ terminates at the time $\overline{t}$ when, by construction, $\dot{h} = 0$ and the target $\mathcal{C}$ is reached.

Remark 6: Observe that $\mu^\star$ is in feedback form: at each $x \in \mathcal{K}$, the set $I^\star(x)$ must be evaluated and a control value in V selected. However, $u^\star$ is an open loop control. Its value and its duration $\overline{t}$ are computed at $t = 0$ based on the initial condition only. One deduces that $u^\star$ is actually a feedback by using the *barrier property* [19]. It says that viable trajectories that start in the boundary of the viability kernel remain in its boundary until reaching the target. In the same spirit as the dynamic programming principle for optimal control, we obtain that $u^\star$ is identically equal to $\mu^\star$ at each point, so it is effectively a state feedback.

Define a function $h^\star : \mathbb{R}^n \to \mathbb{R}$ by

$$h^\star(x) = \max_{i \in I} \{ \, \overline{h}_i(x) \, \} \, .$$

Finally, we define

$$\mathcal{S}^\star := \{ x \in \mathbb{R}^n \mid h^\star(x) \geq 0 \} \, . \tag{6}$$

Assumption 7: $h^\star$ is continuous on $\mathrm{Dom}(h^\star) := \{ x \in \mathbb{R}^n \mid |h^\star(x)| < \infty \}$ and $\mathcal{S}^\star$ is closed.

Note that $\mathcal{S}^\star \subset \mathcal{K}$, because if $x_0 \notin \mathcal{K}$ then $\overline{t}_i(x_0) = \infty$, $\forall i \in I$. Our aim is to show that $\mathcal{S}^\star$ is the viable capture basin solving Problem 1, and we do so in three steps depending on the class of controls: bang controls, bang-bang controls, and measurable controls. Our main theoretical tool is the following characterization of viable capture basins, adapted from [2].

Theorem 8: Let $\mathcal{K}$ and $\mathcal{C}$ be closed sets such that $\mathcal{C} \subset \mathcal{K}$. The viable capture basin $Capt(\mathcal{K}, \mathcal{C})$ is the unique closed subset $\mathcal{D}$ satisfying $\mathcal{C} \subset \mathcal{D} \subset \mathcal{K}$ and

(i) For each $x_0 \in \mathcal{D}$, there exists a control $u(t)$ such that the trajectory starting at x_0 and using control u reaches $\mathcal{C}$ in finite time without first exiting $\mathcal{D}$.

(ii) $\mathcal{D}$ is backward invariant relative to $\mathcal{K}$. That is, for every $x_0 \in \mathcal{D}$ and every solution $\phi(\cdot, x_0)$, if there exists $T > 0$ such that $\phi(t, x_0) \in \mathcal{K}$ for $t \in [-T, 0]$, then $\phi(t, x_0) \in \mathcal{D}$ for $t \in [-T, 0]$.

Remark 9: Theorem 8 is a version of Frankowska's method [14] which gives a unique characterization of viability kernels and capture basins. We use Theorem 8 in the following way. First we show in Lemma 10 that by construction $u^\star$ satisfies condition (i). Second, we replace condition (ii) by equivalent tangential conditions (see [2]) given by:

$$-(f(x) + g(x)u) \in T_{\mathcal{D}}(x)\,, \forall x \in \mathcal{D} \cap \mathcal{K}^\circ,\ \forall u \in U \quad (7)$$

$$-(f(x) + g(x)u) \in T_{\mathcal{D}}(x) \cup T_{\neg\mathcal{K}}(x)\,, \forall x \in \mathcal{D} \cap \partial\mathcal{K},\ \forall u \in U\,. \quad (8)$$

These are then adapted to obtain our main condition which guarantees backward invariance of $\mathcal{S}^\star$ relative to $\mathcal{K}$. Our main condition says that for all $x \in \partial\mathcal{D} \cap \neg\mathcal{C}$ and for all $j \notin I^\star(x)$, $-(f(x)+g(x)v^j) \in T_{\mathcal{D}}(x)$. It is clear that this is a refinement or restriction of (7)-(8). It focuses on a finite set of control values and a restricted set of points in $\mathcal{S}^\star$ where backward invariance must be tested. From this restricted test, (7)-(8) can be deduced to hold. This is useful computationally, because the expression for $h^\star$ is at times simple enough that the backward invariance test can be manually performed, whereas (7)-(8) may be more unwieldy to verify.

