On the Reach Control Indices of Affine Systems on Simplices ⋆

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Abstract: We study the reach control problem for affine systems on simplices, and the focus is on cases when it is known that the problem is not solvable by continuous state feedback. We examine from a geometric viewpoint the structural properties of the system which make continuous state feedbacks fail. This structure is encoded by so-called *reach control indices*, which are defined and developed in the paper.

1. INTRODUCTION

This paper continues the study of the *reach control problem* initiated in [4]. The overall concept of the problem and its setting were introduced in [5] and further developed in [6, 7, 10]. The significance of the problem stems from its capturing the essential features of any reachability problem for control systems: the presence of state constraints and the notion of trajectories reaching a goal in a guided and finite-time manner. The choice to use a simplex on which to formulate such problems is based on the simplex being the simplest geometric object defining bounded n-space. Therefore, a deep study of this problem will inform on related and more complex problems which have not been treated in a systematic, theoretical way.

In [4] it was shown that affine feedback and continuous state feedback are equivalent from the point of view of solvability of the reach control problem. The approach is based, fundamentally, on fixed point theory. The latter allows to deduce that continuous state feedbacks always generate closed-loop equilibria inside the simplex of interest when affine feedbacks do. The current paper departs from these findings, and using a geometric approach, we explore the system structure which gives rise to equilibria. This structure is encoded in so-called *reach control indices*. The goal of this paper is to elucidate these indices, and to show how they isolate closed-loop equilibria.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of $\mathcal{K}$ is $\mathcal{K}^c := \mathbb{R}^n \setminus \mathcal{K}$ and the closure is $\overline{\mathcal{K}}$. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). For a matrix $A \in \mathbb{R}^{n\times n}$, the notation $A \succ 0$ ($A \succeq 0$) means $a_{ij} > 0$ ($a_{ij} \geq 0$) for $1 \leq i, j \leq n$. Notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. Finally, co $\{p_1, p_2, \ldots\}$ denotes the convex hull of a set of points $p_i \in \mathbb{R}^n$.

2. BACKGROUND

Consider an n-dimensional simplex $\mathcal{S}$ with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex not contained). Let h_i, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Let $\mathcal{F}_0$ be the target set in $\mathcal{S}$. Also define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$ (thus, $I_0 = I$).

We consider the following affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + a + Bu\,, \qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$.

Problem 1. (Reach Control Problem (RCP)). Consider system (1) defined on $\mathcal{S}$. Find a feedback control $u(x)$ such that for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\epsilon > 0$ satisfying:

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$;
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$;
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \epsilon)$.

Definition 2. A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ with control type $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a control u of type $\mathbb{U}$ such that properties (i)-(iii) of Problem 1 hold. We write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by control type $\mathbb{U}$ if for every $x_0 \in \mathcal{S}$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$ with control of type $\mathbb{U}$.

Definition 3. The *invariance conditions* require that there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that:

$$h_j \cdot (Av_i + a + Bu_i) \leq 0\,, \quad i \in \{0, \ldots, n\}, \quad j \in I_i\,. \tag{2}$$

The invariance conditions (2) are suitable for affine feedback, but for continuous state feedback, the following stronger conditions must be used.

Definition 4. The *invariance conditions* for state feedback $u(x)$ require that for all $j \in I$ and $x \in \mathcal{F}_j$,

$$h_j \cdot (Ax + Bu(x) + a) \leq 0\,. \tag{3}$$

Lemma 5. ([6]). If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by a continuous state feedback $u(x)$, then $u(x)$ satisfies the invariance conditions (3).

For Problem 1 the following necessary and sufficient conditions have been established for the case of affine feedback.

Theorem 6. [7, 10] Given the system (1) and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \ldots, u_n = u(v_n)$, the closed-loop system satisfies $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ if and only if

(a) The invariance conditions (2) hold.

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(b) There is no equilibrium in $\mathcal{S}$.

It has been shown in [6] that the invariance conditions are also necessary for solvability of RCP by continuous state feedback. In [4] geometric sufficient conditions were identified for solvability of RCP. To understand these conditions, we introduce some geometric constructs.

Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define the closed, convex cone $\mathcal{C}_i$ at $v_i \in V$ by

$$\mathcal{C}_i := \{\ y \in \mathbb{R}^n\ :\ h_j \cdot y \le 0,\ j \in I_i\ \}.$$

Also define

$$\mathrm{cone}(\mathcal{S}) := \mathcal{C}_0 = \mathrm{cone}\{v_1 - v_0, \dots, v_n - v_0\}\,.$$

Next, define

$$\mathcal{O} := \{\ x \in \mathbb{R}^n\ :\ Ax + a \in \mathcal{B}\}\,.$$

It is fairly easy to prove that $\mathcal{O} = \emptyset$ when $\mathrm{Im}(A) \subseteq \mathcal{B}$ and $a \notin \mathcal{B}$; $\mathcal{O} = \mathbb{R}^n$ when $\mathrm{Im}(A) \subseteq \mathcal{B}$ and $a \in \mathcal{B}$; and $\mathcal{O}$ is an affine space, otherwise. Notice that the vector field $Ax + Bu + a$ on $\mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. Define

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O}\,.$$

Associated with $\mathcal{G}$ is its vertex index set $I_{\mathcal{G}} := \{i \mid v_i \in V \cap \mathcal{G}\}$. In the remainder of the paper we make an important assumption concerning the placement of $\mathcal{O}$ with respect to $\mathcal{S}$. The reader is referred to [4] for the motivation for and a method of triangulation of the state space that achieves it.

