Generalized flow conditions for reach control on polytopes

Mohamed K. Helwa and Mireille E. Broucke

Abstract— **The paper studies the reach control problem (RCP) to make an affine system defined on a polytopic state space reach and exit a prescribed facet of the polytope in finite time without first leaving the polytope. We introduce the notion of *generalized flow conditions*, which give a necessary and sufficient condition for closed-loop trajectories to exit the polytope. In analogy with Lyapunov stability theory, the generalized flow condition comprises a functional that decreases along closed-loop trajectories. We provide a set of results to analyze whether an instance of RCP is solved, without resorting to exhaustive simulation of the closed-loop system. This includes a variant of the LaSalle principle tailored to RCP. An open problem is to identify suitable classes of functionals that give rise to a generalized flow condition. We explore functions of the form $V(x) = \max\{V_i(x)\}$, and we give evidence that these functions arise naturally when RCP is solved using continuous piecewise affine feedbacks.**

I. Introduction

We study the reach control problem (RCP) for affine systems on polytopes. The problem is for an affine system defined on a polytopic state space to reach and exit a prespecified facet of the polytope in finite time without first leaving the polytope [7]. The problem sits within a family of reachability problems for hybrid systems [6], [11]. Our interest lies in a subclass, piecewise affine hybrid systems, which consist of a discrete automaton such that each discrete mode is equipped with its own continuous-time affine dynamics defined on a polytope. When the continuous state crosses a facet of a polytope, the system is transferred to a new discrete mode. The reachability analysis for piecewise affine hybrid systems at the continuous level reduces to studying RCP for an affine system on a polytope [7].

The most definitive results on the problem are focused on solvability of RCP on simplices by affine feedback [8], [15], [2]. In this case the so-called flow condition was shown to be a necessary and sufficient condition so that all trajectories initiated in a simplex leave it in finite time [8], [15]. However, in [9], [10] it is shown that for general continuous feedbacks such as continuous piecewise affine (PWA) feedbacks the flow condition is no longer necessary for leaving a polytope $\mathcal{P}$ in finite time. Indeed, we have found many examples in which a given continuous control law does not yield a flow condition on $\mathcal{P}$; nevertheless, simulation results show that it solves RCP. The investigation highlights that a more general test is needed for leaving $\mathcal{P}$ in finite time.

We introduce the notion of a generalized flow condition, which is a necessary and sufficient condition that all trajectories initiated in $\mathcal{P}$ leave it in finite time. Precisely, we seek a scalar function $V(x)$ on $\mathcal{P}$ that decreases along closed-loop trajectories. There are strong analogies with Lyapunov stability theory, but the control objective in RCP is very different. (Indeed, we will see that our function need not be positive definite.) The generalized flow condition is related to barrier certificates [12], [13], which are mainly used in the verification of safety of hybrid systems. Again the considered problems are different, but more importantly, unlike barrier certificates, our generalized flow condition does not encode a safe set. We focus on generalized flow conditions based on locally Lipschitz functions, and we provide a set of results that can be used for analysis of solvability of RCP without the need for calculating the state trajectories. These results include a Lasalle Principle for RCP on polytopes. Then we focus on the class of continuous PWA feedbacks, which have been widely studied [7], [9], [10], [1], [5], [14]. For this feedback class we conjecture and then prove in a special case that a suitable generalized flow condition is based on a functional of the form $V(x) = \max\{V_i(x)\}$. Finally, we provide an LP-based computational method for finding a generalized flow condition of this form.

The paper is organized as follows. Section II provides some preliminaries on nonsmooth analysis. Section III presents RCP. In Section IV we introduce the generalized flow condition. In Section V we present a Lasalle Principle for RCP on polytopes. In Section VI we propose a suitable class of functions to generate a generalized flow condition when using PWA feedback. In Section VII two examples are given illustrating the findings of the paper.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $L_f V(x)$ denotes the Lie derivative of function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to function $f : \mathbb{R}^n \to \mathbb{R}^n$.

II. Background

We require some notions from nonsmooth analysis, particularly Dini derivatives of locally Lipschitz functions [3], [16]. Consider

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. Let $\phi(t, x_0)$ denote the unique solution of (1) starting at x_0. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The *upper right Dini derivative* of $V(\phi(t, x_0))$ with respect to t is

$$D^+ V(\phi(t, x_0)) := \limsup_{\tau \to 0^+} \frac{V(\phi(t+\tau, x_0)) - V(\phi(t, x_0))}{\tau} .$$

The authors are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: mkhelwa@scg.utoronto.ca, broucke@control.utoronto.ca). Supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

We can also define the *upper Dini derivative of V with respect to f* given by

$$D^+_f V(x) := \limsup_{\tau \to 0^+} \frac{V(x+\tau f(x)) - V(x)}{\tau} . \qquad (2)$$

It was shown by Yoshizawa [17] that for V locally Lipschitz $D^+V(\phi(t,x_0)) = D^+_f V(x)$ where $x = \phi(t,x_0)$.