Lemma 10: We are given a system (1), a safe set (2), and a target set (3). Suppose that Assumptions 1, 4, and 7 hold. For each $x_0 \in \mathcal{S}^\star$, the trajectory starting at x_0 and using control $u^\star$ reaches $\mathcal{C}$ in finite time without first exiting $\mathcal{S}^\star$.

Proof: Let $x_0 \in \mathcal{S}^\star$ and let $i \in I^\star(x_0)$. Then we have $h^\star(x_0) =: c_0 \geq 0$ and $\overline{h}_i(\phi_i(t, x_0)) = c_0$ for all $t \in [0, \overline{t}_i(x_0)]$. Thus, $h^\star(\phi_i(t, x_0)) \geq \overline{h}_i(\phi_i(t, x_0)) = c_0 \geq 0$ for all $t \in [0, \overline{t}_i(x_0)]$. This implies that the trajectory $\phi_i(t, x_0)$ is viable in $\mathcal{S}^\star$ until reaching the target $\mathcal{C}$ in finite time $\overline{t}_i(x_0)$. ■

Remark 11: A situation when our method does not apply is when $\mathcal{K} \neq \emptyset$ but $\mathcal{C} = \emptyset$, which can happen, in particular, when $\partial\mathcal{W} = \emptyset$. The computations would yield $\overline{t}_i(x_0) = \infty$ and $\overline{h}_i(x_0) = -\infty$ for all $i \in I$ and $x_0 \in \mathcal{W}$. Therefore, $h^\star(x_0) = -\infty$, $\mathcal{S}^\star = \emptyset$, and $u^\star$ is undefined. Evidently another notion of viability kernel must be considered for this case, such as one that does not require finite termination.

A. Main results

In this section we obtain the main theoretical results when the problem is restricted to bang controls, bang-bang controls, and measurable controls. First, due to the properties of bang controls and the special structure of $\mathcal{S}^\star$, we have the following straightforward result.

Proposition 12: We are given a system (1), a safe set (2), and a target set (3). Suppose that Assumptions 1, 4, and 7 hold. Then $\mathcal{S}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$ under the restriction to bang controls, and $u^\star$ is a viability controller.

Proof: From Lemma 10 we know that for each $x_0 \in \mathcal{S}^\star$, there is a bang control and associated trajectory that reaches $\mathcal{C}$ in finite time without leaving $\mathcal{S}^\star$. Next, consider $x_0 \in \mathcal{K} \setminus \mathcal{S}^\star$. For each $i \in I$, it must be that either the trajectory ϕ_i never reaches $\mathcal{C}$ (and $h^\star(x_0) = -\infty$), or alternatively, ϕ_i reaches $\mathcal{C}$ in a finite time $\overline{t}_i$ with $h(x_0) \geq 0$ and $h(\phi_i(\overline{t}_i, x_0)) \geq 0$ but ϕ_i first exits $\mathcal{K}$ (and again $h^\star(x_0) = -\infty$). Since this holds for all $i \in I$, it is clear that x_0 cannot belong to the viable capture basin, so $\mathcal{S}^\star$ is the unique viable capture basin under the restriction to bang controls. ■

In the previous result, the restrictive assumption that the control may not switch implies that a trajectory starting at $x_0 \in \mathcal{K} \setminus \mathcal{S}^\star$ may not be viable, even if it has the opportunity to enter $\mathcal{S}^\star$ by switching control. Therefore, to obtain a characterization of the viability kernel using bang-bang controls, the backward invariance conditions (7)-(8) have to be introduced. We first expose some useful properties of the boundary $\mathcal{S}^\star \cap \partial\mathcal{K}$ which allow us to state a restricted version of (7)-(8).

Lemma 13: $\mathcal{S}^\star \cap \partial\mathcal{S} \cap \mathcal{W} = \emptyset$.