Assumption 7. Simplex $\mathcal{S}$ and system (1) satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \le \kappa \le n$.

The geometric sufficient conditions of [4] place focus on the relationship between $\mathcal{B}$, $\mathrm{cone}(\mathcal{S})$, and $\mathcal{G}$, and several simple cases in which affine feedbacks exist were identified.

Theorem 8. ([4]). Suppose $\mathcal{G} = \emptyset$. If the invariance conditions are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

Theorem 9. ([4]). Suppose Assumption 7 holds and $\mathcal{G} \neq \emptyset$. If the invariance conditions are solvable and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq \mathbf{0}$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

The primary conclusion of [4] is the following.

Theorem 10. ([4]). Suppose Assumption 7 holds. Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by continuous state feedback if and only if $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

The goal of this paper is to take steps toward solving RCP in cases where continuous state feedback cannot be used. We consider the following assumptions.

Assumption 11. Simplex $\mathcal{S}$ and system (1) satisfy the following conditions.

(A1) $\mathcal{G} = \mathrm{co}\ \{v_1, \dots, v_{\kappa+1}\}$, with $0 \le \kappa < n$.
(A2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
(A3) The maximum number of linearly independent vectors in any set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ is $m^\star$ with $1 \le m^\star \le \kappa + 1$.

Assumption (A1) rules out the application of Theorem 8, and it enforces that $v_0 \notin \mathcal{O}$. The latter requirement is because it was shown in [4] that when $v_0 \in \mathcal{O}$ and (A2) holds, then RCP is not solvable by continuous state feedback; this appears to extend to other classes of controls without difficulty. Assumption (A2) rules out the application of Theorem 9. Finally, (A3) introduces a new condition in terms of the variable $m^\star$, which necessarily satisfies $m^\star \le \kappa + 1$. When $\kappa = m^\star - 1$, an affine feedback solves RCP, as stated below. The remaining cases when $\kappa \ge m^\star$ are the topic of this paper.

Theorem 12. ([4]). Suppose Assumption 11 holds. If the invariance conditions are solvable and $m^\star = \kappa + 1$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

We conclude the background discussion with a brief review of $\mathscr{M}$-matrices. Consider the following family of matrices. Let $1 \le p \le q \le \kappa + 1$ and define

$$M_{p,q} := \begin{bmatrix} (h_p \cdot b_p) & (h_p \cdot b_{p+1}) & \cdots & (h_p \cdot b_q) \\ \vdots & \vdots & & \vdots \\ (h_q \cdot b_p) & (h_q \cdot b_{p+1}) & \cdots & (h_q \cdot b_q) \end{bmatrix} \qquad (4)$$

where $M_{p,q} \in \mathbb{R}^{(q-p+1)\times(q-p+1)}$. Define the matrices

$$H_{p,q} := [h_p \cdots h_q] \in \mathbb{R}^{n\times(q-p+1)}\,,$$
$$Y_{p,q} := [b_p \cdots b_q] \in \mathbb{R}^{n\times(q-p+1)}\,.$$

Then

$$M_{p,q} = H_{p,q}^T Y_{p,q}\,.$$

We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \le 0$ for all $i \neq j$ [2]. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix. Also under the condition that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$, $M_{p,q}$ adopts further algebraic properties. In particular, we require the notion of an $\mathscr{M}$-matrix. The following theorem characterizes non-singular $\mathscr{M}$-matrices (see [2], Ch. 6).

Theorem 13. Let $M \in \mathbb{R}^{k\times k}$ be a $\mathscr{Z}$-matrix. Then the following are equivalent:

(i) M is a non-singular $\mathscr{M}$-matrix.
(ii) $\Re(\lambda) > 0$ for all eigenvalues λ of M.
(iii) There exists a vector $\xi \succeq 0$ in $\mathbb{R}^k$ such that $M\xi \succ 0$.
(iv) The inequalities $y \succeq 0$ and $My \preceq 0$ have only the trivial solution $y = 0$, and M is non-singular.
(v) M is monotone; that is, $My \succeq 0$ implies $y \succeq 0$ for all $y \in \mathbb{R}^k$.
(vi) M is nonsingular and M^{-1} is a non-negative matrix.

Lemma 14. ([4]). Suppose $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$. Let $1 \le p \le q \le \kappa+1$ and suppose $\{b_p, \dots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ are linearly independent. Then $M_{p,q}$ is a non-singular $\mathscr{M}$-matrix.