III. Reach Control Problem

Consider an n-dimensional polytope

$$\mathcal{P} := \text{co } \{v_1, \ldots, v_p\}$$

with vertex set $V := \{v_1, \ldots, v_p\}$ and facets $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$. The *exit facet* is the facet $\mathcal{F}_0$ of $\mathcal{P}$. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J = \{1, \ldots, r\}$, and $J(x) = \{j \in J \mid x \in \mathcal{F}_j\}$. For each $v \in V$, define the closed, convex cone

$$\mathcal{C}(v) := \left\{ \; y \in \mathbb{R}^n \;:\; h_j \cdot y \leq 0, \; j \in J(v) \; \right\}.$$

(Note that h_0 does not appear since $\mathcal{F}_0$ is the exit facet). We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a \,, \qquad\qquad x \in \mathcal{P} \,, \qquad (3)$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and rank$(B) = m$. Let $\mathcal{B} = \text{Im } B$, the image of B. Also, let $\phi_u(t,x_0)$ be the trajectory of (3) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 3.1 (Reach Control Problem (RCP)): Consider system (3) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t,x_0) \in \mathcal{P}$ for all $t \in [0,T]$, $\phi_u(T,x_0) \in \mathcal{F}_0$, and $\phi_u(t,x_0) \notin \mathcal{P}$ for all $t \in (T, T+\gamma)$.

RCP says that trajectories of (3) starting from initial conditions in $\mathcal{P}$ reach and exit the facet $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. We use the shorthand notation $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ to denote Problem 3.1 is solved by some control law. A recently proposed [9], restricted version of RCP will also be discussed.

Problem 3.2 (Monotonic Reach Control Problem (MRCP)): Consider system (3) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t,x_0) \in \mathcal{P}$ for all $t \in [0,T]$, $\phi_u(T,x_0) \in \mathcal{F}_0$, and $\phi_u(t,x_0) \notin \mathcal{P}$ for all $t \in (T, T+\gamma)$.

(ii) There exists $\xi \in \mathbb{R}^n$ such that for all $x \in \mathcal{P}$, $\xi \cdot (Ax + Bu(x) + a) < 0$.

The added condition (ii) not present in RCP is called a *flow condition*. The new problem is called “monotonic” because trajectories are required to flow through the polytope in a common sense with respect to a foliation of parallel hyperplanes with normal vector ξ.

We conclude our discussion on RCP with conditions that guarantee that under continuous PWA feedback, closed-loop trajectories that exit $\mathcal{P}$ do so only through $\mathcal{F}_0$.

Definition 3.1: We say the *invariance conditions are solvable* if for each $v \in V$ there exists $u \in \mathbb{R}^m$ such that

$$Av + Bu + a \in \mathcal{C}(v) \,. \qquad (4a)$$

Equivalently,

$$h_j \cdot (Av + Bu + a) \leq 0 \,, \qquad j \in J(v) \,. \qquad (4b)$$

Equation (4a) or (4b) is referred to as the *invariance conditions*; we use this terminology either to refer to the conditions for a single vertex or referring to the entire collection of conditions for all vertices of the polytope. Solvability of the invariance conditions has been shown to be necessary for solvability of RCP by continuous feedback [7]. For a given feedback $u(x)$, the following stronger invariance conditions guarantee that closed-loop trajectories that exit $\mathcal{P}$ only do so via $\mathcal{F}_0$

$$h_j \cdot (Ax + Bu(x) + a) \leq 0, \qquad \forall x \in \mathcal{F}_j, \; \forall j \in J. \quad (5)$$

IV. Generalized Flow Conditions

In this section we introduce the main ideas of the paper. These ideas are simple, yet, like Lyapunov theory, they have the potential to be far-reaching. We propose a tool for analysis of controllers for solving RCP on polytopes; specifically, a tool that tells us if closed-loop trajectories exit the polytope.

Suppose we are presented with an instance of RCP on a polytope and we have in hand a continuous feedback $u(x)$ as a candidate feedback solution. For information on how to construct candidate feedbacks, the reader is referred to [7], [8], [15], [9], [10]. Since the invariance conditions (5) necessarily are satisfied by any continuous feedback solving RCP [7], we may assume that $u(x)$ already achieves (5). Immediately we conclude that trajectories can only exit $\mathcal{P}$ through $\mathcal{F}_0$. Then to verify if $u(x)$ actually solves RCP on $\mathcal{P}$, we only have to verify whether all trajectories initiated in $\mathcal{P}$ leave it in finite time. Like Lyapunov theory, we hope to avoid a verification by exhaustive simulation.

In the literature on RCP for simplices and affine feedbacks this verification is performed using a *flow condition* comprising a linear functional that strictly decreases along closed-loop trajectories. Since the simplex is compact, the strictly decreasing condition means closed-loop trajectories must exit. Such a linear functional always exists if RCP is solved on a simplex by a given affine feedback [8], [15]. On the other hand, linear functionals are too restrictive as a class when verifying feedback solutions on polytopes [9]. Indeed, we have many examples where a continuous feedback $u(x)$ is verified to solve RCP via exhaustive simulation, but no linear functional exists. These examples highlight the need for a more general tool to verify that trajectories leave $\mathcal{P}$ in finite time.

