Reach Control Problem: Well-posedness and Structural Stability

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***Abstract*— We study well-posedness and structural stability of the reach control problem (RCP) for affine systems. We demonstrate that the affine and piecewise affine feedbacks introduced in previous papers for solving RCP on a simplex are well-posed and structurally stable in the presence of small perturbations of system parameters. We also present general results on well-posedness and structural stability of RCP on polytopes by continuous state feedback. Some results echo classical findings on well-posedness for the robust regulator problem.**

I. Introduction

This paper studies the well-posedness and structural stability of the *reach control problem* (RCP) on a simplex or a polytope. The problem is to find a feedback control that guides closed-loop trajectories of an affine system through a polytopic state space in order to exit from a prespecified exit facet, without first crossing other facets. RCP was introduced in [7] and further developed in [8], [9], [14]. More geometric tools were developed to obtain conditions for solvability with different classes of controls in [3], [6], [5]. The quest has been to identify the smallest feedback control class that solves RCP when it is solvable by open-loop controls. The feedback control classes studied so far are affine feedbacks [8], [14], continuous state feedbacks [3], and piecewise affine feedbacks [9], [6], [5].

Some of these results rely on an assumption about how the triangulation [11] of the state space is performed; particularly, that closed-loop equilibria may only appear on a facet of any simplex [3]. However, in practice the nominal system parameters are not precise, so a triangulation for the nominal system might fail to satisfy this requirement when one goes to the real-world setup. Therefore, the key question here is whether the proposed solution methods of [3], [6], [5] remain valid if system parameters are perturbed. The present work explores this question and verifies the results obtained in [3], [6], [5] subject to small perturbations of system parameters. We study both well-posedness and structural stability, in line with classical developments for the robust regulator problem [15]. Similar to our previous work, we show that affine feedback and continuous state feedback remain equivalent from the point of view of solvability of RCP even under the requirement of a structurally stable synthesis.

One of the themes explored in the present work is that two types of state constraints arise in RCP. We call them *safety constraints* and *non-safety constraints*. Safety constraints define the permitted region in the state-space in which system trajectories may evolve. These conditions are strict, i.e. under no condition can they be violated. For example, in automated anesthesis delivery, the concentration of drug in the patient's body should not exceed a safe threshhold. In contrast, non-safety constraints are state constraints imposed inside the permitted region of evolution. They guarantee that desired sequences of events occur (such as crossing certain facets) or desired sub-regions of the safe region are visited (such as visiting certain simplices). Under perturbation of system parameters, these internal, non-safety constraints may be violated, yet the global system behavior may still be acceptable; whereas if a safety constraint is violated, the controller is deemed to have failed. We further clarify this distinction in Section III. Whether the so-called *invariance conditions* which restrict trajectories from exiting certain facets of a simplex are interpreted as safety or non-safety constraints shapes the nature of results on well-posedness and structural stability. This flexibility of interpretation allows to bypass some of the inherent conservativism built into the mathematical notions of well-posedness and structural stability.

Notation. Let $\mathcal{S} \subset \mathbb{R}^n$ be a set. The relative interior of $\mathcal{S}$ is denoted $\mathrm{ri}(\mathcal{S})$. For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). For a matrix $A \in \mathbb{R}^{n\times n}$, the notation $A \succ 0$ ($A \succeq 0$) means $a_{ij} > 0$ ($a_{ij} \geq 0$) for $1 \leq i, j \leq n$. Notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. Notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$.

II. Reach Control Problem

We consider an n-dimensional simplex $\mathcal{S} := \mathrm{co}\{v_0, v_1, \ldots, v_n\}$ with vertex set $V := \{v_0, v_1, \ldots, v_n\}$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Let $\mathcal{F}_0$ be the exit facet in $\mathcal{S}$. Define the index set $I := \{1, \ldots, n\}$. For $x \in \mathcal{S}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \; y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in I, \; x \in \mathcal{F}_j \; \right\}.$$

We write $\mathrm{cone}(\mathcal{S}) := \mathcal{C}(v_0)$ since $\mathcal{C}(v_0)$ is the tangent cone to $\mathcal{S}$ at v_0.

Consider an affine control system defined on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a \,, \qquad x \in \mathcal{S} \,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B) = m < n$. Let $\phi_u(t, x_0)$ denote the trajectory of (1) under a

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control $u(t)$ starting from $x_0 \in \mathcal{S}$ and evaluated at time t. We assume throughout that only control inputs that provide for unique solutions of (1) are studied. We are interested in studying reachability of $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{S}$. Find a state feedback $u(x)$ such that: for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

A useful shorthand notation is to write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by $u(x)$ if Problem 1 is solved by $u(x)$.