3. SOME PRELIMINARIES

Assumption 11 specifies there is a maximal set of linearly independent vectors in $\mathcal{B}$ available to vertices in $\mathcal{G}$ in terms of the variable $m^\star$. Let

$$\tilde{\mathcal{B}} := \mathrm{sp}\{b_1, \dots, b_{m^\star}\}\,.$$

Condition (A3) implies that for all $v_i \in \mathcal{G}$,

$$\mathcal{B} \cap \mathcal{C}_i \subset \tilde{\mathcal{B}} \cap \mathcal{C}_i\,. \qquad (5)$$

We show that $\mathcal{B}$ can be restricted to $\tilde{\mathcal{B}}$ without affecting solvability of RCP. Thus, a distinction between $m^\star$ and $m = \mathrm{rank}(B)$ is not needed. To that end, suppose we have a simplex $\mathcal{S}$ and a system (1) such that Assumption 11 holds, $m^\star < m$, and the invariance conditions are solvable. We construct a state feedback transformation

$$u = K_1 x + g_1 + G_1 w$$

where $w \in \mathbb{R}^m$ is a new exogenous input. This yields a new system

$$\dot{x} = (A+BK_1)x+(a+Bg_1)+BG_1 w =: \tilde{A}x+\tilde{a}+\tilde{B}w\,. \quad (6)$$

The parameters K_1, g_1, and G_1 are selected so that

$$\tilde{A}v_i + \tilde{a} = 0\,, \qquad i \in I_{\mathcal{G}} \quad (7)$$

$$\tilde{A}v_i + \tilde{a} \in \mathcal{C}_i \setminus \mathcal{B}\,, \qquad i \notin I_{\mathcal{G}} \quad (8)$$

$$\tilde{B} = [\, b_1 \;\, b_2 \;\, \cdots \;\, b_{m^\star} \,]\,. \quad (9)$$

The columns of $\tilde{B}$ are given by (A3) and $\mathrm{rank}(\tilde{B}) = m^\star$. Note that when $w = 0$, the invariance conditions are satisfied at $v_i \notin \mathcal{G}$ and equilibria appear at $v_i \in \mathcal{G}$. The next result shows that the proposed state feedback transformation does not affect $\mathcal{O}$.

Lemma 15. $\mathcal{O} = \tilde{\mathcal{O}} := \{\, x \in \mathbb{R}^n \;:\; \tilde{A}x + \tilde{a} \in \tilde{\mathcal{B}}\}$.

Proof. Let $v_i \in \mathcal{O}$. Then $\tilde{A}v_i + \tilde{a} = 0$, so $\tilde{A}v_i + \tilde{a} \in \tilde{\mathcal{B}}$. Thus, $v_i \in \tilde{\mathcal{O}}$. Conversely, let $v_i \in \tilde{\mathcal{O}}$. Then $\tilde{A}v_i + \tilde{a} \in \tilde{\mathcal{B}} \subset \mathcal{B}$. Now $\tilde{A}v_i + \tilde{a} = Av_i + a + BK_1 v_i + Bg_1$, so $Av_i + a = \tilde{A}v_i + \tilde{a} - B(K_1 v_i + g_1) \in \mathcal{B}$. Therefore, $v_i \in \mathcal{O}$. The result now follows by taking the affine hull of $\{v_1, \ldots, v_{\kappa+1}\}$. $\square$

We claim it is no loss of generality to work with the reduced system, assuming one has fixed an appropriate class of controls. First, if we find $w = f(x)$ such that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ for the reduced system, then we have a solution for the original system (1) by setting

$$u = K_1 x + g_1 + G_1 f(x)\,.$$

For the converse statement we focus on affine feedback; by the results of [4], the argument extends to continuous state feedback.

Proposition 16. If there exists $u = Kx + g$ such that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ for (1), then there exist $u = K_1 x + g_1 + G_1 w$ and $w = K_2 x + g_2$ such that (7)-(9) hold and $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ for (6).

Proof. Let $u_i := Kv_i + g$ for $v_i \in V$. Also fix G_1 such that $\tilde{B} = BG_1$. In the first step we show that for each $v_i \in V$, there exist u_i' and w_i such that

$$Av_i + a + Bu_i = Av_i + a + Bu_i' + \tilde{B}w_i\,,$$

and (7)-(8) hold. First consider $v_i \in \mathcal{G}$. Since $Av_i + a \in \mathcal{B}$, there exists $u_i' \in \mathbb{R}^m$ such that

$$Av_i + a + Bu_i' = 0\,.$$

Set $q_i = u_i - u_i'$. Invoking Theorem 6(a) and (5), we have

$$Av_i + a + Bu_i = Av_i + a + B(u_i' + q_i) = Bq_i \in \mathcal{B} \cap \mathcal{C}_i \subset \tilde{\mathcal{B}} \cap \mathcal{C}_i\,.$$

This implies there exists w_i such that $Bq_i = \tilde{B}w_i$. Therefore we obtain

$$Av_i + a + Bu_i = Av_i + a + Bu_i' + \tilde{B}w_i\,,$$

with $Av_i + a + Bu_i' = 0$. Next, consider $v_i \notin \mathcal{G}$. Then $Av_i + a \notin \mathcal{B}$ and $Av_i + a + Bu_i \in \mathcal{C}_i \setminus \mathcal{B}$. This can be written as

$$Av_i + a + Bu_i = Av_i + a + Bu_i' + \tilde{B}w_i$$

with $u_i' := u_i$ and $w_i := 0$.

In the second step, the affine feedback $u = K_1 x + g_1$ can be obtained by solving the system of equations

$$u_i' = K_1 v_i + g_1\,, \qquad i = 0, \ldots, n$$

for unknowns K_1 and g_1. See [6]. Because of the construction above and by convexity, (7)-(8) easily follow. Similiarly, $w = K_2 x + g_2$ is obtained by solving

$$w_i = K_2 v_i + g_2\,, \qquad i = 0, \ldots, n$$

for unknowns K_2 and g_2. In the final step, we observe that the right-hand sides of the closed-loop systems (1) and (6) are identical, so $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ for (6). $\square$

4. REACH CONTROL INDICES

In light of Lemma 5 and Proposition 16, we make the following standing assumptions.

