REACH CONTROL ON SIMPLICES BY CONTINUOUS STATE FEEDBACK

MIREILLE E. BROUCKE

Abstract. This paper studies a theoretical problem of whether continuous state feedback and affine feedback are equivalent from the point of view of making an affine system defined on a simplex reach a prespecified facet in finite time. We show that the two classes of feedbacks are equivalent. As a byproduct, new necessary and sufficient conditions for solvability based more directly on the problem data are obtained.

1. Introduction

This paper studies a theoretical problem of whether continuous state feedback and affine feedback are equivalent from the point of view of making an affine system defined on a simplex reach a prespecified facet in finite time. In general, such problems have been overlooked in the literature pertaining to reachability problems via feedback control. This contrasts with the situation for stabilization, where it has long been known that linear state feedback is the largest class of feedbacks needed to stabilize a linear system. The study of related questions for nonlinear systems [5] has led to a rich and ongoing inquiry into the precise class of feedbacks needed for stabilization [6, 15], and it is clear that these sorts of questions are at the very heart of the theory of control. Fortunately, the situation for affine systems on simplices with finite-time reachability specifications is rather simple, and the outcome of our work showing that affine feedbacks are the central object of interest falls in line with one's expectations, based on the linear theory. This is because affine systems are geometrically very similar to linear systems, and because the control specification to reach a facet in finite time is, in fact, less demanding on the system than stabilization. The latter observation is evidenced by the fact that controllability is not necessary for solvability of the studied reachability problem. It is also well-known from the linear theory that such results are fragile. Once one considers more complex specifications or classes of systems which have uncertainties or nonlinearities, the picture changes drastically. To cite one example, in [3] it is shown that certain linear systems with fast time-varying uncertain coefficients can be stabilized by nonlinear static state feedback, but not by linear dynamic compensators. Investigations of such theoretical questions have pervaded the literature on mathematical control theory.

The problem studied in this paper is for an affine system to reach a prespecified facet of a simplex in finite time and is taken from [10, 17]. Facet reachability problems on simplices and polytopes, with minor variations in assumptions, were first introduced in [8] and further studied in [9, 13, 14]. While the problem formulation is the same as in [10, 17], the focus and results of this paper are different and new. In [10, 17], two sets of conditions called *invariance conditions* and a *flow condition* were given as necessary and sufficient conditions

Date: December 7, 2012.

for existence of an affine feedback to solve the problem of reaching a facet of a simplex in finite time. The invariance conditions are Nagumo-like conditions restricting the closed-loop vector field to lie in appropriately defined tangent cones at the vertices of the simplex. These conditions can be shown to be necessary for continuous state feedback, as is done in [9]. The flow condition is present to force closed-loop trajectories to exit the simplex, and its necessity for existence of an affine feedback is tied to its direct link to existence of closed-loop equilibria, assuming the closed-loop vector field is convex. Once one relaxes the class of controls to continuous state feedback, convexity is lost, and the necessity of the flow condition becomes problematic to establish.

Therefore, it is the flow condition which is the object of interest. Indeed, solving the flow condition is known to be undesirable [19], although it can be carried out using a series of linear programs whose number increases exponentially with the system dimension [17]. One would like to bypass this condition altogether, and instead combine the invariance conditions, which in any case must be solved for synthesis of the control [9], with conditions that come directly from the problem data. The price to be paid for this simplification is that one is required to triangulate the polytopic state space properly. A key observation is that arbitrary triangulations disguise the structural information encoded in the system, and instead one must use triangulations which are adapted to the system dynamics. Since typically triangulations are performed by standalone software libraries that are not tailored to control problems, our requirement for a proper triangulation is, in principle, no loss of generality.

Finally, our results have implications for the study of piecewise linear and piecewise affine feedbacks to solve more general reachability problems on polytopes and unions of polytopes. Piecewise affine and piecewise linear feedback has been extensively studied by a number of researchers. See for example [7, 1, 4, 20].

2. Contributions

In this section we describe in intuitive terms the contributions of the paper and we discuss some of the techniques introduced in order to be able to solve the main problem of equivalence of continuous state and affine feedback.

We are given a problem of finding a state feedback to drive the state of an affine system $\dot{x} = Ax + Bu + a$ to a prespecified facet $\mathcal{F}_0$ of a simplex $\mathcal{S}$ in finite time, without first exiting the simplex. Our attention is restricted to continuous state feedbacks for which the closed-loop system has unique solutions. First, it is easy to establish (see also [9]) that the above-described invariance conditions (guaranteeing that trajectories do not exit prespecified "restricted facets") are necessary to solve the problem. Attention then turns to the flow condition. To make sense of it, we isolate the part of the state space, called $\mathcal{O}$, where the system can have equilibria - precisely that part where the flow condition might fail. In general for affine systems, $\mathcal{O}$ is an affine space. We impose that the state space has been triangulated so that $\mathcal{S} \cap \mathcal{O}$ is either the empty set, $\mathcal{S}$ itself, or a lower-dimensional face of $\mathcal{S}$. Then we concentrate on analyzing when the flow condition either can be made to hold or made to fail on $\mathcal{O}$.

A first observation and motivation for our choice of triangulation, is that linear and affine control systems naturally exhibit a flow condition off the set $\mathcal{O}$, without any preconditions on the control input. This useful fact has the consequence that the question of existence of a flow condition is decoupled from the selection of controls to satisfy the invariance conditions. Therefore, if it is known that $\mathcal{S} \cap \mathcal{O} = \emptyset$, the problem is trivially solvable by affine feedback. If instead, $\mathcal{S} \cap \mathcal{O} \neq \emptyset$, then we define the convex polytope $\mathcal{G} = \mathcal{S} \cap \mathcal{O}$, and if $\mathcal{G}$ can be made to satisfy a flow condition by proper choice of control, then again the problem is solvable by affine feedback. The only remaining question is to determine the precise mathematical boundary, in terms of the problem data, for when $\mathcal{G}$ has a flow condition.

A fundamental observation concerns the nature of points in $\mathcal{G}$. In order for these to be possible equilibrium points in $\mathcal{S}$, it is necessary that the drift term $Ax + a$ lie in $\mathcal{B}$, the image of B. This means that the closed-loop vector field on $\mathcal{G}$ only involves $\mathcal{B}$. Therefore, what is most relevant is the geometric relationship between subspace $\mathcal{B}$ and the simplex $\mathcal{S}$. A canonical way to describe this relationship is through the cone $\mathrm{cone}(\mathcal{S})$ which is the tangent cone of $\mathcal{S}$ at the vertex v_0 that does not contain the exit facet. For example, if there is a non-zero vector $b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$, then this single vector can be used to resolve all the invariance conditions at all the vertices of $\mathcal{G}$. A flow condition based on b can be trivially established, and we again find that an affine feedback solves the problem. If instead $\mathcal{B} \cap \mathrm{cone}(\mathcal{S})$ only contains the zero vector, then the geometric situation is more delicate. The combined requirements that the invariance conditions hold at $\mathcal{G}$ and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \{0\}$ restricts the orientation of $\mathcal{B}$ with respect to the simplex. In particular, $\mathcal{B}$ must be aligned so that it uses all its available degrees of freedom to span directions in the affine space that contains $\mathcal{G}$. Now, if $\mathcal{B}$ does not have enough degrees of freedom, closed-loop trajectories can become "stuck" in this affine space and an invariant set appears in $\mathcal{G}$. Using a fixed point theorem, one can show this invariant set contains an equilibrium. On the other hand, if $\mathcal{B}$ has extra degrees of freedom, one can show that any velocity vectors which satisfy the invariance conditions on $\mathcal{G}$ can always be perturbed to obtain a flow condition on $\mathcal{G}$. So to speak, there are enough degrees of freedom to "push" trajectories out of $\mathcal{G}$. In that case, the problem is solvable by affine feedback.

We have described the geometric intuition which shapes how the arguments are structured. Further intuitive descriptions, especially discussing the role of sufficient degrees of freedom in $\mathcal{B}$, are given in the text. A useful tool to convert the geometric intuition to algebra is provided by $\mathscr{M}$-matrices [2]. It can be shown that if one assembles all the expressions for the invariance conditions on all the vertices of $\mathcal{G}$ in a matrix and one adds the condition that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \{0\}$, the resulting matrix is an $\mathscr{M}$-matrix.