Proof: Let $x_0 \in \mathcal{S}^\star \cap \partial\mathcal{S} \cap \mathcal{W}$. Then $h(x_0) = 0$ and $L_f h(x_0) < 0$. Thus, there exists $\delta > 0$ sufficiently small such that for all $i \in I$, $h(\phi_i(t, x_0)) < 0$ and $\phi_i(t, x_0) \in \mathcal{W}$, $\forall t \in (0, \delta]$, and $\phi_i(t, x_0) \notin \mathcal{C}$, $\forall t \in [0, \delta]$. This contradicts Lemma 10 which says that for $i \in I^\star(x_0) \neq \emptyset$, $\phi_i(t, x_0) \in \mathcal{S}^\star$ for all $t \in [0, \overline{t}_i(x_0)]$. ■

Let $\partial\mathcal{W} \cap \mathcal{S}$ be partitioned as the disjoint union $\partial\mathcal{W} \cap \mathcal{S} = \partial\mathcal{W}_{1e} \cup \partial\mathcal{W}_{1o} \cup \partial\mathcal{W}_2 \cup \mathcal{C}$ where

$$\begin{aligned}
\partial\mathcal{W}_{1e} &= \{x \in \partial\mathcal{W} \cap \mathcal{S} \cap \neg\mathcal{C} \mid (\exists\ 2 \leq k(x) \leq r-1, k(x) \text{ even}) \\
&\qquad L_f h(x) = \cdots = L_f^{(k(x)-1)} h(x) = 0, L_f^{k(x)} h(x) < 0\} \\
\partial\mathcal{W}_{1o} &= \{x \in \partial\mathcal{W} \cap \mathcal{S} \cap \neg\mathcal{C} \mid (\exists\ 2 < k(x) \leq r-1, k(x) \text{ odd}) \\
&\qquad L_f h(x) = \cdots = L_f^{(k(x)-1)} h(x) = 0, L_f^{k(x)} h(x) < 0\} \\
\partial\mathcal{W}_2 &= \{x \in \partial\mathcal{W} \cap \mathcal{S} \cap \neg\mathcal{C} \mid (\exists\ 2 \leq k(x) \leq r-2) \\
&\qquad L_f h(x) = \cdots = L_f^{(k(x)-1)} h(x) = 0, L_f^{k(x)} h(x) > 0\}\,.
\end{aligned}$$

Note that for $r = 2$, $\partial\mathcal{W}_{1e} = \partial\mathcal{W}_{1o} = \partial\mathcal{W}_2 = \emptyset$, and for $r = 3$, $\partial\mathcal{W}_{1o} = \partial\mathcal{W}_2 = \emptyset$.

Lemma 14: Trajectories arrive at $\mathcal{S} \cap \partial\mathcal{W}_{1e}$ only from $\neg\mathcal{K}$.

Proof: Consider $x_0 \in \partial\mathcal{W}_{1e}$. We have $h(x_0) \geq 0$, $L_f h(x_0) = \cdots = L_f^{(k(x_0)-1)} h(x_0) = 0$ and $L_f^{(k(x_0))} h(x_0) < 0$, with $k(x_0)$ even. This implies there exists $\delta > 0$ sufficiently small such that for all $t \in (-\delta, 0)$ and for all controls u, $L_f^j h(\phi_u(t, x_0)) > 0$ for j odd and $1 \leq j \leq k(x_0) - 1$. Also, $L_f^j h(\phi_u(t, x_0)) < 0$ for j even and $2 \leq j \leq k(x_0)$. In particular, for $j = 1$, $L_f h(\phi_u(t, x_0)) > 0$, $\forall t \in (-\delta, 0)$. That is, $\phi_u(t, x_0) \in \neg\mathcal{K}$, $\forall t \in (-\delta, 0)$. ■

Lemma 15: $\mathcal{S}^\star \cap \partial\mathcal{W}_2 = \emptyset$.

Proof: Consider $x_0 \in \mathcal{S}^\star \cap \partial\mathcal{W}_2$. We have $h(x_0) \geq 0$, $h^\star(x_0) \geq 0$, $L_f h(x_0) = \cdots = L_f^{(k(x_0)-1)} h(x_0) = 0$, and $L_f^{k(x_0)} h(x_0) > 0$. Using the fact that $\neg\mathcal{C}$ is open, there exists $\delta > 0$ such that for all $t \in (0, \delta)$ and for all $i \in I$, $\phi_i(t, x_0) \in \neg\mathcal{C}$ and $L_f h(\phi_i(t, x_0)) > 0$. Thus, $\phi_i(t, x_0) \in \neg\mathcal{C} \cap \neg(\overline{\mathcal{W}}) \subset \neg\mathcal{K}$, $\forall t \in (0, \delta)$, and $x_0 \in \neg\mathcal{C}$. This contradicts Lemma 10 which says that for $i \in I^\star(x_0)$, $\phi_i(t, x_0) \in \mathcal{S}^\star$, $\forall t \in [0, \overline{t}_i(x_0)]$. ■

Remark 16: Lemma 13 and 15 show that, moreover, for all $x_0 \in (\partial\mathcal{S} \cap \mathcal{W}) \cup \partial\mathcal{W}_2$ and for all trajectories $\phi_u(t, x_0)$, there exists $\delta > 0$ such that $\phi_u(t, x_0) \in \neg\mathcal{K}$, $\forall t \in (0, \delta)$.