At the same time there are well-known results in the literature providing general tests for trajectories to leave compact sets. For example, Proposition 3.5, Chapter 7, of [16] gives the following condition: let $\mathcal{P}$ be a compact set and V a continuously differentiable ($\mathcal{C}^1$) function defined on a neighborhood of $\mathcal{P}$. If $\dot{V}(\phi(t,x_0)) \neq 0$, then all trajectories $\phi(t,x_0)$ starting in $\mathcal{P}$ leave it in finite time.

In sum, on the one hand, we have specific forms of the flow condition matching specific forms of the feedback, in the same way that quadratic Lyapunov functions fit with linear systems and feedbacks. On the other hand, we have general forms of the flow condition requiring only certain differentiability assumptions. A *generalized flow condition* will comprise a general functional that strictly decreases along closed-loop trajectories. An open problem is to identify the most useful classes of functionals for RCP. In this paper we make some headway on this open problem.

We begin with the most general context. Suppose we have a feedback $u(x)$ such that the closed-loop vector field is locally Lipschitz. Suppose we have a functional $V(x)$ bounded from below on $\mathcal{P}$ and satisfying

$$V(\phi_u(t,x_0)) \leq V(x_0) - t \tag{6}$$

for all $x_0 \in \mathcal{P}$ and $t \geq 0$ such that $\phi_u(\tau,x_0) \in \mathcal{P}$, $\tau \in [0,t]$. It is obvious that trajectories must exit $\mathcal{P}$ in finite time. Conversely, suppose that using $u(x)$, all trajectories leave $\mathcal{P}$ in finite time. Then for each $x_0 \in \mathcal{P}$, there exist $T_{x_0} \geq 0$ and $\gamma_{x_0} > 0$ such that $\phi_u(t,x_0) \in \mathcal{P}$ for all $t \in [0,T_{x_0}]$, and $\phi_u(t,x_0) \notin \mathcal{P}$ for all $t \in (T_{x_0}, T_{x_0}+\gamma_{x_0})$. Define the map $T : \mathcal{P} \mapsto \mathbb{R}_+$ by $T(x) := T_x$, $x \in \mathcal{P}$. By uniqueness of solutions, T is a well-defined (i.e. single-valued) function. Also $T(x) \geq 0$ on $\mathcal{P}$. By the semi-group property, $T(\phi_u(t,x_0)) = T(x_0) - t,\ t \in [0, T(x_0)]$. Thus, we have proved the following straightforward but fundamental result showing that existence of a generalized flow condition satisfying (6) is a necessary and sufficient condition.

Theorem 4.1: Consider the system (3) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field is locally Lipschitz. All trajectories starting in $\mathcal{P}$ leave it in finite time if and only if there exists $V : \mathcal{P} \to \mathbb{R}$ such that $V(x)$ is bounded from below on $\mathcal{P}$ and (6) holds.

Next consider the case when V is locally Lipschitz; here only sufficient conditions can be obtained.

Theorem 4.2: Consider the system (3) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. All trajectories starting in $\mathcal{P}$ leave it in finite time if there exists $V : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on a neighborhood of $\mathcal{P}$ and satisfies

$$D_f^+ V(x) \leq -1\,, \qquad x \in \mathcal{P}\,. \tag{7}$$

Now we focus on a particular form of V that appears to have special relevance to RCP. Let $\mathcal{I}_0 = \{1, 2, \cdots, L\}$, and suppose for each $i \in \mathcal{I}_0$, $V_i(x) : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^1$ function. Define

$$V(x) := \max_{i \in \mathcal{I}_0} V_i(x)\,. \tag{8}$$

Lemma 4.3 ([4]): Consider the system (1) and let $V(x)$ be as in (8). Then $V(x)$ is locally Lipschitz and

$$D_f^+ V(x) = \max_{i \in \mathcal{I}(x)} L_f V_i(x)$$

where $\mathcal{I}(x) = \{i \in \mathcal{I}_0 \mid V_i(x) = V(x)\}$.

With this choice of V the condition (7) can be further relaxed.

Theorem 4.4: Consider the system (3) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Let V be as in (8). All trajectories starting in $\mathcal{P}$ leave it in finite time if $D_f^+ V(x) < 0,\ x \in \mathcal{P}$.

Note that the above results are generic in the sense that they are also true for compact non-convex sets. This fact is useful in solving examples (See Example 7.2). In Section VI, we explore in greater depth the properties of polytopes, and we provide a suitable class of generalized flow conditions for RCP on polytopes by PWA feedbacks.