The paper is organized as follows. Section 3 presents the problem statement and Section 4 presents the relevant results from [10, 17]. Section 5 describes a suitable triangulation method based on the placing triangulation [12]. Section 6 gives two different sufficient conditions for existence of an affine feedback to solve the problem, and the set of $\mathscr{M}$ matrices associated with the problem is introduced. Section 8 identifies two cases when equilibria arise using continuous feedback, or equivalently two necessary conditions for existence of affine feedback. Section 9 collects all the above results to resolve the boundary between continuous state feedback and affine feedback.

Notation. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). For a matrix $A \in \mathbb{R}^{n \times n}$, the notation $A \succ 0$ ($A \succeq 0$) means $a_{ij} > 0$ ($a_{ij} \geq 0$) for $1 \leq i, j \leq n$. The notation $\{0\}$ will denote the subset of $\mathbb{R}^n$ containing only the zero vector.

3. Problem Statement

Consider an n-dimensional simplex $\mathcal{S}$ with vertices $v_0, v_1, \ldots, v_n$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ such that the index of each facet is determined by the vertex it does not contain. Let h_i, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Let $\mathcal{F}_0$ be the target set in $\mathcal{S}$.

We consider the following affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + a + Bu =: f(x, u), \quad x \in \mathcal{S}, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n\times m}$ with $\mathrm{rank}(B) = m$. Let $\phi_u(t, x_0)$ be the trajectory of (3.1) under a control u starting from $x_0 \in \mathcal{S}$ and evaluated at time t.

We are interested in studying reachability of the target $\mathcal{F}_0$ from $\mathcal{S}$ by way of feedback control. A number of results on finding feedbacks to solve reachability specifications on simplices have already appeared in the literature. In particular, the following problem was proposed in [10, 17].

Problem 3.1. Consider system (3.1) defined on $\mathcal{S}$. Find an affine feedback control $u = Kx+g$ such that for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\epsilon > 0$ satisfying:

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$;
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$;
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \epsilon)$.

Remark 3.1. Condition (iii) is interpreted to mean that the closed-loop dynamics on $\mathcal{S}$ are extended to a neighborhood of $\mathcal{S}$. This condition plays an important role in ruling out undesirable pathological cases in which, for instance, equilibria appear on $\mathcal{F}_0$, even though trajectories starting in $\mathcal{S}$ reach $\mathcal{F}_0$ in finite time.

In this paper, we extend the problem to consider continuous state feedback. Thus, we formulate the following *reach control problem*.

Problem 3.2. Consider system (3.1) defined on $\mathcal{S}$. Find a continuous state feedback $u(x)$ such that for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\epsilon > 0$ satisfying:

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$;
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$;
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \epsilon)$.

4. Background

The following notation will be used. Define the set of vertices of $\mathcal{S}$ to be $V := \{v_0, \ldots, v_n\}$ and define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$. Define the closed, convex cone $\mathcal{C}_i$ at $v_i \in V$ by

$$\mathcal{C}_i := \left\{ \ y \in \mathbb{R}^n \ : \ h_j \cdot y \leq 0, \ j \in I_i \ \right\}.$$

Also define

$$\mathrm{cone}(\mathcal{S}) := \mathcal{C}_0 = \mathrm{cone}\{v_1 - v_0, \ldots, v_n - v_0\}\,.$$

If w.l.o.g. we take $v_0 = 0$, then $\mathrm{cone}(\mathcal{S})$ is the cone generated by the points in $\mathcal{S}$, motivating the choice of notation.

Definition 4.1. A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ by continuous state feedback, denoted $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a continuous state feedback $u(x)$ such that properties (i)-(iii) of Problem 3.2 hold. A set $\mathcal{S}' \subseteq \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ by continuous state feedback, denoted by $\mathcal{S}' \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a continuous state feedback such that for every $x_0 \in \mathcal{S}'$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$.

Let $\mathcal{B}$ denote the m-dimensional subspace spanned by the column vectors of B (namely, $\mathcal{B} = \mathrm{Im}(B)$, the image of B). Define the set

$$\mathcal{O} := \{ \ x \in \mathbb{R}^n \ : \ Ax + a \in \mathcal{B} \ \} .$$

It is fairly easy to prove that $\mathcal{O} = \emptyset$ when $\mathrm{Im}(A) \subseteq \mathcal{B}$ and $a \notin \mathcal{B}$; $\mathcal{O} = \mathbb{R}^n$ when $\mathrm{Im}(A) \subseteq \mathcal{B}$ and $a \in \mathcal{B}$; and $\mathcal{O}$ is an affine space, otherwise. Notice that vector field $f(x,u)$ can vanish on $\mathcal{O}$ for an appropriate choice of u, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. Define

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O}.$$

Associated with $\mathcal{G}$ is its vertex index set

$$I_{\mathcal{G}} := \{i : v_i \in V \cap \mathcal{G}\} .$$

Definition 4.2. The *invariance conditions* require that there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that:

$$h_j \cdot (Av_i + a + Bu_i) \leq 0 \,, \qquad i \in \{0, \ldots, n\}, \qquad j \in I_i \,. \tag{4.1}$$

Example 4.1. Figure 1 illustrates the definitions so far for the case $n = 3$ and $m = 2$. We have a simplex $\mathcal{S}$ with normal vectors h_i to each facet $\mathcal{F}_i$. Depicted by a shaded section is cone($\mathcal{S}$), the tangent cone at v_0. The space $\mathcal{B}$ is copied to v_0, and in this view we see that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \{0\}$. That is, $\mathcal{B}$ does not "dip" into the tangent cone at v_0. The set $\mathcal{O}$ intersects $\mathcal{S}$ along the face $\overline{v_1 v_2}$, and this forms $\mathcal{G}$. It is interpreted as the set of possible equilibria of the system. We know that in $\mathcal{G}$, the only velocity vectors available to the closed loop system are vectors in $\mathcal{B}$. This is depicted by placing copies of $\mathcal{B}$ at each of the vertices of $\mathcal{G}$. Two velocity vectors b_1 and b_2 are shown, and these clearly satisfy the invariance conditions at v_1 and v_2, respectively. At vertices not in $\mathcal{G}$, the drift term $Ax + a$ becomes relevant, and the figure depicts closed-loop velocity vectors at $v_0, v_3 \notin \mathcal{G}$ which satisfy their respective invariance conditions. The invariance conditions can be interpreted in terms of the cones $\mathcal{C}_i$. Consider vertex v_3 where $\mathcal{C}_3$ is depicted by a shaded region. This cone is shaped like an open book whose spine is parallel to the face $\overline{v_0 v_3}$ and whose cover and back cover lie in $\mathcal{F}_2$ and $\mathcal{F}_1$, respectively. The invariance condition at v_3 is satisfied if the closed-loop velocity vector $Av_3 + Bu_3 + a$ lies in $\mathcal{C}_3$.

For Problem 3.1 the following necessary and sufficient conditions have been established.

Theorem 4.1. [10, 17] *Given the system* (3.1) *and an affine feedback* $u(x) = Kx + g$, *with* $K \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$, *and* $u_0 = u(v_0), \ldots, u_n = u(v_n)$, *the closed-loop system satisfies* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *if and only if*

(a) *The invariance conditions* (4.1) *hold.*
(b) *The closed-loop system has no equilibrium in* $\mathcal{S}$.