Assumption 17. Simplex $\mathcal{S}$ and system (1) satisfy the following conditions.

- (R1) $\mathcal{G} = \mathrm{co}\,\{v_1, \ldots, v_{\kappa+1}\}$, where $m \le \kappa < n$.
- (R2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
- (R3) $\mathcal{B} = \mathrm{sp}\{b_1, \ldots, b_m \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$.
- (R4) $\mathcal{B} \cap \mathcal{C}_i \neq 0\,, \quad i \in I_{\mathcal{G}}$.

From condition (R3) we have a preferred set of vectors $\{b_1, \ldots, b_m \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ which span $\mathcal{B}$. Of course there are many possible choices for this basis, but we assume for the following development that these vectors have been fixed. By condition (R1), we have

$$\kappa + 1 = m + p$$

for some $p \ge 1$. Using (R4), if we select any set $\{b_{m+1}, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ and we use $\{b_1, \ldots, b_m\}$ as in (R3), then p denotes the number of linearly dependent vectors in the set $\{b_1, \ldots, b_{\kappa+1}\}$. Consider now $\mathcal{B} \cap \mathcal{C}_{m+1}$. It satisfies $b_{m+1} \in \mathrm{sp}\{b_1, \ldots, b_m\}$ for every $b_{m+1} \in \mathcal{B} \cap \mathcal{C}_{m+1}$. Using (R4), there exists $1 \le \rho_1 \le m$ such that w.l.o.g. (reordering indices $1, \ldots, m$),

$$\mathcal{B} \cap \mathcal{C}_{m+1} \subset \mathrm{sp}\{b_1, \ldots, b_{\rho_1}\}$$

and $\mathrm{sp}\{b_1, \ldots, b_{\rho_1}\}$ is the smallest such subspace generated by the basis $\{b_1, \ldots, b_m\}$. One can think of $\mathrm{sp}\{b_1, \ldots, b_{\rho_1}\}$ as a *container subspace* for the cone $\mathcal{B} \cap \mathcal{C}_{m+1}$, and the dimension of $\mathcal{B} \cap \mathcal{C}_{m+1}$ is between 1 and ρ_1.

This argument can be repeated for each of the remaining cones $\mathcal{B} \cap \mathcal{C}_i$, $i = m+2, \ldots, m+p$ to obtain indices $\pi_k \le \rho_k$, $k = 1, \ldots, p$, with $\pi_1 = 1$, such that

$$\mathcal{B} \cap \mathcal{C}_{m+1} \subset \mathrm{sp}\{b_{\pi_1}, \ldots, b_{\rho_1}\}\,, \quad (10)$$

$$\mathcal{B} \cap \mathcal{C}_{m+2} \subset \mathrm{sp}\{b_{\pi_2}, \ldots, b_{\rho_2}\}\,, \quad (11)$$

$$\vdots$$

$$\mathcal{B} \cap \mathcal{C}_{m+p} \subset \mathrm{sp}\{b_{\pi_p}, \ldots, b_{\rho_p}\}\,. \quad (12)$$

Moreover, for the fixed basis $\{b_1, \ldots, b_m\}$, these subspaces represent the minimal subspaces that contain the respective cones. This has the interpretation that every basis vector b_j in a container subspace contributes to generating at least one vector in the associated cone $\mathcal{B} \cap \mathcal{C}_k$. The following result establishes that one can always find a vector in each cone which depends all on the basis vectors of its container subspace.

Lemma 18. Suppose Assumption 17 and (10)-(12) hold. For each $k = 1, \ldots, p$, there exists $b_{m+k} \in \mathcal{B} \cap \mathcal{C}_{m+k}$ such that

$$b_{m+k} = c_{\pi_k} b_{\pi_k} + \cdots + c_{\rho_k} b_{\rho_k} , \qquad c_i \neq 0 . \tag{13}$$

Proof. We consider only the case $k = 1$; the other cases follow by reordering indices $k = 1, \ldots, p$. Suppose w.l.o.g. that for all $b \in \mathcal{B} \cap \mathcal{C}_{m+1}$,

$$b = c_1 b_1 + \cdots + c_{\rho_1 - 1} b_{\rho_1 - 1} .$$

That is, no $b \in \mathcal{B} \cap \mathcal{C}_{m+1}$ depends on b_{ρ_1}. Then $\mathcal{B} \cap \mathcal{C}_{m+1} \subset \mathrm{sp}\{b_1, \ldots, b_{\rho_1 - 1}\}$, which contradicts the definition of ρ_1. Therefore, for each $i \in \{1, \ldots, \rho_1\}$ there exists $\beta_i \in \mathcal{B} \cap \mathcal{C}_{m+1}$ such that

$$\beta_i = \mu_1^i b_1 + \cdots + \mu_{\rho_1}^i b_{\rho_1}$$

with $\mu_i^i \neq 0$. Now we show inductively that b satisfying (13) exists. Let $j = 1$. Set

$$b = \beta_1 \in \mathcal{B} \cap \mathcal{C}_{m+1} .$$

Then b satisfies $c_1 := \mu_1^1 \neq 0$. Next suppose there exists $b \in \mathcal{B} \cap \mathcal{C}_{m+1}$ such that