V. LASALLE PRINCIPLE FOR RCP

In this section we study the case where a generalized flow condition has not been found, but we have identified a locally Lipschitz function V satisfying $D_f^+ V(x) \leq 0$ for all $x \in \mathcal{P}$. The question is whether this information is enough to deduce that closed-loop trajectories exit $\mathcal{P}$. For this we use an argument similar to the LaSalle principle, but we use it in the opposite way to how the LaSalle principle is normally applied. The LaSalle principle is used in Lyapunov theory in the case when a positive definite Lyapunov function is not available, but some function that is non-increasing along solutions is available. It allows to show that trajectories tend to an invariant set. Instead, we use the LaSalle principle in the case when a generalized flow condition is not available, but some function that is non-increasing along solutions is available. We use this information to show that trajectories exit from $\mathcal{P}$ if there is no invariant set in a particular subset of $\mathcal{P}$. An example is given in Section VII. Thus, the novelty and the contribution are in showing that a LaSalle principle is meaningful in the context of RCP, despite RCP imposing the opposite requirement of Lyapunov stability. As such, the proof method is almost identical to the standard LaSalle principle, so it is omitted.

Theorem 5.1 (LaSalle): Consider the system (3) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Suppose there exists $V : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on a neighborhood of $\mathcal{P}$ and satisfies $D_f^+ V(x) \leq 0,\ x \in \mathcal{P}$. Let $\mathcal{M} := \{x \in \mathcal{P} \mid D_f^+ V(x) = 0\}$. If $\mathcal{M}$ does not contain an invariant set, then all trajectories starting in $\mathcal{P}$ leave it in finite time.

VI. PWA FEEDBACK

In this section we focus on (continuous) PWA feedback which is widely used to solve RCP on polytopes [7], [8], [9]. There are currently two techniques to solve RCP on polytopes by PWA feedback: MRCP [9] and simplex methods [8], [15]. MRCP imposes that the closed-loop system satisfies a linear flow condition, like the case of simplices with affine feedback, but it does not require that all the invariance conditions of individual simplices of the triangulation are satisfied by the feedback. On the other hand, simplex

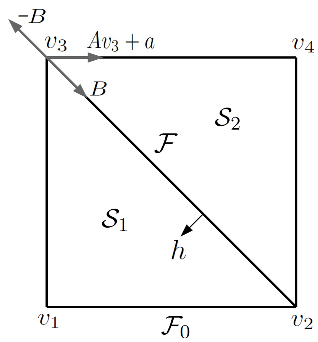

Fig. 1. Invariance conditions of $\mathcal{S}_1$ are not solvable at v_3.

methods relax the requirement for a linear flow condition, but they require that the invariance conditions of each simplex in the triangulation be satisfied. We have found examples in which both techniques fail; nevertheless via exhaustive simulation we verify that a PWA feedback solves RCP on a polyope [9], [10]. Evidently existing techniques are not general enough to explain why a given continuous PWA feedback solves RCP.

In comparing MRCP with the proposed approach, it is clear that MRCP is merely a special case when the generalized flow condition is a linear functional. More interesting is the question of the relationship between generalized flow conditions and simplex methods. That is, what class of generalized flow conditions emerges when RCP is solved by simplex methods? The answer may give clues about the preferred classes of generalized flow conditions for PWA feedback. Because this problem has not been formulated before, we study it in the simplest possible context. We consider the case when $\mathcal{P} \subset \mathbb{R}^n$ consists of two simplices $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $\mathbb{T} = \{\mathcal{S}_1, \mathcal{S}_2\}$ denote the triangulation of $\mathcal{P}$. See Figure 1. Saying that RCP is solved by simplex methods using a PWA feedback $u(x)$ on $\mathbb{T}$ means that

- $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ using $u_1(x) = K_1 x + g_1$, and
- $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ using $u_2(x) = K_2 x + g_2$, where $\mathcal{F} = \mathcal{S}_1 \cap \mathcal{S}_2$.

Moreover, the controller

$$u(x) = \begin{cases} u_1(x), & x \in \mathcal{S}_1 \\ u_2(x), & x \in \mathcal{S}_2 \setminus \mathcal{S}_1 \end{cases}$$

is continuous. Define the closed-loop vector field $f(x) := Ax + Bu(x) + a$. What form does the generalized flow condition take in this case?

Theorem 6.1: Consider a polytope $\mathcal{P}$, a triangulation $\mathbb{T} = \{\mathcal{S}_1, \mathcal{S}_2\}$ of $\mathcal{P}$, and a continuous PWA feedback $u(x)$ defined on $\mathbb{T}$. If $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ using $u(x)$, then there exist affine functions $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(x) = \max\{V_1(x), V_2(x)\} \tag{9}$$

satisfies $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$.