Theorem 4.2. [10, 17] *We have* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by affine feedback if and only if there exist* $u_0, \ldots, u_n \in \mathbb{R}^m$ *and* $\xi \in \mathbb{R}^n$ *such that*

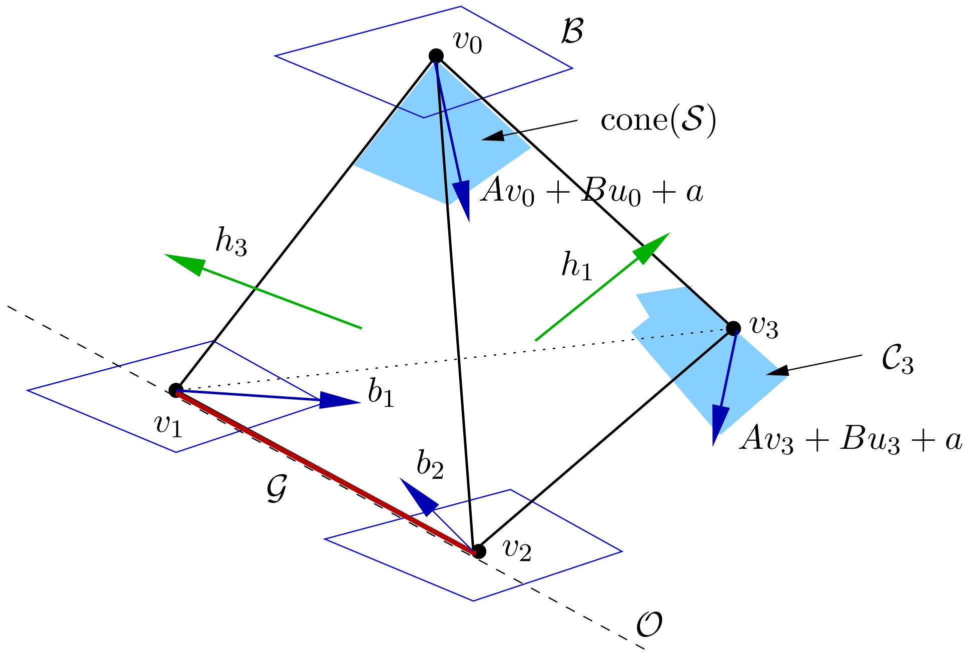

FIGURE 1. Geometric constructs for the reach control problem.

(a) *The invariance conditions* (4.1) *hold.*
(b) *The flow condition holds:* $\quad \xi \cdot (Av_i + Bu_i + a) < 0\,, \qquad i \in \{0, \ldots, n\}\,.$

Theorem 4.1 is of theoretical interest but does not provide a practical procedure. Theorem 4.2 can be viewed as a computational solution to the problem and a linear programming based solution is presented in [17]. If the invariance and flow conditions can be solved simultaneously for the unknowns $\xi \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$, then an affine feedback can be constructed by the procedure of [9].

The invariance conditions (4.1) are suitable for affine feedback, but for continuous state feedback, the following stronger conditions must hold.

Definition 4.3. The *invariance conditions* for state feedback $u(x)$ require that for all $j \in I$ and $x \in \mathcal{F}_j$,

$$h_j \cdot (Ax + Bu(x) + a) \leq 0\,. \tag{4.2}$$

The following result is easily proved (see the analogous result in [9] for conditions (4.1)) and forms the starting point for our investigation of continuous state feedback.

Lemma 4.3. *If* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by a continuous state feedback* $u(x)$*, then* $u(x)$ *satisfies the invariance conditions* (4.2)*.*

Finally, we collect some useful properties about simplices. Properties (1)-(5) follow directly from our conventions on indices. Properties (6)-(7) follow from the fact that any simplex is affinely isomorphic to the simplex $\mathrm{conv}(\{0, e_1, \ldots, e_n\})$, where e_i, $i = 1, \ldots, n$ are the Euclidean coordinate vectors.

Lemma 4.4. *Let* $\mathcal{S}$ *be a simplex. Then the following hold:*

(1) *If* $x \in \operatorname{conv}\{v_1, \dots, v_k\}$*, then* $x \in \mathcal{F}_j$*, for* $k+1 \le j \le n$*.*
(2) $h_j \cdot (v_i - v_0) = 0$ *for all* $1 \le i, j \le n$ *and* $j \ne i$*.*
(3) $h_i \cdot (v_i - v_0) < 0$*, for all* $1 \le i \le n$*.*
(4) $h_j \cdot (v_i - x) > 0$ *for all* $x \in \mathcal{S} \setminus \mathcal{F}_j$ *and* $1 \le i, j \le n$ *and* $i \ne j$*.*
(5) $h_0 \cdot (v_i - v_0) > 0$ *for all* $1 \le i \le n$*.*
(6) *The vectors* $\{v_1 - v_0, \dots, v_n - v_0\}$ *are a basis for* $\mathbb{R}^n$*.*
(7) *The vectors* $\{h_1, \dots, h_n\}$ *are a basis for* $\mathbb{R}^n$*.*

5. Triangulation with Respect to $\mathcal{O}$

In this section we describe in more detail our proposal for triangulating the state space of the system (3.1). Suppose that the state space of (3.1) is presented as a polytope $\mathcal{P}$, set $\mathcal{O}$ is an affine space of dimension less than n, and $\mathcal{P} \cap \mathcal{O}$ is a polytope with vertices $V_{\mathcal{O}} := \{o_1, \dots, o_r\}$. First, we define an ordered point set $V := \{v_1, \dots, v_p\}$ such that $\mathcal{P} = \operatorname{conv}(V)$ and the first r points of V are $V_{\mathcal{O}}$. Note that not every element of $V_{\mathcal{O}}$ need be a vertex of $\mathcal{P}$. Now we propose a triangulation of $\mathcal{P}$ which will have the feature that $\mathcal{O}$ can only lie in lower dimensional faces of simplices of the triangulation. We use a standard procedure called the *placing triangulation* (see [12, 11]). To describe this triangulation method we need a few definitions.

Suppose V is a finite set of points such that $\mathcal{P} = \operatorname{conv}(V)$ is an n-dimensional polytope. A subdivision of V is a finite collection $\mathbb{S} = \{\mathcal{P}_1, \dots, \mathcal{P}_q\}$ of n-dimensional polytopes such that: (1) The vertices of each $\mathcal{P}_i$ are drawn from V (though not every point in V need be used); (2) $\mathcal{P} = \cup_i \mathcal{P}_i$; (3) If $i \ne j$, then $\mathcal{P}_i \cap \mathcal{P}_j$ is a common (possibly empty) face of the boundaries of $\mathcal{P}_i$ and $\mathcal{P}_j$.

Definition 5.1. Let $x \in \mathbb{R}^n$, $\mathcal{P}$ an n-dimensional polytope, and $\mathcal{F}$ a facet of $\mathcal{P}$. The hyperplane $\mathcal{H} = \operatorname{aff}(\mathcal{F})$ defines an open half-space containing $\operatorname{int}(\mathcal{P})$. If x is contained in the opposite open half-space, then $\mathcal{F}$ is said to be *visible* from x. (If $\mathcal{P}$ is a k-dimensional polytope in $\mathbb{R}^n$ with $k < n$ and $x \in \operatorname{aff}(\mathcal{P})$, then the ambient space is viewed to be $\operatorname{aff}(\mathcal{P})$.)

Now we can describe what it means to place a vertex. Let $\mathbb{S} = \{\mathcal{P}_1, \dots, \mathcal{P}_q\}$ be a subdivision of V and $v \in \mathbb{R}^n$ such that $v \notin V$.

Definition 5.2. The subdivision $\mathbb{T}$ of $V \cup \{v\}$ that results from *placing* v is obtained as follows:

(1) If $v \notin \operatorname{aff}(V)$, then for each $\mathcal{P}_i \in \mathbb{S}$, include $\operatorname{conv}(\mathcal{P}_i \cup \{v\})$ in $\mathbb{T}$.
(2) If $v \in \operatorname{aff}(V)$, then for each $\mathcal{P}_i \in \mathbb{S}$, $\mathcal{P}_i \in \mathbb{T}$ and if $\mathcal{F}$ is a facet of $\mathcal{P}_i$ that is contained in a facet of $\operatorname{conv}(V)$ visible from v, then $\operatorname{conv}(\mathcal{F} \cup \{v\}) \in \mathbb{T}$.

Theorem 5.1. [12] *Suppose* V *is a finite set of points such that* $V_{\mathcal{O}} \subset V$ *and* $\mathcal{P} = \operatorname{conv}(V)$ *is an* n*-dimensional polytope. If the points of* V *are ordered such that* $\{o_1, \dots, o_r\}$*, the vertices of* $\mathcal{P} \cap \mathcal{O}$*, are listed first and if* $\mathbb{T}$ *is the subdivision obtained by placing the points of* V *in order, then* $\mathbb{T}$ *is a triangulation of* V *such that for every* n*-dimensional simplex* $\mathcal{S} \in \mathbb{T}$*,* $\operatorname{int}(\mathcal{S}) \cap \mathcal{O} = \emptyset$ *and if* $\mathcal{S} \cap \mathcal{O} \ne \emptyset$*, then* $\mathcal{S} \cap \mathcal{O}$ *is a face of* $\mathcal{S}$*.*

6. Existence of Linear Affine Feedback

As we have seen in Theorem 4.2, the invariance conditions by themselves are generally not enough to establish that the reach control problem is solvable by affine feedback. However, there is one extreme case when the invariance conditions are also sufficient to solve the problem. These depend on combining Theorem 4.1 with the fact that $\mathcal{O}$ is the only place in the state space where equilibria can appear. See also [17].