$$b = c_1 b_1 + \cdots + c_j b_j + \cdots + c_{\rho_1} b_{\rho_1} ,$$

and $c_i \neq 0$, for $i = 1, \ldots, j$. Moreover, by reordering indices $j+1, \ldots, \rho_1$, it can be assumed that $c_{j+1} = 0$ (for if this is not possible, then b satisfies (13) and the induction is finished). Let $\overline{c} := \min_{i \in \{1, \ldots, j\}} |c_i| > 0$. Consider $\beta_{j+1} = \mu_1^{j+1} b_1 + \cdots + \mu_{\rho_1}^{j+1} b_{\rho_1}$, with $\mu_{j+1}^{j+1} \neq 0$. We choose $\alpha \in (0, 1)$ such that

$$\alpha |\mu_i^{j+1}| < \overline{c} , \qquad i = 1, \ldots, j . \tag{14}$$

Now let

$$b' := b + \alpha \beta_{j+1} \in \mathcal{B} \cap \mathcal{C}_{m+1} .$$

Then $b' = c_1' b_1 + \cdots + c_{\rho_1}' b_{\rho_1}$, where, using (14),

$$c_i' = c_i + \alpha \mu_i^{j+1} \neq 0 , \qquad i = 1, \ldots, j$$
$$c_{j+1}' = \alpha \mu_{j+1}^{j+1} \neq 0 .$$

Therefore b' replaces b and the induction step is complete. □

The dependency of cones on a limited number of vectors in $\mathcal{B}$ places restrictions on the orientation of those vectors with respect to $\mathcal{S}$.

Lemma 19. Suppose Assumption 17 and (10)-(12) hold. For each $k = 1, \ldots, p$ and for any $b_{m+k} \in \mathcal{B} \cap \mathcal{C}_{m+k}$, we have

$$h_j \cdot b_i = 0 , \quad i = \pi_k, \ldots, \rho_k, m+k , \quad j = I \setminus \{\pi_k, \ldots, \rho_k, m+k\} \tag{15}$$

Proof. We consider only the case $k = 1$; the other cases follow by reordering indices $k = 1, \ldots, p$. Also let $\rho := \rho_1$. Following Lemma 18, select $b_{m+1} \in \mathcal{B} \cap \mathcal{C}_{m+1}$ such that

$$b_{m+1} = c_1 b_1 + \cdots + c_\rho b_\rho , \qquad c_i \neq 0 .$$

Define $c := (c_1, \ldots, c_\rho)$. Since $\{b_1, \ldots, b_\rho\}$ are linearly independent and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$, by Lemma 14, $M_{1,\rho}$ is a non-singular $\mathscr{M}$-matrix. Consider the following invariance conditions for v_{m+1}:

$$H_{1,\rho}^T b_{m+1} = H_{1,\rho}^T Y_{1,\rho} c = M_{1,\rho} c \preceq 0 .$$

By Theorem 13(v) and the fact that $c_i \neq 0$, we obtain $c \prec 0$. Now consider the remaining invariance conditions for v_{m+1}:

$$h_j \cdot b_{m+1} = h_j \cdot (c_1 b_1 + \cdots + c_\rho b_\rho) \leq 0 , \quad j = \rho+1, \ldots, m, m+2, \ldots, n .$$

Every term in the sum is non-negative, since $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and $c_i < 0$, and so we obtain

$$h_j \cdot b_i = 0 , \quad i = 1, \ldots, \rho, m+1 , j = \rho+1, \ldots, m, m+2, \ldots, n$$

as desired. □

Using the restrictions placed in $\mathcal{B}$ by Lemma 19, it can be shown that the container subspaces in (10)-(12) split into independent subspaces, thus yielding a decomposition of $\mathcal{B}$ with respect to $\mathcal{G}$.

Lemma 20. Suppose Assumption 17 and (10)-(12) hold. Then there exists an ordering of indices $\{1, \ldots, m\}$ such that

$$\rho_k < \pi_{k+1} , \qquad k = 1, \ldots, p-1 .$$

Proof. We consider only the case $k = 1$; the other cases follow by reordering indices. Suppose by way of contradiction that $\rho_1 \geq \pi_2$. Applying Lemma 19 we obtain

$$h_j \cdot b_i = 0 , \quad i = 1, \ldots, \rho_1 , \quad j = \rho_1 + 1, \ldots, m, m+2, \ldots, n \tag{16}$$
$$h_j \cdot b_i = 0 , \quad i = \pi_2, \ldots, \rho_2 , j = 1, \ldots, \pi_2 - 1, \rho_2 + 1, \ldots, m+1, m+3, \ldots, n . \tag{17}$$