Proof: Let $\mathcal{S}_1 = \text{co } \{v_1, \cdots, v_{n+1}\}$, $\mathcal{S}_2 = \text{co } \{v_2, \cdots, v_{n+2}\}$, h be the unit normal vector to $\mathcal{F}$ pointing out of $\mathcal{S}_2$, and define $\alpha := h \cdot x$, $x \in \mathcal{F}$. By [15] there exists $\xi_1 \in \mathbb{R}^n$ such that

$$\xi_1 \cdot (Ax + Bu(x) + a) < 0\,, \qquad x \in \mathcal{S}_1\,. \tag{10}$$

We choose $V_1(x) = \xi_1 \cdot x$. Second, because $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$, the invariance conditions hold at v_{n+2}. By the geometry of the simplex, h is a negative linear combination of the outward normal vectors of the facets contains v_{n+2}. Thus, $(-h) \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0$. Also, because $u(x)$ is continuous and $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$

$$(-h) \cdot (Av_j + Bu(v_j) + a) \le 0\,, \qquad v_j \in \mathcal{F}\,. \tag{11}$$

Now define

$$\xi_2 := \xi_1 - ch \tag{12}$$

where $c > 0$ is selected sufficiently large such that $\xi_2 \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0$. Using (10), (11), and (12), we get $\xi_2 \cdot (Av_j + Bu(v_j) + a) < 0$, $v_j \in \mathcal{F}$. Since $u(x)$ is affine on $\mathcal{S}_2$, we get $\xi_2 \cdot (Ax + Bu(x) + a) < 0$, $x \in \mathcal{S}_2$. We choose $V_2(x) = \xi_2 \cdot x + c\alpha$ and let $V(x)$ be as in (9).

It remains to show $D_f^+ V(x) < 0$ for $x \in \mathcal{P}$. From above, $L_f V_1(x) < 0$, $x \in \mathcal{S}_1$ and $L_f V_2(x) < 0$, $x \in \mathcal{S}_2$. Recall that $ch \cdot x = c\alpha$ for $x \in \mathcal{S}_1 \cap \mathcal{S}_2$ and h points outside of $\mathcal{S}_2$. Thus $ch \cdot x = (\xi_1 - \xi_2) \cdot x \le c\alpha$, $x \in \mathcal{S}_2$, and $ch \cdot x = (\xi_1 - \xi_2) \cdot x \ge c\alpha$, $x \in \mathcal{S}_1$. Define the sets $\Gamma_1 := \{x \in \mathcal{P} \mid V_1(x) \ge V_2(x)\} = \{x \in \mathcal{P} \mid \xi_1 \cdot x \ge \xi_2 \cdot x + c\alpha\}$ and $\Gamma_2 := \{x \in \mathcal{P} \mid V_2(x) \ge V_1(x)\} = \{x \in \mathcal{P} \mid \xi_1 \cdot x \le \xi_2 \cdot x + c\alpha\}$. Clearly we have $\Gamma_i = \mathcal{S}_i$, $i = 1, 2$. Finally, we apply Lemma 4.3 to get $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$. ■

The goal of the previous result was to discover a form of the generalized flow condition that naturally arises from solving RCP via simplex methods. The result appears to be primarily of theoretical interest because if we know that $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$, then we know that RCP is solved. However, the result is of practical interest when simplex methods fail, yet a flow condition of the form (9) may still be relevant. A typical scenario is when simplex methods fail because the invariance conditions of $\mathcal{S}_1$ are not solvable at some vertices on $\mathcal{F}$. For instance, in Figure 1 the invariance conditions of $\mathcal{P}$ are solvable at v_3. However, for any $u(v_3)$ that we select, the velocity vector $Av_3 + Bu(v_3) + a$ will point outside $\mathcal{S}_1$, and so $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ always fails for any PWA feedback $u(x)$ on $\mathbb{T}$. Despite this failure, the overall problem to exit the polytope may still be solved by the same $u(x)$ and by verifying a generalized flow condition of the form (9). The next result gives a computational test that explicitly depends on the form of generalized flow condition given in the proof of Theorem 6.1.

Corollary 6.2: Consider a polytope $\mathcal{P}$ and a triangulation $\mathbb{T} = \{\mathcal{S}_1, \mathcal{S}_2\}$ of $\mathcal{P}$, where $\mathcal{S}_1 = \text{co } \{v_1, \cdots, v_{n+1}\}$ and $\mathcal{S}_2 = \text{co } \{v_2, \cdots, v_{n+2}\}$. Let $u(x)$ be a continuous PWA feedback on $\mathbb{T}$ that satisfies invariance conditions of $\mathcal{P}$, and does not achieve invariance conditions of $\mathcal{S}_1$ at vertices $v_k, \cdots, v_{n+1} \in \mathcal{F}$, where $2 < k \le n + 1$. Suppose that the following linear programming (LP) problem is solvable

$$\begin{bmatrix} f(v_1)^T & 0 \\ \vdots & \vdots \\ f(v_{n+1})^T & 0 \\ f(v_k)^T & -h\cdot f(v_k) \\ \vdots & \vdots \\ f(v_{n+2})^T & -h\cdot f(v_{n+2}) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ c \end{bmatrix} < 0\,. \qquad (13)$$

Then there exists a function $V(x)$ of the form (9) such that $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$.