Theorem 6.1. *Suppose $\mathcal{G} = \emptyset$. If the invariance conditions are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.*

Proof. Select the control $u_i \in \mathbb{R}^m$ for each vertex $v_i \in V$ to satisfy the invariance conditions. Using the method of [9], since the vertices are affinely independent one can find unique K and g corresponding to the affine feedback $u(x) = Kx + g$ such that $u(v_i) = u_i$, $0 \le i \le n$. We obtain the affine closed-loop system $\dot{x} = (A + BK)x + (a + Bg)$. Since $\mathcal{O} \cap \mathcal{S} = \emptyset$, we know the closed-loop system has no equilibria in $\mathcal{S}$. Therefore, applying Theorem 4.1, the result is obtained. $\square$

In general it is difficult to extend results such as Theorem 6.1. However, if one propitiously chooses a triangulation of the state space which respects the underlying structure of the system, then new necessary and sufficient conditions for solvability of the reach control problem are obtainable and, moreover, the boundary between affine and continuous state feedback can be clarified. We propose the following triangulation.

Assumption 6.1. Simplex $\mathcal{S}$ and system (3.1) satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \le \kappa \le n$.

Remark 6.1. We have discussed that there are three possibilities for $\mathcal{O}$. If $\mathcal{O} = \emptyset$, then one applies Theorem 6.1. If $\mathcal{O}$ is the entire state space then we will see in Remark 8.1 that there are easily derived necessary and sufficient conditions for solvability. The only interesting case is when $\mathcal{O}$ is a κ-dimensional affine subspace with $\kappa < n$. This case arises, for example, when (A, B) is controllable, and then the placing triangulation can be applied.

Based on the proposed triangulation, we can find several new sufficient conditions for existence of affine feedback.

Theorem 6.2. *Suppose Assumption 6.1 holds and $\mathcal{G} \neq \emptyset$. Suppose the following conditions hold.*

(1) *The invariance conditions* (4.1) *are solvable.*
(2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq \{0\}$.

Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

Proof. Let $\mathcal{G} = \mathrm{conv}\{v_{i_1}, \ldots, v_{i_{\kappa+1}}\}$, a κ-dimensional facet of $\mathcal{S}$ where $0 \le \kappa \le n$. Thus, $I_{\mathcal{G}} = \{i_1, \ldots, i_{\kappa+1}\}$. Let $b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$, $b \neq 0$, and select control values u_i such that $y(v_i) = Av_i + Bu_i + a = b$ for all $i \in I_{\mathcal{G}}$ (notice this is always achievable for $v_i \in \mathcal{O}$). Clearly, by the assumption that $b \in \mathrm{cone}(\mathcal{S})$, $y(v_i)$ satisfies the invariance conditions for $v_i \in V \cap \mathcal{O}$. We can select the remaining controls u_i for $i \in \{0, \ldots, n\} \setminus I_{\mathcal{G}}$ such that $y(v_i) \neq 0$ (since $v_i \notin \mathcal{O}$) and $y(v_i)$ satisfies the invariance conditions. Finally, using $\{u_0, \ldots, u_n\}$ and the synthesis procedure in [9], construct the affine feedback $u(x) = Kx + g$.

Define the closed-loop vector field $y(x) := (A + BK)x + Bg + a$. We show it satisfies a flow condition on $\mathcal{G}$. Let $\beta := -b$. We have $\beta^T y(v_i) = -\|b\|^2 < 0$ for all $i \in I_{\mathcal{G}}$. By the convexity of $y(x)$, this implies a flow condition holds on $\mathcal{G}$. Clearly $y(x) \neq 0$ for $x \in \mathcal{G}$.

Also, $y(x) \neq 0$ for all $x \in \mathcal{S} \setminus \mathcal{G}$ since equilibria only lie in $\mathcal{O}$. Applying Theorem 4.1, the result is obtained. □

One can also obtain sufficient conditions for existence of affine feedback even when $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$. Of course, this will only be possible if $v_0 \notin \mathcal{G}$ (see Remark 8.1). This relies on the idea that there are enough degrees of freedom in $\mathcal{B}$ with respect to $\mathcal{G}$. We make the following assumptions.

Assumption 6.2.

(A1) W.l.o.g. $\mathcal{G} = \operatorname{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < m$.
(A2) $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$.
(A3) There exists a linearly independent set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$.

The important new assumption is (A3) which says that $\mathcal{B}$ and $\mathcal{G}$ are arranged with respect to each other so that there are enough degrees of freedom in $\mathcal{B}$ both to span a $\kappa+1$-dimensional subspace of $\mathcal{B}$ and at the same time satisfy all the invariance conditions for the vertices of $\mathcal{G}$. For this to work, it is of course necessary that $\kappa < m$. It is helpful to obtain some intuition as to why linear independence is a central property which determines whether or not an affine feedback exists in the case when $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$. Also this intuition will help motivate the developments of Section 8.

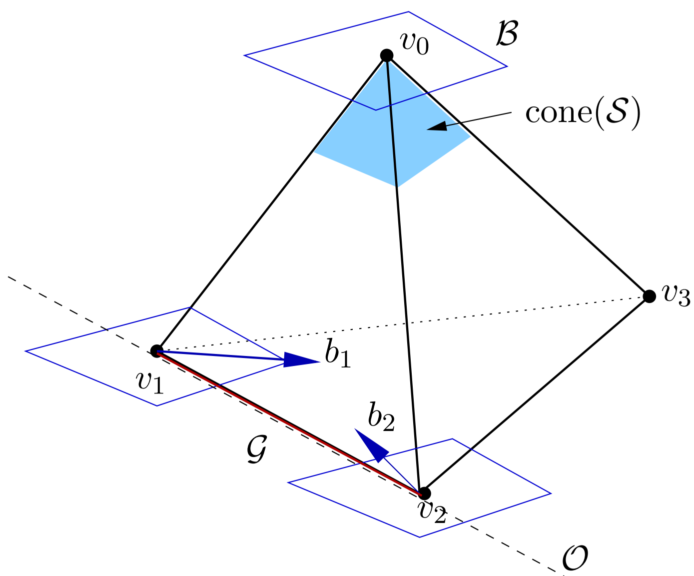

FIGURE 2. Continuous Assignment Along $\mathcal{G}$ of Vectors in $\mathcal{B}$.

Consider Figure 2 in which $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$ and $\mathcal{G} = \overline{v_1 v_2}$. Suppose we find a linearly independent set $\{b_1, b_2 \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ as shown in the figure (a copy of $\mathcal{B}$ is attached at each vertex). Then along $[v_1, v_2]$ we can always choose a feedback $u(x)$ such that the closed-loop vector field

$$y(x) := Ax + Bu(x) + a = c_1(x)b_1 + c_2(x)b_2, \qquad c_i(x) \geq 0 \tag{6.1}$$

continuously interpolates between $y(v_1) = b_1$ and $y(v_2) = b_2$. This is also evident from examining the figure. Indeed with this controller, the invariance conditions are guaranteed to hold not only at v_1 and v_2 (by the definition of $\mathcal{C}_i$) but also on the open interval (v_1, v_2). Namely, because $c_i(x) \geq 0$,

$$h_j \cdot y(x) \leq 0\,, \qquad\qquad j = 3, \ldots, n\,.$$

Now consider the opposite situation as depicted in Figure 3. Again we have $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$ and $\mathcal{G} = \overline{v_1 v_2}$, but in this case $m = 1$. Thus, for every choice of $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i = 1, 2$, we obtain that $\{b_1, b_2\}$ are linearly dependent. Pick any $b_1 \in \mathcal{B} \cap \mathcal{C}_1$ with $b_1 \neq 0$, as shown. Then we know that for any $b_2 \in \mathcal{B} \cap \mathcal{C}_2$,

$$b_2 = c_1 b_1\,, \qquad\qquad c_1 \in \mathbb{R}.$$

Now check the invariance conditions at v_2. In particular, we have the invariance condition

$$h_1 \cdot b_2 = c_1 (h_1 \cdot b_1) \leq 0\,. \tag{6.2}$$

If we have assumed that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$, then it must be that $h_1 \cdot b_1 > 0$, for otherwise we would have $0 \neq b_1 \in \mathcal{B} \cap \text{cone}(S)$. Then from (6.2) we obtain that $c_1 \leq 0$. This is illustrated in Figure 3, where b_2 points in the opposite direction of b_1. Consider a continuous vector field $y(x)$ on $\mathcal{S}$ and suppose we assign $y(v_1) = b_1$. Then we know that $y(x)$ is irrevocably constrained to be $y(v_2) = c_1 b_1$ with $c_1 \leq 0$. Now suppose that $y(x)$ continuously interpolates between $y(v_1)$ and $y(v_2)$ along $[v_1, v_2]$ using only $\{b_1, b_2\}$. Then along $[v_1, v_2]$, $y(x)$ has the form:

$$y(x) = c(x) b_1\,,$$

where $c(x)$ is a continuous function of $x \in [v_1, v_2]$ with $c(v_1) > 0$ and $c(v_2) \leq 0$. By the Intermediate Value Theorem, there exists $\overline{x} \in [v_1, v_2]$ such that $c(\overline{x}) = 0$. Thus, there is an equilibrium along $[v_1, v_2]$ for the closed loop system.