Consider

$$M_{\pi_2,\rho_1} = H_{\pi_2,\rho_1}^T Y_{\pi_2,\rho_1} ,$$

where $Y_{\pi_2,\rho_1} = [b_{\pi_2} \cdots b_{\rho_1}]$. Since $\{b_{\pi_2}, \ldots, b_{\rho_1}\}$ are linearly independent, by Lemma 14, M_{π_2,ρ_1} is a non-singular $\mathscr{M}$-matrix. Applying Theorem 13(iii) there exists $c' = (c_{\pi_2}', \ldots, c_{\rho_1}')$ such that $c' \preceq 0$ and $M_{\pi_2,\rho_1} c' \prec 0$. Define $\beta := Y_{\pi_2,\rho_1} c' \neq 0$. The statement $H_{\pi_2,\rho_1}^T \beta = M_{\pi_2,\rho_1} c' \prec 0$ gives

$$h_j \cdot \beta < 0 , \qquad j = \pi_2, \ldots, \rho_1 .$$

Also, by (16)-(17), for $j = 1, \ldots, \pi_2 - 1, \rho_1 + 1, \ldots, n$

$$h_j \cdot \beta = h_j \cdot \left(c_{\pi_2}' b_{\pi_2} + \cdots + c_{\rho_1}' b_{\rho_1} \right) = 0 .$$

This implies $\beta \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$. By Assumption (R2) we obtain $\beta = 0$, a contradiction. □

We can show that the decomposition obtained so far extends to all cones $\mathcal{B} \cap \mathcal{C}_i$ associated with $\mathcal{G}$. This is based on the observation that the selection of basis vectors $\{b_1, \ldots, b_m\}$ is arbitrary. For example $\{b_2, \ldots, b_{m+1}\}$, with b_{m+1} as in Lemma 18, could have been used as a basis for $\mathcal{B}$ and all foregoing results would apply. In recognition of the fact that no special role is played by vertices $v_{m+1}, \ldots, v_{m+p}$, we define a new set of indices which place dependent cones within consecutively numbered container subspaces. To do so, we first reorder indices so that the container subspaces have basis elements which are consecutively numbered:

$$\{b_1, \ldots, b_{\rho_1}\} , \{b_{\rho_1+1}, \ldots, b_{\rho_2}\} , \ldots , \{b_{\rho_{p-1}+1}, \ldots, b_{\rho_p}\} .$$

Now adjoint to these lists the vectors b_{m+k}, $k = 1, \ldots, p$, obtained from Lemma 18, to get the new lists

$$\{b_1, \ldots, b_{\rho_1}, b_{m+1}\} , \{b_{\rho_1+1}, \ldots, b_{\rho_2}, b_{m+2}\} , \ldots , \{b_{\rho_{p-1}+1}, \ldots, b_{\rho_p}, b_{m+p}\} .$$

Finally, define new indices so that also these lists are consecutive. That is, there exist indices $r_1, \ldots, r_p$ such that the previous list becomes

$$\{b_1, \ldots, b_{r_1}\} , \{b_{r_1+1}, \ldots, b_{r_1+r_2}\} , \ldots , \{b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_r\} ,$$

where

$$r := r_1 + \cdots + r_p\,.$$

Using (10)-(12) and Lemma 20 with our new numbering, we get

$$\mathcal{B} \cap \mathcal{C}_{r_1} \subset \mathrm{sp}\{b_1, \ldots, b_{r_1}\}\,, \tag{18}$$

$$\mathcal{B} \cap \mathcal{C}_{r_1+r_2} \subset \mathrm{sp}\{b_{r_1+1}, \ldots, b_{r_1+r_2}\}\,, \tag{19}$$

$$\vdots$$

$$\mathcal{B} \cap \mathcal{C}_{r_1+\cdots+r_p} \subset \mathrm{sp}\{b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_r\}\,. \tag{20}$$

Theorem 21. Suppose Assumption 17 holds. Then there exist integers $r_1, \ldots, r_p \geq 0$ and a decomposition of $\mathcal{B}$ into p subsets such that

$$\mathcal{B} \cap \mathcal{C}_i \;\subset\; \mathrm{sp}\{b_1, \ldots, b_{r_1}\}\,, i = 1, \ldots, r_1\,, \tag{21}$$

$$\mathcal{B} \cap \mathcal{C}_i \;\subset\; \mathrm{sp}\{b_{r_1+1}, \ldots, b_{r_1+r_2}\}\,, i = r_1+1, \ldots, r_1+r_2 \tag{22}$$

$$\vdots$$

$$\mathcal{B} \cap \mathcal{C}_i \;\subset\; \mathrm{sp}\{b_{r_1+\cdots+r_{p-1}+1}, \ldots, b_r\}\,, \quad i = r_1 + \cdots + r_{p-1}+1, \ldots, r\,. \tag{23}$$

The proof relies on a preliminary lemma proved only for $k = 1$.

Lemma 22. Suppose Assumption 17 and (18)-(20) hold. There does not exist $\beta \in \mathrm{sp}\{b_{r_1+1}, \ldots, b_{m+p}\}$, $\beta \neq 0$, such that $\{b_1, \ldots, b_{r_1-1}, \beta\}$ are linearly independent and

$$h_j \cdot \beta \leq 0\,, \qquad j = r_1+1, \ldots, n\,. \tag{24}$$

Proof. Since $\{b_1, \ldots, b_{r_1-1}\}$ are linearly independent and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$, by Lemma 14, M_{1,r_1-1} is a non-singular $\mathscr{M}$-matrix. Applying Theorem 13(iii) there exists $c' = (c'_1, \ldots, c'_{r_1-1})$ such that $c' \preceq 0$ and $M_{1,r_1-1}c' \prec 0$. Define $b'_{r_1} := Y_{1,r_1-1}c'$. The vector $H^T_{1,n} b'_{r_1} \in \mathbb{R}^n$ has the following sign pattern:

$$(-, \ldots, -, *, 0, \ldots, 0) \tag{25}$$

where the $*$ appears in the (r_1)th component and the zero components are due to Lemma 19. In particular $b'_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}$ and the first $r_1 - 1$ invariance conditions are strictly negative. Now suppose we find a non-zero vector $\beta \in \mathrm{sp}\{b_{r_1+1}, \ldots, b_{m+p}\}$ such that (24) holds and $\{b_1, \ldots, b_{r_1-1}, \beta\}$ are linearly independent. Then for $\alpha > 0$ we can form

$$b''_{r_1} := b'_{r_1} + \alpha\beta\,.$$

Using (25) and (24), α can be selected sufficiently small so that $h_j \cdot b''_{r_1} \leq 0$ for all $j = 1, \ldots, r_1-1, r_1+1, \ldots, n$. That is, $b''_{r_1} \in \mathcal{B} \cap \mathcal{C}_{r_1}$. Moreover, with $\beta \neq 0$,

$$\{b_1, \ldots, b_{r_1-1}, b''_{r_1}\}$$

is a linearly independent set. This contradicts (18) in which b_{r_1} depends on $\{b_1, \ldots, b_{r_1-1}\}$. ☐

Proof. [Proof of Theorem 21] We consider only $k = 1$. We must show $\mathcal{B} \cap \mathcal{C}_i \subset \mathrm{sp}\{b_1, \ldots, b_{r_1}\}$, for $i = 1, \ldots, r_1$. (By (18) we already have $\mathcal{B} \cap \mathcal{C}_{r_1} \subset \mathrm{sp}\{b_1, \ldots, b_{r_1-1}\}$). Consider any $i \in \{1, \ldots, r_1\}$ and any $\beta_i \in \mathcal{B} \cap \mathcal{C}_i$ such that

$$\beta_i = c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta$$

where $c_i \in \mathbb{R}$ and $\beta \in \mathrm{sp}\{b_{r_1+1}, \ldots, b_{m+p}\}$. W.l.o.g. assume that β is independent of $\{b_1, \ldots, b_{r_1-1}\}$ (otherwise the c_i's can be redefined so that $\beta = 0$ and the result is immediately obtained). From the invariance conditions associated with v_i and by Lemma 19, we have

$$h_j \cdot \beta_i = h_j \cdot (c_1 b_1 + \cdots + c_{r_1} b_{r_1} + \beta) = h_j \cdot \beta \leq 0\,,$$

for $j = r_1+1, \ldots, n$. By Lemma 22, $\beta = 0$. Hence, for any $i \in \{1, \ldots, r_1\}$ and any $\beta_i \in \mathcal{B} \cap \mathcal{C}_i$, $\beta_i \in \mathrm{sp}\{b_1, \ldots, b_{r_1}\}$, as desired. ☐

The lists in (21)-(23) have the property that any vector in a list on the right is dependent on all the other vectors in its list. Also, if any vector is removed from a list, the remaining vectors are linearly independent. In particular, the kth list has $r_k - 1$ linearly independent vectors of $\mathcal{B}$. We can say that $\mathcal{B}$ has been decomposed into p independent *cycles of dependency.* Also, because of (R4), $p \leq m$. Thus, in order for (R4) to hold it is necessary that

$$m \geq \frac{\kappa + 1}{2}\,.$$

This condition is interpreted to say that RCP is only solvable if there are sufficient inputs, assuming that (R4) is necessary for solvability. Since each of the p lists comprises $r_k - 1$ independent vectors in $\mathcal{B}$ and there are a total of m such vectors, we deduce that

$$r - p \leq m\,.$$

Definition 23. The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (1) with respect to simplex $\mathcal{S}$.

From the previous developments, it is straightforward to show the following.

Proposition 24. The reach control indices are: (i) invariant under affine feedback transformations $u = Kx + g + w$; (ii) unique up to ordering of indices; and (iii) independent of choice of maximal linearly independent set in $\{b_i \in \mathcal{B} \cap \mathcal{C}_i, i \in I_{\mathcal{G}}\}$.

The importance of the reach control indices stems from their ability to isolate closed-loop equilibria when using continuous state feedback. Define for $k = 1, \ldots, p$

$$m_k := r_1 + \cdots + r_{k-1} + 1\,,$$

$$\widehat{\mathcal{S}}_k := \mathrm{co}\left\{v_{m_k}, \ldots, v_{m_k+r_k-1}\right\}.$$

The following lemma gives the direct consequence of Lemma 22 in the case of continuous state feedback.

Lemma 25. Suppose Assumption 17 holds. Let $u(x)$ be a continuous state feedback satisfying the invariance conditions (3). Then for each $k = 1, \ldots, p$ and $x \in \widehat{\mathcal{S}}_k$,

$$h_j \cdot y(x) = 0\,, \qquad j \in I \setminus \{m_k, \ldots, m_k + r_k - 1\}\,.$$