Corollary 6.2 provides a simple tool for verifying that all closed-loop trajectories initiated in $\mathcal{P}$ leave it in finite time for the case where existing techniques fail.

VII. Examples

Example 7.1: In this example we show how to use the generalized flow condition to check if a given locally Lipschitz control law $u(x)$ solves RCP on $\mathcal{P}$. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -1 \\ 1 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 0.5 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (14)$$

defined on a polytope $\mathcal{P}$. The polytope is shown in Figure 2. The vertices of $\mathcal{P}$ are $v_0 = (0,0,0)$, $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$, and $v_4 = (1,1,1)$. The exit facet is $\mathcal{F}_0 = \text{co }\{v_1, v_3, v_4\}$. Let $\mathcal{S}_1 := \text{co }\{v_1, v_2, v_3, v_4\}$ and $\mathcal{S}_2 := \text{co }\{v_0, v_1, v_2, v_3\}$.

Suppose that the following continuous PWA feedback is used on $\mathcal{P}$

$$u(x) = \begin{cases} \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} x + 1\,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} x + 0\,, & x \in \mathcal{S}_2\,. \end{cases}$$

It is required to verify that $u(x)$ solves RCP on $\mathcal{P}$. Let $f(x) := Ax + Bu(x) + a$. Then $f(v_0) = (1,1,1)$, $f(v_1) = (1,0,2)$, $f(v_2) = (0,-1,1)$, $f(v_3) = (0,0,-1)$, and $f(v_4) = (-1,-1,-1)$. It is easily verified that the $f(v_j)$ satisfy the invariance conditions (4). As continuous PWA feedback is used, invariance conditions (4) imply (5), and so trajectories that leave $\mathcal{P}$ do so only through $\mathcal{F}_0$. Then, we check that $\mathcal{P}$ does not contain closed-loop equilibria. It can be verified that $0 \notin \text{co }\{f(v_1), f(v_2), f(v_3), f(v_4)\}$, and $0 \notin \text{co }\{f(v_0), f(v_1), f(v_2), f(v_3)\}$. Therefore, there exists a linear flow condition on each simplex $\mathcal{S}_i$ [15]. However, the no-equilibrium condition is not sufficient to conclude that all trajectories initiated in $\mathcal{P}$ leave it in finite time [9].

To verify that closed-loop trajectories leave $\mathcal{P}$, we first check if MRCP is satisfied. We compute $0.5f(v_0) + 0.5f(v_4) = 0$ or $0 \in \text{co }\{f(v_0), f(v_1), \ldots, f(v_4)\}$, and so there does not exist $\xi \in \mathbb{R}^n$ such that $\xi \cdot f(x) < 0$, $x \in \mathcal{P}$. So $u(x)$ does not solve MRCP. Next we check if $u(x)$ solves RCP using simplex methods. Let $\mathcal{F} := \mathcal{S}_1 \cap \mathcal{S}_2$, and h be the unit normal vector to $\mathcal{F}$ pointing toward $\mathcal{S}_1$. We compute $h = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $h \cdot f(v_3) < 0$; hence, the velocity vector $f(v_3)$ dips inside $\mathcal{S}_2$. Therefore, $u(x)$ does

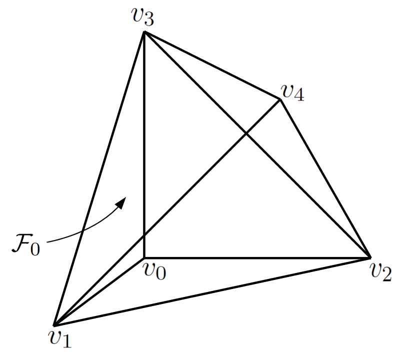

Fig. 2. $\mathcal{P}$ for Examples 7.1 and 7.2

not achieve $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$, and $u(x)$ does not solve RCP by simplex methods.

We conclude RCP is not solved by $u(x)$ via any existing technique. Now we check if a generalized flow condition exists. It is verified that v_3 is the only vertex on $\mathcal{F}$ at which the velocity vector $f(v_3)$ does not satisfy the invariance conditions of $\mathcal{S}_1$. Based on Corollary 6.2 we check the existence of a generalized flow condition with V of the form (9) by solving the LP

$$\begin{bmatrix} f(v_1)^T & 0 \\ f(v_2)^T & 0 \\ f(v_3)^T & 0 \\ f(v_4)^T & 0 \\ f(v_3)^T & -h\cdot f(v_3) \\ f(v_0)^T & -h\cdot f(v_0) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ c \end{bmatrix} < 0\,.$$

A solution of this LP is $\xi_1 = (-167.855, 150.962, 66.055)$, and $c = 71.303$. Then, we calculate $\xi_2 = \xi_1 - ch = (-209.023, 109.795, 24.888)$. We define $V(x) = \max(\xi_1 \cdot x, \xi_2 \cdot x + c\alpha)$. By Corollary 6.2, $D_f^+ V(x) < 0$, $x \in \mathcal{P}$. By Theorem 4.4, all closed-loop trajectories exit $\mathcal{P}$ in finite time. Since the invariance conditions of $\mathcal{P}$ hold, they do so only through $\mathcal{F}_0$. We conclude that $u(x)$ solves RCP on $\mathcal{P}$.