One of the important contributions of [8] was a set of inequality conditions that guarantee that closed-loop trajectories do not exit $\mathcal{S}$ from non-exit facets $\mathcal{F}_1, \ldots, \mathcal{F}_n$. We say the *invariance conditions are solvable* if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that

$$Av_i + Bu_i + a \in \mathcal{C}(v_i)\,, \qquad i \in \{0, \ldots, n\}\,. \tag{2}$$

Conditions (2) are called the *invariance conditions*, and they are used to construct affine feedbacks [8]. For general continuous state feedbacks, stronger conditions (also called invariance conditions) are needed. We say a state feedback $u(x)$ *satisfies the invariance conditions* if

$$Ax + Bu(x) + a \in \mathcal{C}(x)\,, \qquad x \in \mathcal{S}\,. \tag{3}$$

III. Well-posedness and Structural Stability

We review some facts about well-posedness and structural stability [15]. Consider the system data (A, B, a), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $a \in \mathbb{R}^n$. If the matrix elements are listed in arbitrary order, then (A, B, a) can be regarded as a point λ in vector space $\mathbb{R}^N$, where $N = n^2 + n * m + n$. We say that RCP is *well-posed at* λ if it is solvable at all points in some neighborhood of λ in $\mathbb{R}^N$. It is also of interest to characterize well-posedness for a specific control class. For instance, we say RCP by continuous state feedback is well-posed at λ if it is solvable by continuous state feedback at all points in some neighborhood of λ in $\mathbb{R}^N$. Similarly, we say RCP is *structurally stable at* $\lambda = (A, B, a)$ if it is solvable by a state feedback $u(x)$ at $\lambda \in \mathbb{R}^N$, and it remains solvable with the same feedback $u(x)$ at all points in a neighborhood of λ in $\mathbb{R}^N$.

Now we introduce the main ideas of the paper on well-posedness and structural stability of RCP in the general setting of polytopes rather than simplices, and we also begin with continuous state feedbacks. These main ideas are then ramified for the special case of simplices and for other classes of feedbacks: affine feedbacks and (discontinuous) piecewise affine feedbacks. We begin by motivating the problem studied in this section.

On polytopes RCP will typically be solved by continuous PWA feedback [8] or discontinuous PWA feedback [6], [5]. It is not uncommon that the underlying triangulation of the polytope is chosen arbitrarily, and the precise behavior on simplices is not of interest. Rather, the desired global behavior on the polytope is that the exit facet is reached in finite time without first exiting $\mathcal{P}$ from other facets. In this context, the safety constraints are those encoded in the invariance conditions of the polytope; instead "internal" invariance conditions associated with simplices of a triangulation might be violated when system parameters are perturbed. Consider the example in Figure 1 showing a polytope $\mathcal{P}$ with two simplices. A piecewise affine feedback is designed so that trajectories only exit $\mathcal{P}$ through $\mathcal{F}_0$, and moreover they only flow from $\mathcal{S}_2$ to $\mathcal{S}_1$. The figure on the left shows the closed-loop velocity vectors at the vertices of $\mathcal{P}$ for the nominal system. The invariance conditions of the polytope (safety constraints) and the invariance conditions of the simplices (non-safety constraints) are both satisfied. Now suppose the system is perturbed, so the closed-loop velocity vectors are shifted as in the figure on the right. In particular, the invariance condition of $\mathcal{S}_1$ at v_1 is now violated. Now closed-loop trajectories no longer flow only from $\mathcal{S}_2$ to $\mathcal{S}_1$. Nevertheless, from the point of view of solving RCP on $\mathcal{P}$, this behavior is still acceptable.

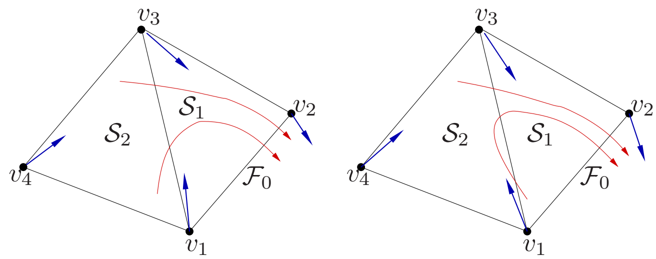

Fig. 1. Safety constraints guarantee that trajectories only exit from $\mathcal{F}_0$. Non-safety constraints guarantee that trajectories only flow from $\mathcal{S}_2$ to $\mathcal{S}_1$.

In summary, RCP on a specific simplex may fail under perturbation of system parameters, whereas RCP remains solved on $\mathcal{P}$ under the same perturbations. A notion of well-posedness or structural stability should account for this flexibility in internal dynamics. For this reason, we first study well-posedness and structural stability for a polytope.

To that end, we consider an n-dimensional polytope $\mathcal{P} := \text{co}\{v_1, \ldots, v_q\}$ with facets $\mathcal{F}_0, \ldots, \mathcal{F}_p$. As before, let h_i be the unit normal vector of facet $\mathcal{F}_i$ pointing outside of $\mathcal{P}$. Let $\mathcal{F}_0$ be the exit facet in $\mathcal{P}$. All definitions in Sections II are analogously defined on polytope $\mathcal{P}$ rather than simplex $\mathcal{S}$. To ensure that RCP remains solvable under perturbation of parameters, we introduce strict invariance conditions on $\mathcal{P}$.