Theorem 17: We are given a system (1), a safe set (2), and a target set (3). Suppose that Assumptions 1, 4, and 7 hold. In addition, suppose that for all $x \in \partial\mathcal{S}^\star \cap \neg\mathcal{C}$ and for all

$j \notin I^\star(x)$,
$$-(f(x) + g(x)v^j) \in T_{\mathcal{S}^\star}(x)\,. \qquad (9)$$

Then $\mathcal{S}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$ under the restriction to bang-bang controls, and $u^\star$ is a viability controller.

Proof: We show conditions (i) and (ii) of Theorem 8 hold for bang-bang controls, which is equivalent to verifying they hold for bang controls. Condition (i) holds by Lemma 10, so we only need to verify backward invariance relative to $\mathcal{K}$. We consider the following types of points of $\mathcal{S}^\star$: $(\mathcal{S}^\star)^\circ$, $\partial\mathcal{S}^\star \cap (\mathcal{K})^\circ$, and $\mathcal{S}^\star \cap \partial\mathcal{K}$. By Lemmas 13 and 15

$$\begin{aligned}
\mathcal{S}^\star \cap \partial\mathcal{K} &= (\mathcal{S}^\star \cap \partial W \cap \mathcal{S}) \cup (\mathcal{S}^\star \cap \partial\mathcal{S} \cap \mathcal{W}) \\
&= (\mathcal{S}^\star \cap \partial W \cap \mathcal{S}) \\
&= (\mathcal{S}^\star \cap \partial\mathcal{W}_{1e}) \cup (\mathcal{S}^\star \cap \partial\mathcal{W}_{1o}) \cup (\mathcal{S}^\star \cap \partial\mathcal{C})\,.
\end{aligned}$$

Thus, there are five cases.

(a) Let $x_0 \in (\mathcal{S}^\star)^\circ$. Then $T_{\mathcal{S}^\star} = \mathbb{R}^n$ and the result is immediate.

(b) Let $x_0 \in \partial\mathcal{S}^\star \cap \mathcal{S}^\circ \cap \mathcal{W}$. In light of condition (9), we must only verify that for all $i \in I^\star(x_0)$, $-(f(x_0)+g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0)$. Note that $h^\star(x_0) = \overline{h}_i(x_0) \geq 0$. Consider the bang trajectory ϕ_i which arrives at x_0 at $t=0$. On some interval $(-\delta, 0]$, $\delta > 0$, the segment $\phi_i|_{(-\delta,0]} \subset \mathcal{S}^\circ \cap \mathcal{W}$. Also, $h^\star(\phi_i(t,x_0)) \geq \overline{h}_i(\phi_i(t,x_0)) = 0$, for all $t \in (-\delta, 0]$. Therefore, for all $t \in (-\delta, 0]$, $\phi_i(t,x_0) \in \mathcal{S}^\star$. This implies, by the continuity of $f(x) + g(x)v^i$, that $-(f(x_0) + g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0)$.

(c) Let $x_0 \in \mathcal{S}^\star \cap \mathcal{C}$. Let $i \in I$ and consider the bang trajectory ϕ_i that arrives at x_0 in a finite time. First, consider the case when there exists a segment of ϕ_i that reaches $\mathcal{C}$ from $\mathcal{K}$ for the first time at x_0. That is, there exists $\delta > 0$ such that for all $t \in [-\delta, 0)$, $\phi_i(t,x_0) \in \mathcal{K} \setminus \mathcal{C}$. This means that $\overline{h}_i$ is defined and constant along the segment $\phi_i|_{[-\delta,0]}$. In particular, $\overline{h}_i(\phi_i(t,x_0)) = \overline{h}_i(x_0) = h(x_0) \geq 0$, $\forall t \in [-\delta, 0]$. Thus, $h^\star(\phi_i(t,x_0)) \geq 0$, for all $t \in [-\delta, 0]$, or $\phi_i|_{[-\delta,0]} \in \mathcal{S}^\star$. Therefore, by continuity of $f(x) + g(x)v^i$, $-(f(x_0) + g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0)$.
Second, suppose ϕ_i reaches $x_0 \in \mathcal{C}$ and there exists a sequence of times $\{\epsilon^k < 0\}$ with $\epsilon^k \to 0$ such that $\phi_i(\epsilon^k, x_0) \in \mathcal{C} \subset \mathcal{S}^*$. Then again we have $-(f(x_0) + g(x_0)v^i) \in T_{\mathcal{S}^*}(x_0)$.
Finally, suppose ϕ_i reaches $\mathcal{C}$ for the first time at x_0 and there exists a sequence of times $\{\epsilon^k < 0\}$ with $\epsilon^k \to 0$ such that $\phi_i(\epsilon^k, x_0) \in \neg\mathcal{K}$. Then $-(f(x_0) + g(x_0)v^i) \in T_{\neg\mathcal{K}}(x_0)$.