Example 7.2: In this example RCP is not solvable by continuous PWA feedback using any existing technique. However, we show using the results presented in this paper that there exists a continuous PWA feedback solving RCP on $\mathcal{P}$. Consider the system

$$\dot{x} = \begin{bmatrix} 1.25 & 3 & 0 \\ -1 & -11.5 & -1 \\ -1.25 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ -1 \\ -10 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (15)$$

defined on a polytope $\mathcal{P}$. The polytope is shown in Figure 2. The vertices of $\mathcal{P}$ are $v_0 = (0,0,0)$, $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$, and $v_4 = (1,1,1)$. The exit facet is $\mathcal{F}_0 = co\,\{v_1, v_3, v_4\}$. The control objective in this example is to solve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by continuous PWA feedback.

First, it can be verified using Corollary 4.5 of [9] that MRCP is not solvable by continuous PWA feedback. Secondly, it can be shown using an argument similar to the one used in Example 6.2 of [9] that RCP is not sovable by

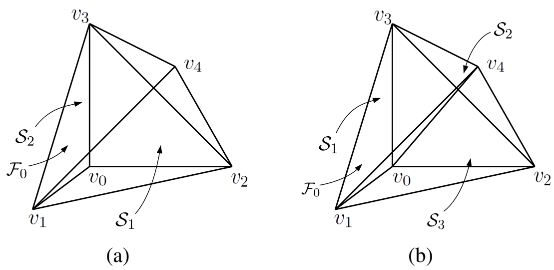

Fig. 3. Two triangulations of $\mathcal{P}$ for Example 7.2

simplex methods for any choice of triangulation of $\mathcal{P}$.

Hence, RCP is not solvable by continuous PWA feedback using any known technique. Now we use the results presented in this paper to show there exists a continuous PWA feedback that solves RCP on $\mathcal{P}$. First, we construct a candidate feedback solution, which is a continuous PWA feedback that satisfies the necessary conditions for solvability of RCP. Following Proposition 4.4 of [9], we select the $B-$extremal control values: $u(v_0) = 0,\ u(v_1) = -0.13158,\ u(v_2) = 0.1,\ u(v_3) = 0$, and $u(v_4) = 0.3158$. This assignment achieves the invariance conditions (4) at all vertices. If we triangulate $\mathcal{P}$ as shown in Figure 3(a) ($\mathcal{S}_1 = \text{co } \{v_1, v_2, v_3, v_4\}$ and $\mathcal{S}_2 = \text{co } \{v_0, v_1, v_2, v_3\}$), then $\mathcal{P}$ will contain a closed-loop equilibrium point since $0 \in \text{co } \{f(v_1), f(v_2), f(v_3), f(v_4)\}$. Instead, we triangulate $\mathcal{P}$ as shown in Figure 3(b) ($\mathcal{S}_1 = \text{co } \{v_0, v_1, v_3, v_4\}$, $\mathcal{S}_2 = \text{co } \{v_0, v_2, v_3, v_4\}$, and $\mathcal{S}_3 = \text{co } \{v_0, v_1, v_2, v_4\}$), and construct the affine feedback on each simplex [7]. We get $u(x) = K_i x + g_i,\ x \in \mathcal{S}_i$, where $K_1 = [-0.1316\ 0.4474\ 0]$, $g_1 = 0$, $K_2 = [0.2158\ 0.1\ 0]$, $g_2 = 0$, $K_3 = [-0.1316\ \ 0.1\ \ 0.3474]$, $g_3 = 0$. It can be checked that $0 \notin \text{co } \{f(v_0), f(v_1), f(v_3), f(v_4)\}$, $0 \notin \text{co } \{f(v_0), f(v_2), f(v_3), f(v_4)\}$, and $0 \notin \text{co } \{f(v_0), f(v_1), f(v_2), f(v_4)\}$. So, using $u(x)$ $\mathcal{P}$ does not contain closed-loop equilibrium points. Now we show, using the results obtained in this paper, that the above continuous PWA feedback $u(x)$ solves RCP on $\mathcal{P}$.

Proposition 7.1: Given the polytope $\mathcal{P}$ and system (15), $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ using $u(x)$.