Therefore, it is clear that $y(x)$ cannot simply interpolate between $\{b_1, b_2\}$ along $[v_1, v_2]$ and other directions in $\mathcal{B}$ must be invoked. This argument can now be carried on inductively to higher dimensions and in each dimension one finds that more degrees of freedom are needed in $\mathcal{B}$ to carry out the continuous assignment of the vector field. Finally, the procedure either terminates with exhausting all the vertices of $\mathcal{G}$, without first exhausting the degrees of freedom in $\mathcal{B}$ or instead one first exhausts all the usable degrees of freedom in $\mathcal{B}$. This question of which is exhausted first determines a sharp boundary between existence of affine feedbacks and existence of equilibria.

Theorem 6.3. *Suppose Assumption 6.1 holds and* $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{\kappa+1}\}$, *with* $0 \leq \kappa < m$. *Suppose the following conditions hold.*

1. *The invariance conditions* (4.1) *are solvable.*
2. *There exists a linearly independent set* $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$.

Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by affine feedback.*

Proof. If $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \{0\}$ then the result follows from Theorem 6.2. Therefore, we assume that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$. For all $i \in I_{\mathcal{G}}$, we assign control values u_i such that $y(v_i) = Av_i + Bu_i + a = b_i$. We can select the remaining controls u_i for $i \in \{0, \ldots, n\} \setminus I_{\mathcal{G}}$ such that $y(v_i) \neq 0$ (since $v_i \notin \mathcal{O}$) and $y(v_i)$ satisfies the invariance conditions. Finally, using $\{u_0, \ldots, u_n\}$ and the synthesis procedure in [9], construct the affine feedback $u(x) = Kx + g$.

Now we observe that a flow condition holds for the closed loop vector field $y(x) := (A + BK)x + Bg + a$ on $\mathcal{G}$. In particular, since $\{b_1, \ldots, b_{\kappa+1}\}$ are linearly independent, $0 \notin \text{conv}\{b_1, \ldots, b_{\kappa+1}\}$. Let $\mathcal{C}_1 := \{0\}$ and $\mathcal{C}_2 := \text{conv}\{b_1, \ldots, b_{\kappa+1}\}$. Note that both $\mathcal{C}_1$ and $\mathcal{C}_2$

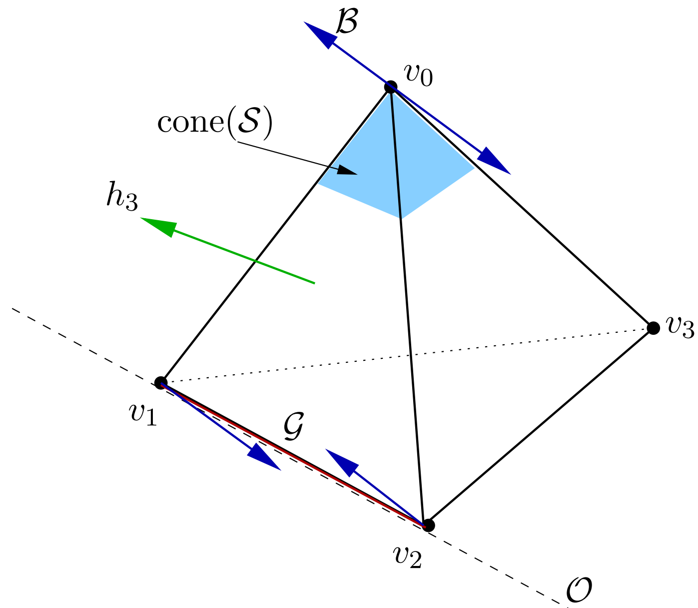

FIGURE 3. Not enough degrees of freedom in $\mathcal{B}$.

are compact and convex and by assumption $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. By the Separating Hyperplane Theorem [16], there exists a hyperplane $\mathcal{H}$ with normal vector β pointing to the side containing $\mathcal{C}_1$. By the convexity of $y(x)$, we get $\beta \cdot y(x) < 0$ for all $x \in \mathcal{G}$. Clearly $y(x) \neq 0$ for $x \in \mathcal{G}$. Also, $y(x) \neq 0$ for all $x \in \mathcal{S} \setminus \mathcal{G}$ since equilibria only lie in $\mathcal{O}$. Applying Theorem 4.1, the result is obtained. $\square$

7. $\mathscr{M}$-MATRICES

We introduce a family of matrices used to concisely characterize the invariance conditions on $\mathcal{G}$. Let $1 \leq p \leq q \leq \kappa + 1$ and define

$$M_{p,q} := \begin{bmatrix} (h_p \cdot b_p) & (h_p \cdot b_{p+1}) & \cdots & (h_p \cdot b_q) \\ \vdots & \vdots & & \vdots \\ (h_q \cdot b_p) & (h_q \cdot b_{p+1}) & \cdots & (h_q \cdot b_q) \end{bmatrix} \in \mathbb{R}^{(q-p+1)\times(q-p+1)} . \tag{7.1}$$

Define the matrices

$$H_{p,q} := [h_p \cdots h_q] \in \mathbb{R}^{n\times(q-p+1)} , \qquad Y_{p,q} := [b_p \cdots b_q] \in \mathbb{R}^{n\times(q-p+1)} .$$

Then

$$M_{p,q} = H_{p,q}^T Y_{p,q} .$$

We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \leq 0$ for all $i \neq j$ [2]. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix. Also under the condition that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$, $M_{p,q}$ adopts further algebraic properties. In particular, we require the notion of an $\mathscr{M}$-matrix. The following theorem characterizes non-singular $\mathscr{M}$-matrices (see [2], Ch. 6).

Theorem 7.1. *Let $M \in \mathbb{R}^{k\times k}$ be a $\mathscr{Z}$-matrix. Then the following are equivalent:*

(i) *M is a non-singular $\mathscr{M}$-matrix.*

(ii) $\Re(\lambda) > 0$ *for all eigenvalues* λ *of* M.
(iii) *There exists a vector* $\xi \succeq 0$ *in* $\mathbb{R}^k$ *such that* $M\xi \succ 0$.
(iv) *The inequalities* $y \succeq 0$ *and* $My \preceq 0$ *have only the trivial solution* $y = 0$, *and* M *is non-singular.*
(v) M *is monotone; that is,* $My \succeq 0$ *implies* $y \succeq 0$ *for all* $y \in \mathbb{R}^k$.
(vi) M *is nonsingular and* M^{-1} *is a non-negative matrix.*

Lemma 7.2. *Suppose* $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$. *Let* $1 \le p \le q \le \kappa+1$ *and suppose* $\{b_p, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ *are linearly independent. Then* $M_{p,q}$ *is a non-singular* $\mathscr{M}$*-matrix.*