(d) Let $x_0 \in \mathcal{S}^\star \cap \partial\mathcal{W}_{1e}$. We must verify that for all $i \in I$, $-(f(x_0)+g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0) \cup T_{\neg\mathcal{K}}(x_0)$. By Lemma 14 and the continuity of $f(x)+g(x)u$ we immediately obtain that for all $u \in U$, $-(f(x_0) + g(x_0)u) \in T_{\neg\mathcal{K}}(x_0)$.

(e) Let $x_0 \in \mathcal{S}^\star \cap \partial\mathcal{W}_{1o}$. We must verify that for all $i \in I$, $-(f(x_0) + g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0) \cup T_{\neg\mathcal{K}}(x_0)$. We know there exists an odd number $2 < k(x_0) \leq r-1$ such that $L_f h(x_0) = \cdots = L_f^{(k(x_0)-1)} h(x_0) = 0$, and $L_f^{(k(x_0))} h(x_0) < 0$. It follows that there exists $\delta > 0$ sufficiently small such that for all $t \in (-\delta, 0)$ and for all controls $u(\cdot)$, $L_f^j h(\phi_u(t,x_0)) < 0$ for j odd and $1 \leq j \leq k(x_0)$. Also, $L_f^j h(\phi_u(t,x_0)) > 0$ for j even and $2 \leq j \leq k(x_0) - 1$. Moreover, because $h(x_0) \geq 0$, we also have $h(\phi_u(t,x_0)) > 0$, $\forall t \in (-\delta, 0)$, $\forall u$. Therefore, $\forall t \in (-\delta, 0)$, $\forall u$, $\phi_u(t,x_0) \in \mathcal{S}^\circ \cap \mathcal{W}$.
Now let $i \in I^\star(x_0)$ and note that $h^\star(x_0) \geq 0$. Since $\phi_i(t,x_0) \in \mathcal{S}^\circ \cap \mathcal{W}, \forall t \in (-\delta, 0)$, we have $h^\star(\phi_i(t,x_0)) \geq \overline{h}_i(\phi_i(t,x_0)) = \overline{h}_i(x_0) = h^\star(x_0) \geq 0$, for all $t \in (\delta, 0]$. Therefore, for all $t \in (-\delta, 0]$, $\phi_i(t,x_0) \in \mathcal{S}^\star$. This implies, by the continuity of $f(x) + g(x)v^i$, that $-(f(x_0) + g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0)$. Instead, suppose $i \notin I^\star(x_0)$. Then by (9), $-(f(x_0)+g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0)$.

■

Finally, we extend the previous results to show that $\mathcal{S}^\star$ is the viable capture basin even when measurable controls are permitted. The proof is adapted from Theorem 3.2.4, p. 85, in [1].

Theorem 18: We are given a system (1), a safe set (2), and a target set (3). Suppose that Assumptions 1, 4, and 7 hold. In addition, suppose that for all $x \in \partial\mathcal{S}^\star \cap \neg\mathcal{C}$ and for all $j \notin I^\star(x)$, (9) holds. Then $\mathcal{S}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$ and $u^\star$ is a viability controller.

Proof: We apply Theorem 8. By Lemma 10, condition (i) holds, so we only have to verify condition (ii). Cases (a) and (d) are already proved in Theorem 17.