Proof: First, we show that all closed-loop trajectories initiated in $\mathcal{P}$ leave it in finite time. Let $V(x) = \max(V_1(x), V_2(x))$, where $V_1(x) = x_3 - x_1$ and $V_2(x) = 0$. We study $D_f^+ V(x)$ on $\mathcal{P}$. In this example we have $\Gamma_1 = \{x \in \mathcal{P} : x_3 - x_1 \geq 0\}$ and $\Gamma_2 = \{x \in \mathcal{P} : x_3 - x_1 \leq 0\}$. Then, we study $L_f V_1(x)$ on Γ_1. We have $L_f V_1(x) = -2.5x_1 - 2x_2 - 2x_3 - 20u(x)$. It can be verified that $L_f V_1(x) \leq 0,\ x \in \Gamma_1$, with equality holding only at v_0. Also, it is clear that $L_f V_2(x) = 0,\ x \in \Gamma_2$. So, for all $i = 1, 2$ we have $L_f V_i(x) \leq 0,\ x \in \Gamma_i$. Using Lemma 4.3, it follows that $D_f^+ V(x) \leq 0,\ x \in \mathcal{P}$. Also, equality holds for all $x \in \mathcal{P}$ satisfying $x_3 - x_1 \leq 0$. This gives $\mathcal{M} = \{x \in \mathcal{P} : x_3 - x_1 \leq 0\}$. Now to apply Theorem 5.1, it remains to show $\mathcal{M}$ does not contain invariant sets. The set $\mathcal{M}$ is compact, and it can be verified that $\mathcal{M} \subset \mathcal{S}_1 \cup \mathcal{S}_3$. Consider the function $W(x) := f(x)^T P f(x)$ defined on $\mathcal{M}$, where P is a symmetric matrix determined by solving the following set of linear matrix inequalities (LMIs)

$$(A + BK_i)^T P + P(A + BK_i) < 0, \quad i = 1, 3.$$

The problem is feasible, and we get

$$P = \begin{bmatrix} 0.2066 & 0.0941 & 0.0791 \\ 0.0941 & 0.086 & 0.0001 \\ 0.0791 & 0.0001 & 0.1538 \end{bmatrix}.$$

The function $W(x)$ is locally Lipschitz, but not C^1 on $\mathcal{M}$. Using Proposition 1.5 in Chapter 2 of [3], it can be shown that there exists $\epsilon > 0$ such that $D_f^+ W(x) < -\epsilon,\ x \in \mathcal{M}$. By rescaling W we can apply Theorem 4.2 (which clearly also applies to nonconvex sets) to obtain that all trajectories initiated in $\mathcal{M}$ leave it in finite time. Therefore, $\mathcal{M}$ does not contain invariant sets. Then by Theorem 5.1, all trajectories initiated in $\mathcal{P}$ leave it in finite time. As $u(x)$ satisfies the invariance conditions of $\mathcal{P}$, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ using $u(x)$. ∎

REFERENCES

[1] A. Bemporad, A. Oliveri, T. Poggi, and M. Storace. Ultra-fast stabilizing model predictive control via canonical piecewise affine approximations. *IEEE Transactions on Automatic Control*. Vol. 56, issue 12, pp. 2883-2897, Dec. 2011.

[2] M.E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. vol. 48, issue 5, pp. 3482-3500, February 2010.

[3] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer-Verlag New York Inc., 1998.

[4] J. M. Danskin. The theory of max–min, with applications. *SIAM Journal on Applied Mathematics*. Vol. 14, no. 4, pp. 641–664, 1966.

[5] B.A.G. Genuit, L. Lu, W.P.M.H. Heemels. Approximation of PWA control laws using regular partitions: an ISS approach. *18th IFAC World Congress*. Milano, Italy, August 2011, pp. 4540-4545.

[6] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*. Vol. 29, no. 2, pp. 28–93, April 2009.

[7] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.

[8] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control*. no. 51, pp. 938–948, 2006.

[9] M. K. Helwa and M. E. Broucke. Monotonic Reach Control on Polytopes. IEEE Conf. Decision and Control and European Control Conf. pp. 4741-4746, December 2011.

[10] M. K. Helwa and M. E. Broucke. Monotonic Reach Control on Polytopes. *IEEE Trans. Automatic Control*. Accepted, April 2012.

[11] Z. Lin and M. E. Broucke. On a reachability problem for affine hypersurface systems on polytopes. *Automatica*, vol. 47, issue 4, pp. 769–775, April 2011.

[12] S. Prajna and A. Jadbabaie. Safety verification of hybrid systems using barrier certificates. *The 7th International Conf. Hybrid Systems: Computation and Control*, pp. 477–492, 2004.

[13] S. Prajna, A. Rantzer. Convex programs for temporal verification of nonlinear dynamical systems. *SIAM Journal on Control and Optimization*. Vol. 46, no. 3, pp. 999–1021, 2007.

[14] L. Rodrigues and S. Boyd. Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization. *Systems and Control Letters*. 54(9), pp. 835–853, September 2005.

[15] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. vol. 42, no. 11, pp. 1913–1918, November 2006.

[16] N. Rouche, P. Habets, M. Laloy. *Stability Theory by Liapunov's Direct Method*. Springer-Verlag, New York Inc., 1977.

[17] T. Yoshizawa. *Stability Theory by Lyapunov's Second Method*. Mathematical Society of Japan, 1966.