Proof. First, since $\operatorname{rank}(H_{p,q}) = q - p + 1$, $\operatorname{rank}(Y_{p,q}) = q - p + 1$, and $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$, we have that $M_{p,q}$ is non-singular. Next, we claim that $M_{p,q}$ has a positive diagonal; that is, $(M_{p,q})_{ii} > 0$ for $i = 1, \ldots, q - p + 1$. For if not, we would have $h_j \cdot b_{p+i-1} \le 0$ for all $j = 1, \ldots, n$, which implies $0 \ne b_{p+i-1} \in \mathcal{B} \cap \operatorname{cone}(\mathcal{S})$, a contradiction. Now suppose there exists $c \in \mathbb{R}^{q-p+1}$ with $c \ne 0$ and $c \succeq 0$ such that $M_{p,q}c \preceq 0$. Define the vector $\bar{y} = Y_{p,q}c \in \mathcal{B}$. Note that $\bar{y} \ne 0$ because $\{b_p, \ldots, b_q\}$ are linearly independent. Then $M_{p,q}c = H_{p,q}^T Y_{p,q} c = H_{p,q}^T \bar{y} \preceq 0$ implies $h_j \cdot \bar{y} \le 0$ for $j = p, \ldots, q$. Also,

$$h_j \cdot \bar{y} = \sum_{i=p}^{q} c_i (h_j \cdot b_i) \le 0\,, \qquad\qquad j \notin \{p, \ldots, q\}\,.$$

This implies $0 \ne \bar{y} \in \mathcal{B} \cap \operatorname{cone}(\mathcal{S})$, a contradiction. Therefore, $M_{p,q}$ has the property that the only solution of the inequalities $c \succeq 0$ and $M_{p,q}c \preceq 0$ is $c = 0$. By Theorem 7.1(iv) this implies that $M_{p,q}$ is a non-singular $\mathscr{M}$-matrix. □

8. Existence of Equilibria

In this section we explore cases when equilibria appear on $\mathcal{G}$ when an assignment of a continuous state feedback $u(x)$ is made on $\mathcal{S}$, so that the reach control problem is not solvable by continuous state feedback. Particular attention is given to the case when $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$. Let $u(x)$ be a continuous state feedback defined on $\mathcal{S}$. We restrict our attention to such controls which yield unique solutions on $\mathcal{S}$ and which satisfy the invariance conditions (4.2) on $\mathcal{S}$. Define the closed-loop system

$$\dot{x} = Ax + Bu(x) + a =: y(x)\,. \tag{8.1}$$

First we consider an obvious necessary condition for the problem to be solvable, which is that one must be able to assign $y(v_i) \ne 0$ at each vertex $v_i \in \mathcal{G}$.

Proposition 8.1. *Suppose Assumption 6.1 holds and let* $u(x)$ *be a continuous state feedback such that the closed-loop system has unique solutions and the invariance conditions* (4.2) *hold. If at some* $i \in I_{\mathcal{G}}$, $\mathcal{B} \cap \mathcal{C}_i = \{0\}$, *then the closed-loop system* $\dot{x} = Ax + Bu(x) + a$ *has an equilibrium point at* $v_i \in \mathcal{G}$.

Proof. The only way to satisfy the invariance conditions at v_i, $i \in I_{\mathcal{G}}$, when $\mathcal{B} \cap \mathcal{C}_i = \{0\}$ is for that vertex to be an equilibrium of the closed-loop system. □

Remark 8.1. When $v_0 \in \mathcal{G}$, then Proposition 8.1 immediately implies that a necessary condition for existence of a continuous state feedback is that $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) \ne \{0\}$.

From Proposition 8.1 a necessary condition for a solution is that there exists a set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid b_i \neq 0, i \in I_{\mathcal{G}}\}$. In the special case of $v_0 \in \mathcal{G}$ this completely settles the question of necessary conditions since in that case we require that $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \{0\}$. More generally, if $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \{0\}$, the question is settled because of Theorem 6.2. Therefore, other necessary conditions for a solution are studied in this section under the following assumptions.

Assumption 8.1.

- (E1) W.l.o.g. $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$.
- (E2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$.
- (E3) The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ is $m^\star$ with $1 \leq m^\star \leq \kappa$.

Asssumption (E3) says there does not exist a full linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ as in Assumption 6.2. This automatically holds true when $\kappa = m$, in which case (E3) could simply be removed. We remark that $m^\star$ is well-defined (for $\dim(\text{sp}\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}) \in \{0, \ldots, \kappa + 1\}$ defines a finite set of integers for which the maximum exists).

Given $1 \leq m^\star \leq \kappa$ as above, w.l.o.g. fix

$$\{b_1, \ldots, b_{m^\star} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$$

to be one such maximal linearly independent set. By construction, every $b_j \in \mathcal{B} \cap \mathcal{C}_j$ for $j = m^\star + 1, \ldots, \kappa + 1$ satisfies

$$b_j \in \text{sp}\{b_1, \ldots, b_{m^\star}\} .$$

Indeed for each $j \in \{m^\star + 1, \ldots, \kappa + 1\}$ there exists $1 \leq \kappa_j \leq m^\star$ such that w.l.o.g. (reordering indices $1, \ldots, m^\star$),

$$\mathcal{B} \cap \mathcal{C}_j \subset \text{sp}\{b_1, \ldots, b_{\kappa_j}\} ,$$

and $\text{sp}\{b_1, \ldots, b_{\kappa_j}\}$ is the smallest such subspace in $\mathcal{B}$ generated by $\{b_1, \ldots, b_{m^\star}\}$. Now consider $\mathcal{B} \cap \mathcal{C}_{m^\star+1}$. Following the arguments above and w.l.o.g. (reordering indices $1, \ldots, m^\star$), let $\kappa^\star$ be such that

$$\mathcal{B} \cap \mathcal{C}_{m^\star+1} \subset \text{sp}\{b_1, \ldots, b_{\kappa^\star}\} .$$

If $\kappa^\star < m^\star$, swap the indices $m^\star + 1 \iff \kappa^\star + 1$. (The index swap is to make incrementing of indices easier below). Finally select any vectors $\beta_i \in \mathcal{B}$, $i = \kappa^\star + 1, \ldots, m$ such that

$$\mathcal{B} = \text{sp}\{b_1, \ldots, b_{\kappa^\star}, \beta_{\kappa^\star+1}, \ldots, \beta_m\} . \tag{8.2}$$

With our reordering of indices we have that for all $b_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$

$$b_{\kappa^\star+1} = c_1 b_1 + \cdots + c_{\kappa^\star} b_{\kappa^\star} .$$

Also define

$$\mathcal{G}^\star := \text{conv}\{v_1, \ldots, v_{\kappa^\star+1}\} .$$

The following results will show that there exists an equilibrium in $\mathcal{G}^\star$ for any closed-loop vector field $y(x)$ satisfying the invariance conditions on $\mathcal{S}$. We begin by isolating the defect in available degrees of freedom in $\mathcal{B}$ with respect to $\mathcal{G}^\star$.

Proposition 8.2. *Suppose Assumptions 6.1 and 8.1 hold. Suppose that the closed-loop system $\dot{x} = y(x)$ satisfies the invariance conditions* (4.2)*. Then for all $x \in \mathcal{G}^\star$,*

$$h_j \cdot y(x) = 0 , \qquad j = \kappa^\star + 2, \ldots, n .$$

Proof. W.l.og. let a basis of $\mathcal{B}$ be as in (8.2) and select $b_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$ such that

$$b_{\kappa^\star+1} = c_1 b_1 + \cdots + c_{\kappa^\star} b_{\kappa^\star}\,, \qquad\qquad c_i \neq 0\,.$$

(Such a vector exists by the definition of $\kappa^\star$ and the convexity of $\mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$). Define $c := (c_1, \ldots, c_{\kappa^\star})$. Since $\{b_1, \ldots, b_{\kappa^\star}\}$ are linearly independent and $\mathcal{B} \cap \operatorname{cone}(\mathcal{S}) = \{0\}$, by Lemma 7.2, $M_{1,\kappa^\star}$ is a non-singular $\mathscr{M}$-matrix. Consider the following invariance conditions

$$H^T_{1,\kappa^\star} b_{\kappa^\star+1} = H^T_{1,\kappa^\star} Y_{1,\kappa^\star} c = M_{1,\kappa^\star} c \preceq 0\,.$$

By Theorem 7.1(v) and the fact that $c_i \neq 0$, we obtain $c \prec 0$. Now consider the invariance conditions

$$h_j \cdot b_{\kappa^\star+1} = h_j \cdot (c_1 b_1 + \cdots + c_{\kappa^\star} b_{\kappa^\star}) \leq 0\,, \qquad\qquad j = \kappa^\star + 2, \ldots, n\,.$$

Every term in the sum is non-negative, since $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and $c_i < 0$, and so we obtain

$$h_j \cdot b_i = 0\,, \qquad\qquad i = 1, \ldots, \kappa^\star + 1, \quad j = \kappa^\star + 2, \ldots, n\,. \tag{8.3}$$