(b) Let $x_0 \in \partial\mathcal{S}^\star \cap \mathcal{S}^\circ \cap \mathcal{W}$. In the proof of Theorem 17 we showed that for all $i \in I$, $-(f(x_0)+g(x_0)v^i) \in T_{\mathcal{S}^\star}(x_0)$. It follows by convexity that for all $x \in \partial\mathcal{S}^\star \cap \mathcal{S}^\circ \cap \mathcal{W}$,
$$-(f(x)+g(x)u) \in \overline{\text{co}}(T_{\mathcal{S}^\star}(x))\,, \qquad u \in U\,. \quad (10)$$
We must show $-(f(x_0) + g(x_0)u) \in T_{\mathcal{S}^\star}(x_0)$ for all $u \in U$. Fix $u_0 \in U$ and define $v_0 := -(f(x_0)+g(x_0)u_0)$. Let $v \in T_{\mathcal{S}^\star}(x_0)$ achieve the distance between v_0 and $T_{\mathcal{S}^\star}(x_0)$: $\|v_0 - v\| = \inf_{\xi \in T_{\mathcal{S}^\star}(x_0)} \|v_0 - \xi\|$. Let $w = \frac{v_0+v}{2}$. Since $v \in T_{\mathcal{S}^\star}(x_0)$ there exist sequences $\{h_n > 0\}$, $h_n \to 0$, and $\{v_n\}$, $v_n \to v$ such that $x_0 + h_n v_n \in \mathcal{S}^\star$ for all $n \geq 0$. Let $x_n \in \Pi_{\mathcal{S}^\star}(x_0 + h_n w)$ be the projection of $x_0 + h_n w$ onto $\mathcal{S}^\star$. Set $z_n := \frac{x_n - x_0}{h_n}$. Observe that $w - z_n = \frac{1}{h_n}(x_0 + h_n w - x_n)$ so by Proposition 3.2.3 of [1]
$$\langle w - z_n, \xi \rangle \leq 0\,, \qquad \xi \in T_{\mathcal{S}^\star}(x_n)\,. \quad (11)$$
Since $x_n \to x_0$, for $h_n > 0$ sufficiently small, $x_n \in \mathcal{S}^\star \cap \mathcal{S}^\circ \cap \mathcal{W}$. Therefore, by (10), $y_n := -(f(x_n)+g(x_n)u_0) \in \overline{\text{co}}(T_{\mathcal{S}^\star}(x_n))$. Using (11) and convexity,
$$\langle w - z_n, y_n \rangle \leq 0\,. \quad (12)$$
Since $x_n \to x_0$, we have $y_n \to v_0$. Also, by the same argument in [1], pp. 86-87, we have $z_n \to v$. Now consider again (12) and passing to the limit, we get $\langle w - v, v_0 \rangle \leq 0$. However, $w - v = \frac{v_0 - v}{2}$, so $\langle v_0 - v, v_0 \rangle \leq 0$. Since $T_{\mathcal{S}^\star}(x_0)$ is a cone and $v \in T_{\mathcal{S}^\star}(x_0)$ is the projection of v_0 onto $T_{\mathcal{S}^\star}(x_0)$, we also have $\langle v_0 - v, v \rangle = 0$. Therefore, $\|v_0 - v\|^2 = \langle v_0 - v, v_0 \rangle - \langle v_0 - v, v \rangle \leq 0$. We conclude $v_0 = v \in T_{\mathcal{S}^\star}(x_0)$.

(c,e) Let $x_0 \in (\mathcal{S}^\star \cap \mathcal{C}) \cup (\mathcal{S}^\star \cap \partial\mathcal{W}_{1o})$. From the proof of Theorem 17 we showed that for all $i \in I$, $-(f(x_0) + g(x_0)v^i) \in T_{\mathcal{S}^\star \cup \neg\mathcal{K}}(x_0)$. Furthermore, if we collect all of the results of Theorem 17, we note that for $x \in \neg\mathcal{K}$,

$T_{\neg\mathcal{K}}(x) = \mathbb{R}^n$, and we apply convexity, we obtain that for all $x \in \mathcal{S}^\star \cup \neg\mathcal{K}$,

$$-(f(x) + g(x)u) \in \overline{\text{co}}(T_{\mathcal{S}^\star \cup \neg\mathcal{K}}(x)), \qquad u \in U. \tag{13}$$

We must show $-(f(x_0)+g(x_0)u) \in T_{\mathcal{S}^\star \cup \neg\mathcal{K}}(x_0)$ for all $u \in U$. The proof is now the same as for (b) except that we work with the set $\mathcal{S}^\star \cup \neg\mathcal{K}$ (instead of $\mathcal{S}^\star$), and we invoke (13) (instead of (10)). We conclude $-(f(x_0) + g(x_0)u) \in T_{\mathcal{S}^\star \cup \neg\mathcal{K}}(x_0) = T_{\mathcal{S}^\star}(x_0) \cup T_{\neg\mathcal{K}}(x_0)$, for all $u \in U$.