Proof. We prove the result only for $k = 1$. Let $y(x) = Ax + Bu(x) + a$ be the closed-loop vector field on $\mathcal{S}$ satisfying the invariance conditions (3). For $x \in \widehat{\mathcal{S}}_1$, we have

$$y(x) = c_1(x) b_1 + \cdots + c_{r_1}(x) b_{r_1} + \beta(x)\,, \tag{26}$$

where $\beta(x) \in \mathrm{sp}\{b_{r_1+1}, \ldots, b_{m+p}\}$. We may assume w.l.o.g. (as in the proof of Theorem 21) that $\beta(x)$ is independent of $\{b_1, \ldots, b_{r_1}\}$. From (3) we know

$$h_j \cdot y(x) \leq 0\,, \qquad j = r_1+1, \ldots, n\,.$$

Using (26) and Lemma 19, these conditions become

$$h_j \cdot \beta(x) \leq 0\,, \qquad j = r_1+1, \ldots, n\,.$$

By Lemma 22, this implies $\beta(x) = 0$. Therefore, for each $x \in \widehat{\mathcal{S}}_1$,

$$h_j \cdot y(x) = 0 \,, \qquad j = r_1 + 1, \ldots, n \,.$$

□

Theorem 26. Suppose Assumption 17 holds. Let $u(x)$ be a continuous state feedback satisfying the invariance conditions (3). Then each $\widehat{\mathcal{S}}_k$ contains an equilibrium of the closed-loop system.

A more restrictive version of this result was proved in [4] based on Sperner's lemma. The present proof is a standard argument based on Brouwer fixed point theorem [3].

Proof. We consider only $k = 1$; the other cases follow by reordering indices. We begin by performing an affine coordinate transformation such that $\mathcal{S}$ is redefined as the standard simplex, co $\{e_0, \ldots, e_n\}$, where e_i are the Euclidean coordinate vectors. Let $\widehat{I}_1 := \{1, \ldots, r_1\}$ and define the closed-loop vector field (in transformed coordinates) by

$$y(x) := Ax + Bu(x) + a \,, \qquad x \in \mathcal{S} \,.$$

From Lemma 25 we have that for all $x \in \widehat{\mathcal{S}}_1$

$$h_j \cdot y(x) = 0 \,, \qquad j \in I \setminus \widehat{I}_1 \,. \tag{27}$$

Define the sets:

$$\mathcal{Q}_i := \left\{ x \in \widehat{\mathcal{S}}_1 \mid h_i \cdot y(x) \geq 0 \right\} , \qquad i \in \widehat{I}_1 \,.$$

Consider any $I' \subset \widehat{I}_1$ and let $\mathcal{S}' :=$ co $\{e_i \mid i \in I'\}$. Then by (3), for all $x \in \mathcal{S}'$

$$h_j \cdot y(x) \leq 0 \,, \qquad j \in I \setminus I' \,.$$

If, in addition,

$$h_j \cdot y(x) \leq 0 \,, \qquad j \in I' \,,$$

then $y(x) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$, and the result is obtained.

Instead, suppose that for all $I' \subset \widehat{I}_1$ and for all $x \in$ co $\{e_i \mid i \in I'\}$, there exists $j \in I'$ such that $h_j \cdot y(x) > 0$. Then we have that for all $I' \subset I$,

$$\text{co}\,\{e_i \mid i \in I'\} \subset \bigcup_{i \in I'} \mathcal{Q}_i \,. \tag{28}$$

Now suppose that $\bigcap_{i \in \widehat{I}_1} \mathcal{Q}_i = \emptyset$. Then $\{\mathcal{Q}_i^c\}$ is a finite open cover of $\widehat{\mathcal{S}}_1$, and there exists a partition of unity $\{\phi_i(x)\}$ subordinate to $\{\mathcal{Q}_i^c\}$. Define the map $f : \widehat{\mathcal{S}}_1 \to \widehat{\mathcal{S}}_1$ by

$$f(x) := \sum_{i \in \widehat{I}_1} \phi_i(x) e_i \,.$$

Map f is continuous, so by the Brouwer fixed point theorem, there exists $x^\star$ such that $f(x^\star) = x^\star$. Let $I^\star := \{i \in \widehat{I}_1 \mid \phi_i(x^\star) > 0\}$. Then

$$x^\star = \sum_{i \in I^\star} \phi_i(x^\star) e_i \,.$$

That is, $x^\star \in$ co $\{e_i \mid i \in I^\star\}$. Also, by definition of the partition of unity,

$$x^\star \notin \mathcal{Q}_i \,, \qquad i \in I^\star \,.$$

This contradicts (28).

We conclude that there exists a point $\widehat{x} \in \widehat{\mathcal{S}}_1$ such that $\widehat{x} \in \bigcap_{i \in \widehat{I}_1} \mathcal{Q}_i$. That is,

$$h_j \cdot y(\widehat{x}) \geq 0 \,, \qquad j \in \widehat{I}_1 \,.$$

Combined with (27), we obtain that $-y(\widehat{x}) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$, which implies $\widehat{x} \in \widehat{\mathcal{S}}_1$ is an equilibrium of the closed-loop system $\dot{x} = y(x)$.

□

5. CONCLUSION

The paper introduces *reach control indices* to aid in the development of control strategies to solve the reach control problem for affine systems on simplices. The indices have the role to catalog the degeneracies caused by insufficient inputs that lead to the appearance of equilibria in a simplex $\mathcal{S}$ whenever continuous state feedback (satisfying appropriate invariance conditions) is applied.

Any control method that overcomes the limits of continuous state feedback must confront this degeneracy. Such a method will necessarily draw upon the degrees of freedom in $\mathcal{B}$ provided to the set of possible equilibria $\mathcal{G}$, which are inscribed by the indices. Our future work will explore various avenues for synthesis of discontinuous feedbacks.

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