Now by Theorem 7.1(iii) there exists $c' = (c'_1, \ldots, c'_{\kappa^\star})$ such that $c' \preceq 0$ and $M_{1,\kappa^\star} c' \prec 0$. Define $b'_{\kappa^\star+1} := Y_{1,\kappa^\star} c'$. The vector $H^T_{1,n} b'_{\kappa^\star+1} \in \mathbb{R}^n$ has the following sign pattern:

$$(-, \ldots, -, *, 0, \ldots, 0) \tag{8.4}$$

where the $*$ appears in the $(\kappa^\star+1)$th component. In particular $b'_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$ and the first $\kappa^\star$ invariance conditions are strictly negative. Now suppose we find a non-zero vector $\beta \in \operatorname{sp}\{\beta_{\kappa^\star+1}, \ldots, \beta_m\}$ such that

$$h_j \cdot \beta \leq 0\,, \qquad\qquad j = \kappa^\star + 2, \ldots, n\,. \tag{8.5}$$

Then for $\alpha > 0$ we can form

$$b''_{\kappa^\star+1} := b'_{\kappa^\star+1} + \alpha\beta\,.$$

Using (8.4) and (8.5), α can be selected sufficiently small so that $h_j \cdot b''_{\kappa^\star+1} \leq 0$ for all $j = 1, \ldots, \kappa^\star, \kappa^\star + 2, \ldots, n$. That is, $b''_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$. Moreover, with $\beta \neq 0$,

$$\{b_1, \ldots, b_{\kappa^\star}, b''_{\kappa^\star+1}\}$$

is a linearly independent set. This contradicts that $\mathcal{B} \cap \mathcal{C}_{\kappa^\star+1} \subset \operatorname{sp}\{b_1, \ldots, b_{\kappa^\star}\}$. The conclusion is that there does not exist $\beta \in \operatorname{sp}\{\beta_{\kappa^\star+1}, \ldots, \beta_m\}$, $\beta \neq 0$, satisfying (8.5).

Now let $y(x)$ be any continuous closed-loop vector field on $\mathcal{S}$ satisfying the invariance conditions (4.2). Using (8.2), for $x \in \mathcal{G}^\star$, let

$$y(x) = c_1(x) b_1 + \cdots + c_{\kappa^\star}(x) b_{\kappa^\star} + \beta(x)\,, \tag{8.6}$$

where $\beta(x) \in \operatorname{sp}\{\beta_{\kappa^\star+1}, \ldots, \beta_m\}$. From (4.2) we know that for each $x \in \mathcal{G}^\star$

$$h_j \cdot y(x) \leq 0\,, \qquad\qquad j = \kappa^\star + 2, \ldots, n\,.$$

Using (8.3) and (8.6) these conditions become

$$h_j \cdot \beta(x) \leq 0\,, \qquad\qquad j = \kappa^\star + 2, \ldots, n\,,$$

but we have just shown that no such non-zero β exists, so it must be that $\beta(x) = 0$. Therefore for each $x \in \mathcal{G}^\star$,

$$h_j \cdot y(x) = 0\,, \qquad\qquad j = \kappa^\star + 2, \ldots, n\,,$$

as desired. □

Remark 8.2. Proposition 8.2 has the following intuitive meaning. For simplicity suppose $v_0 = 0$. We know from the geometry of the simplex (see Lemma 4.4) that the state space can be decomposed as follows:

$$\mathbb{R}^n = \mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\} \oplus \mathrm{sp}\{h_{\kappa^\star+2}, \ldots, h_n\} \simeq \mathbb{R}^{\kappa^\star+1} \oplus \mathbb{R}^{n-\kappa^\star-1} . \tag{8.7}$$

Therefore, Proposition 8.2 says that

$$\mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}\} \subset \mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\} .$$

Moreover, for all $x \in \mathcal{G}^\star$,

$$y(x) \in \mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}\} .$$

Geometrically, $\mathcal{G}^\star$ lies in $\mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\}$, a $\kappa^\star + 1$ dimensional affine space in $\mathbb{R}^n$, and it is itself a $\kappa^\star$-dimensional simplex in this space. Meanwhile, $\mathcal{B}$ provides to $\mathcal{G}^\star$ only $\kappa^\star$ usable directions (which also lie in $\mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\}$) to resolve all its invariance conditions. We will see that this is not enough to establish a flow condition on $\mathcal{G}^\star$.

Consider again Figure 3 in which $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \{0\}$. We have that $n = 3$, $m = 1$, and $\kappa = \kappa^\star = 1$. Thus, $\mathcal{G} = \overline{v_1 v_2}$. Also, $\mathcal{B} = \mathrm{sp}\{b_1 \mid b_1 \in \mathcal{B} \cap \mathcal{C}_1\}$. We can see that in order not to have any $\mathcal{B} \cap \mathcal{C}_i = \{0\}$, for $i = 1, 2$, it must be that $\mathcal{B} \subset \mathrm{aff}\{v_0, v_1, v_2\}$, which is the affine (linear) space that contains the convex hull of $\mathcal{G}$ and v_0. Notice that this space is equivalently definable by $\{y \mid h_3 \cdot y = 0\}$. Proposition 8.2 then says that all velocity vectors available to $\mathcal{G}$ lie in $\mathrm{aff}\{v_0, v_1, v_2\}$.

Proposition 8.2 describes the fundamental geometric property that forces the existence of an equilibrium. The proof that an equilibrium exists can now be executed in a number of ways, including index theory and the Brouwer Fixed Point Theorem. An efficient proof is based on Sperner's Lemma [18].

Let $\mathbb{T}$ be a triangulation of n-dimensional simplex $\mathcal{S}$. A *proper labeling* of the vertices of $\mathbb{T}$ is as follows:

- (P1) Vertices of the original simplex $\mathcal{S}$ have $n + 1$ distinct labels.
- (P2) Vertices of $\mathbb{T}$ on a face of $\mathcal{S}$ are labeled using only the labels of the vertices forming the face.

Given a properly labeled triangulation of $\mathcal{S}$, we say a simplex in $\mathbb{T}$ is *distinguished* if its vertices have all $n+1$ labels. Sperner's lemma says that every properly labeled triangulation of $\mathcal{S}$ has an odd number of distinguished simplices.

Example 8.1. By way of example, consider the simplex $\mathcal{S}$ in Figure 4 and suppose the possible labels are a (blue), b (red), or c (green). The vertices each have a distinct label, so condition (P1) is met. Also, for the shown triangulation of $\mathcal{S}$, (P2) is satisfied. For example, along the left edge, vertices are labelled only by a or b. Consequently there exists at least one distinguished subsimplex, shaded in the figure, with vertices with all three labels.

Theorem 8.3. *Suppose Assumptions 6.1 and 8.1 hold. Let $u(x)$ be a continuous state feedback such that the closed-loop system $\dot{x} = Ax + Bu(x) + a = y(x)$ has unique solutions and the invariance conditions* (4.2) *hold. Then the closed-loop system has an equilibrium point in $\mathcal{G}$.*

Proof. By Assumption 8.1, $\mathcal{G} = \mathrm{conv}\{v_1, \ldots, v_{\kappa+1}\}$. If $\kappa > m$, redefine $\mathcal{G}$ as $\mathcal{G} = \mathrm{conv}\{v_1, \ldots, v_{m+1}\}$. Define the simplex $\mathcal{G}^\star$ using the construction above and let $I^\star := \{1, \ldots, \kappa^\star + 1\}$. Now we

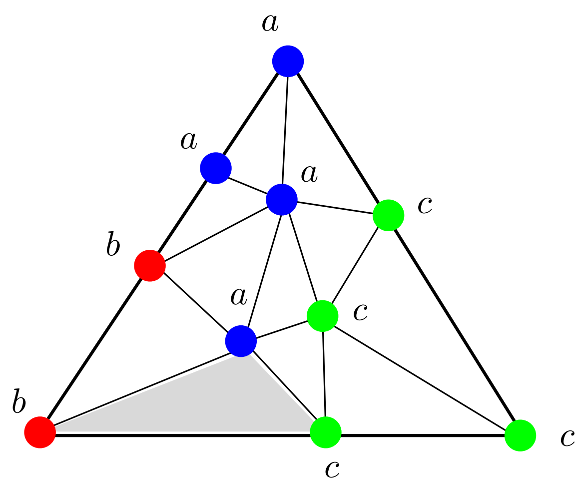

FIGURE 4. Example of Sperner's Lemma

show how to obtain a proper labeling of $\mathcal{G}^\star$. We begin by defining the sets:

$$\mathcal{Q}_i^\star := \{x \in \mathcal{G}^\star \mid h_i \cdot y(x) > 0\}\,, \qquad\qquad i \in I^\star\,.$$