■

IV. Example

We present an example adapted from [3] of managing a fishery. The model captures the effect of fishing activity on a prey-predator system. Let x_1 denote the population level of a prey species, let x_2 denote the population level of a predator species and let x_3 denote the effort expended by humans in fishing the predator species. We assume that in the absence of any predation, the prey population follows an exponential growth model with intrinsic growth rate $r_1 > 0$ (see [21]). Similarly, in the absence of any fishing activity, the predator population follows an exponential growth model with intrinsic growth rate $r_2 > 0$. We do not assume any carrying capacity limitations (see [21]) on either the prey or predator populations. The system model is given by

$$\begin{aligned}\dot{x}_1 &= (r_1 - x_2)x_1\\ \dot{x}_2 &= (r_2 - x_3)x_2\\ \dot{x}_3 &= u\end{aligned}$$

where $x \in \mathbb{R}^3$ and $U := [-1, 1] \subset \mathbb{R}$. Let $v^1 = -1$ and $v^2 = 1$. The viability problem is to keep the stock level of the prey above some positive level $c > 0$. We define $h(x) = x_1 - c$, so $L_f h(x) = (r_1 - x_2)x_1$ and $\mathcal{S} = \{x \in \mathbb{R}^3 \mid x_1 - c \geq 0\}$. Assumption 1 holds with $r = 3$, so $\mathcal{W} = \{\ x \in \mathbb{R}^3 \mid (r_1 - x_2)x_1 < 0\ \}$. If $x_0 \in \mathcal{S} \cap \mathcal{W}$, then $x_1(0) \geq c > 0$ and $x_2(0) > r_1 > 0$. Thus, we compute

$$\begin{aligned}\mathcal{C}^+ &= \{x\ :\ x_1 \geq c, x_2 \leq r_1, (r_1 - x_2)^2 x_1 - (r_2 - x_3)x_1 x_2 \geq 0\}\\ \mathcal{C} &= \{x\ :\ x_1 \geq c, x_2 = r_1, x_3 \geq r_2\}.\end{aligned}$$

Using the expression for $\mathcal{C}$ it can be easily verified that Assumption 4 holds with $u_p = 1$.

Define the functions $m_1(t) := \int_0^t e^{(r_2 - x_3(0))\tau + \frac{1}{2}\tau^2}\, d\tau$ and $m_2(t) := \int_0^t e^{(r_2 - x_3(0))\tau - \frac{1}{2}\tau^2}\, d\tau$. Note that these are expressible in terms of the error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. For constant values of u we have that

$$x_1(t) = \begin{cases} x_1(0)e^{r_1 t - x_2(0)m_1(t)}, & \text{if } u = v^1\\ x_1(0)e^{r_1 t - x_2(0)m_2(t)}, & \text{if } u = v^2. \end{cases} \tag{14}$$

$$x_2(t) = x_2(0)e^{(r_2 - x_3(0))t - \frac{1}{2}ut^2} \tag{15}$$

$$x_3(t) = ut + x_3(0). \tag{16}$$

To compute $\bar{t}_i$, we remark that for $u = \pm 1$ the set $\{x \in \mathbb{R}^3 \mid x_1 = 0\}$ is an asymptote of the system and hence the $x_1 = 0$ component of $\partial\overline{\mathcal{W}}$ cannot be reached in finite time. Therefore, we must consider (15) to determine if there exists a time $\bar{t}_i$ such that $x_2(\bar{t}_i) = r_1$. Substituting $x_2(\bar{t}_i) = r_1$ in (15) and solving for $\bar{t}_i$ we get

$$\bar{t}_1 = -(r_2 - x_3(0)) - \sqrt{(r_2 - x_3(0))^2 + 2\ln\frac{r_1}{x_2(0)}}. \tag{17}$$

$$\bar{t}_2 = (r_2 - x_3(0)) + \sqrt{(r_2 - x_3(0))^2 - 2\ln\frac{r_1}{x_2(0)}}. \tag{18}$$

The analysis shows that for $u = \pm 1$, the set of initial conditions in $\mathcal{S} \cap \overline{\mathcal{W}}$ that can reach $\mathcal{C}$ in finite time are:

$$\begin{aligned}\mathcal{X}_1 &= \left\{\ x \in \mathcal{S} \cap \overline{\mathcal{W}}\ \middle|\ x_3 \geq r_2 + \sqrt{-2\ln\frac{r_1}{x_2}}\ \right\}.\\ \mathcal{X}_2 &= \mathcal{S} \cap \overline{\mathcal{W}}.\end{aligned}$$

Finally, substituting (17) and (18) into the expression for h we get

$$\begin{aligned}\overline{h}_1(x_0) &= x_1(0)e^{r_1\bar{t}_1 - x_2(0)m_1(\bar{t}_1)} - c,\\ \overline{h}_2(x_0) &= x_1(0)e^{r_1\bar{t}_2 - x_2(0)m_2(\bar{t}_2)} - c.\end{aligned}$$

It can be shown [23] that

$$h^\star(x) = \overline{h}_2(x), \qquad \forall x \in \mathcal{S} \cap \overline{\mathcal{W}}.$$

Therefore, $u^\star = 1$. The final step of the design is to verify condition (9). For all $x \in \partial\mathcal{S}^\star \cap \mathcal{W}$, we have that $I^\star(x) = \{2\}$. Therefore, for all $x \in \mathcal{S} \cap \overline{\mathcal{W}}$, the boundary of the viable capture basin is given by $\overline{h}_2(x) = 0$ and since $\overline{h}_2$ is differentiable, condition (9) reduces to verifying that for all $x \in \partial\mathcal{S}^\star \cap \neg\mathcal{C}$,

$$\nabla\overline{h}_2(x) \cdot (f(x) + g(x)v^1) \leq 0.$$

We obtain

$$\nabla\overline{h}_2(x) \cdot (f(x) + g(x)v^1) = \left((r_1 - x_2) - (r_2 - x_3)x_2 m_2(\bar{t}_2)\right) 2c.$$

For $x \in \mathcal{S} \cap \overline{\mathcal{W}}$, we have that $x_2 \geq r_1 > 0$. Moreover, since $erf(\cdot)$ is an increasing function, the value of $m_2(\bar{t}_2)$ is always nonnegative (this is also obvious from the integral definition of $m_2(t)$). Therefore, if $(r_2 - x_3) \geq 0$ the result follows immediately. Now, if $(r_2 - x_3) < 0$, then

$$\begin{aligned}&(r_1 - x_2) - (r_2 - x_3)x_2 m_2(\bar{t}_2)\\ &\leq (r_1 - x_2) - (r_2 - x_3)x_2 \int_0^{\bar{t}_2} e^{(r_2 - x_3)\tau}\, d\tau\\ &= (r_1 - x_2) - (r_2 - x_3)x_2 \frac{1}{(r_2 - x_3)}\left(e^{(r_2 - x_3)\bar{t}_2} - 1\right)\\ &= (r_1 - x_2) + x_2\left(1 - e^{(r_2 - x_3)\bar{t}_2}\right)\\ &= r_1 - x_2 e^{(r_2 - x_3)\bar{t}_2}\\ &\leq r_1 - x_2 e^{(r_2 - x_3)\bar{t}_2 - \frac{1}{2}\bar{t}_2^2}\\ &= r_1 - x_2 e^{\ln\frac{r_1}{x_2}} = 0.\end{aligned}$$

Therefore condition (9) is satisfied.

Remark 19: In the fisheries example the viability problem reduces to increasing (or decreasing) the growth of a particular species, for which the obvious solution is to decrease (or increase) the effort spent on fishing the particular species. Therefore, it is fairly obvious what control strategy should be implemented to maintain viability. However, a viability analysis provides relevant information for policy development in the fisheries industry because the viability procedure provides a mathematical proof that can be used to *guarantee* the results of a particular policy. Moreover, the results provide an explicit characterization for the boundary of the viable capture basin ($h^\star(x) = 0$) which can be used to analyze the current status of the fishery (with respect to long term viability).

V. Conclusion

The paper proposes and solves a viability problem for control affine systems. The problem formulation is based on the notion of viable capture basins, it is shaped by the practical concern to be able to conclude execution of the viability controller in a finite time, and it is relevant in many nonlinear control applications of current interest. An explicit formula for the viability kernel and a viability controller are derived, and these formulas are shown to be valid using the Frankowska method, which provides the essential backward invariance condition to obtain the result. A next step would be to extend the results to multi-output systems.

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