Observe that $v_i \in \mathcal{Q}_i^\star$ and $v_i \notin \mathcal{Q}_j^\star$, $i, j \in I^\star$, $i \neq j$, for otherwise, we would have $y(v_i) \in \mathcal{B} \cap \text{cone}(\mathcal{S})$ which either contradicts that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$ or implies $y(v_i)$ is an equilibrium. Therefore, either the proof concludes with an equilibrium on a vertex of $\mathcal{G}^\star$, or we can infer that inclusion in a set $\mathcal{Q}_i^\star$ provides a distinct label for the vertices $v_i \in \mathcal{G}^\star$. This satisfies (P1) of a proper labeling of $\mathcal{G}^\star$. Next, let $\mathbb{T}$ be any triangulation of $\mathcal{G}^\star$ and consider a vertex v of $\mathbb{T}$ which is not a vertex of $\mathcal{G}^\star$ and lies in $\partial\mathcal{G}^\star$. W.l.o.g. let $v \in \text{conv}\{v_1, \ldots, v_{l+1}\}$ for some $1 \leq l < \kappa^\star$. Then it must be that $v \in \mathcal{Q}_k^\star$ for some $1 \leq k \leq l+1$, by the same reasoning that otherwise $y(v) \in \mathcal{B} \cap \text{cone}(\mathcal{S})$. Clearly this labeling of v satisfies the second condition (P2) for a proper labeling. Finally, for vertices v of $\mathbb{T}$ in the interior of $\mathcal{G}^\star$, any label $\mathcal{Q}_i^\star$ such that $h_i \cdot y(v) > 0$ can be used (at least one such exists because if all $h_i \cdot y(v) \leq 0$, $i \in I^\star$, it implies $h_i \cdot y(v) \leq 0$ for all $i = 1, \ldots, n$ or $y(v) \in \mathcal{B} \cap \text{cone}(\mathcal{S})$).

Now for each $k > 0$, $k \in \mathbb{Z}$, define a triangulation $\mathbb{T}^k$ of $\mathcal{G}^\star$ such that each simplex of $\mathbb{T}^k$ has diameter $\frac{1}{k}$. Apply Sperner's lemma for each $\mathbb{T}^k$ to obtain a distinguished simplex $\text{conv}\{v_1^k, \ldots, v_{\kappa^\star+1}^k\}$ and its baricenter x^k. $\{x^k\}$ defines a bounded sequence in $\mathcal{G}^\star$ which has a convergent subsequence, again denoted $\{x^k\}$. We have $\lim_{k\to\infty} x^k = \overline{x} \in \mathcal{G}^\star$, since $\mathcal{G}^\star$ is closed. Also, by construction $v_i^k \to \overline{x}$, $i \in I^\star$. By Sperner's lemma we know that $h_i \cdot y(v_i^k) > 0$, $i \in I^\star$, so by continuity of $y(x)$ this implies $h_i \cdot y(\overline{x}) \geq 0$, $i \in I^\star$. Combined with Proposition 8.2, we obtain that $-y(\overline{x}) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$, which implies $\overline{x} \in \mathcal{G}^\star$ is an equilibrium of the closed-loop system $\dot{x} = y(x)$. □

9. EXISTENCE OF CONTINUOUS STATE FEEDBACK

In this section we collect the previous results to resolve the boundary between continuous state feedback and affine feedback.

Theorem 9.1. *Suppose Assumption 6.1 holds. Then the following statements are equivalent:*

(1) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by affine feedback.*
(2) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by continuous state feedback.*

Proof. (1) $\Longrightarrow$ (2) is obvious.
(2) $\Longrightarrow$ (1) Suppose there exists a continuous state feedback $u(x)$ such that the closed loop system (8.1) has a unique solution for each initial condition in $\mathcal{S}$ and Problem 3.2 is solved using $u(x)$. By Lemma 4.3 the invariance conditions (4.2) are satisfied, implying (4.1) are solvable. Suppose $\mathcal{G} = \emptyset$. Then by Theorem 6.1, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose $\mathcal{G} \neq \emptyset$. Also, suppose $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \{0\}$. Then by Theorem 6.2, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Instead suppose $\mathcal{G} \neq \emptyset$ and $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \{0\}$. Suppose $v_0 \in \mathcal{G}$. Then by Proposition 8.1, the closed-loop system has an equilibrium point $v_0 \in \mathcal{S}$, a contradiction. Instead suppose $v_0 \notin \mathcal{G}$ and w.l.o.g. $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$. Suppose there does not exist a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$. Then by Theorem 8.3 the closed-loop system has an equilibrium point $x_0 \in \mathcal{S}$, a contradiction. Instead suppose there does exist a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$. Then by Theorem 6.3, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. □

References

[1] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*. vol. 35, pp. 407–428, March 1999.
[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York. 1979.
[3] F. Blanchini, A. Megretski. Robust state feedback control of LTV systems: nonlinear is better than linear. *IEEE Trans Automatic Control*. vol 44, issue 4, April 1999, pp. 802–807.
[4] F. Blanchini and F.A. Pellegrino. Relatively optimal control: A static piecewise affine solution. *SIAM Journal on Control and Optimization* vol. 46, issue 2, p585–603, 2007.
[5] R.W. Brockett. Asymptotic stability and feedback stabilization. In R.W. Brockett, R.S. Millmann, and H.J. Sussmann, Eds., *Progress in Mathematics*. Birkhauser, Boston, MA, 1983.
[6] F.H. Clarke, Yu.S. Ledyaev, E.D. Sontag, and A.I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Trans. Automat. Control.* 42 (1997), pp. 1394–1407.
[7] P.J. Collins, L.C.G.J.M. Habets, A. Kuut, M. Nool, M. Petreczky, and J.H. van Schuppen. ConPAHS - A software package for control of piecewise-affine hybrid systems. *Proc. 2006 IEEE Conference on Computer Aided Control System Design* (CACSD '06). (Munich, Germany, October 4-6, 2006), IEEE Press, New York, pp. 76–81, 2006.
[8] L.C.G.J.M. Habets and J.H. van Schuppen. Control of piecewise-linear hybrid systems on simplices and rectangles, in: M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli (Eds.) *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science*. Springer Verlag, vol. 2034, pp. 261–274, 2001.
[9] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.
[10] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control*. no. 51, pp. 938–948, 2006.
[11] J. Hudson. *Piecewise Linear Topology*. University of Chicago Lecture Notes. W.A. Benjamin Inc. 1969.
[12] C. W. Lee. Subdivisions and triangulations of polytopes. *Handbook of Discrete and Computational Geometry*. CRC Press Series Discrete Math. Appl., pp. 271–290, 1997.
[13] Z. Lin and M.E. Broucke. Resolving control to facet problems for affine hypersurface systems on simplices. *IEEE Conference on Decision and Control* (CDC '06). December 2006.
[14] Z. Lin and M.E. Broucke. Reachability and control of affine hypersurface systems on polytopes. *IEEE Conference on Decision and Control* (CDC '07). December 2007.

[15] L. Rifford. Semiconcave control-Lyapunov functions and stabilizing feedbacks. *SIAM J. Control Optim.* vol. 41, no. 3, pp. 659-681, 2002.
[16] R.T Rockafellar. *Convex Analysis*. Princeton University,Princeton, New Jersey 1970.
[17] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. vol. 42, no. 11, pp. 1913–1918, November 2006.
[18] Y.A. Shashkin. *Fixed Points*. American Mathematical Society, Providence, R.I., 1991.
[19] O. Toker and H. Özbay. On the $\mathcal{N}P$-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback. *Proceedings of the 1995 American Control Conference*, Seattle, Washington, 1995.
[20] M. L. Tyler and M. Morari. Propositional logic in control and monitoring problems. Automatica, vol. 35, no. 4, pp. 565–582, 1999.
[21] G. M. Zeigler. *Lectures on Polytopes*. Graduate Texts in Mathematics. Springer, 1995.

Dept. of Electrical and Computer Engineering, University of Toronto, Toronto ON Canada M5S 3G4

E-mail address: `broucke@control.utoronto.ca`