Definition 1: We say a state feedback $u(x)$ satisfies the *strict invariance conditions* if for all $j \in \{1, \ldots, p\}$ and $x \in \mathcal{F}_j$,

$$h_j \cdot (Ax + Bu(x) + a) < 0\,. \tag{4}$$

Notice that for an n-dimensional polytope, (4) is equivalent to saying $Ax + Bu(x) + a \in \text{int}(\mathcal{C}(x))$. Also note that for a simplex, $p = n$. The strict invariance conditions have been introduced because their solvability is necessary for well-posedness of RCP.

Lemma 2: Consider polytope $\mathcal{P}$ and system (1) with system parameters $\lambda = (A, B, a)$. If RCP by continuous state feedback is well-posed at λ, then the strict invariance conditions (4) are solvable.

Proof: Let $x \in \mathcal{P}$ and define $\mathcal{Y}(x) := \{Ax + Bu + a \mid u \in \mathbb{R}^m\}$. $\mathcal{Y}(x)$ is a non-empty affine space in $\mathbb{R}^n$. For a n-dimensional polytope $\mathcal{P}$, $\text{int}(\mathcal{C}(x))$ is a non-empty, open, convex set in $\mathbb{R}^n$. Suppose by way of contradiction that the strict invariance conditions are not solvable at x. That is, $\mathcal{Y}(x) \cap \text{int}(\mathcal{C}(x)) = \emptyset$. By Theorem 11.2 of [13], there exists a hyperplane $\mathcal{H}$ with unit normal vector ξ that contains $\mathcal{Y}(x)$, and $\xi \cdot z < 0$ for $z \in \text{int}(\mathcal{C}(x))$ and $\xi \cdot z \leq 0$ for $z \in \mathcal{C}(x)$. Consider perturbed system parameters $\tilde{\lambda} = (\tilde{A}, \tilde{B}, \tilde{a})$ with $\tilde{A} = A$, $\tilde{B} = B$, and $\tilde{a} = a + \epsilon\xi$, with $\epsilon > 0$. We can choose $\epsilon > 0$ arbitrarily small so that $\tilde{\lambda}$ is arbitrarily close to λ. Also, for all $u \in \mathbb{R}^m$, $\xi \cdot (\tilde{A}x + \tilde{B}u + \tilde{a}) = \epsilon > 0$. That is, $\mathcal{H}$ strongly separates $\mathcal{C}(x)$ and $\tilde{\mathcal{Y}}(x) = \{\tilde{A}x + \tilde{B}u + \tilde{a} \mid u \in \mathbb{R}^m\}$. In other words, $\mathcal{C}(x) \cap \tilde{\mathcal{Y}}(x) = \emptyset$, so the invariance conditions (3) are not solvable at x. By Proposition 3.1 of [8], RCP by continuous state feedback is not solvable for $\tilde{\lambda}$. Since $\tilde{\lambda}$ is arbitrarily close to λ, RCP is not well-posed at λ. ∎

The previous result can be strengthened for simplices.

Lemma 3: Consider simplex $\mathcal{S}$ and system (1) with system parameters $\lambda = (A, B, a)$. If RCP by open-loop controls is well-posed at λ, then the strict invariance conditions (4) are solvable.

Proof: The proof is the same as above. However, instead of invoking Proposition 3.1 from [8], we invoke Theorem 6 from [6] which says that solvability of the invariance conditions is necessary for solvability of RCP by open-loop controls on a simplex. ∎

When speaking about structural stability for RCP, one must make a distinct between mere solvability of the invariance conditions and the invariance conditions actually holding for a candidate feedback over a neighborhood of system parameters.

Lemma 4: Consider polytope $\mathcal{P}$ and system (1) with system parameters $\lambda = (A, B, a)$. If RCP by continuous state feedback $u(x)$ is structurally stable, then the strict invariance conditions (4) hold.

Proof: Suppose by way of contradiction there is a facet $\mathcal{F}_j \neq \mathcal{F}_0$ and a point $x \in \mathcal{F}_j$ such that $h_j \cdot (Ax + Bu(x) + a) \geq 0$. Consider perturbed system parameters $\tilde{\lambda} = (\tilde{A}, \tilde{B}, \tilde{a})$ with $\tilde{A} = A$, $\tilde{B} = B$, and $\tilde{a} = a + \epsilon h_j$, with $\epsilon > 0$. We can choose $\epsilon > 0$ arbitrarily small so that $\tilde{\lambda}$ is arbitrarily close to λ. Also, $h_j \cdot (\tilde{A}x + \tilde{B}u(x) + \tilde{a}) \geq \epsilon > 0$. By Proposition 3.1 of [8], RCP is not solvable using $u(x)$ for the system $\tilde{\lambda}$. This contradicts that RCP by $u(x)$ is structurally stable. ∎

The following result describes general conditions for a structurally stable synthesis on a polytope.

Theorem 5: Consider polytope $\mathcal{P}$ and system (1) with system parameters $\lambda = (A, B, a)$. RCP by continuous state feedback is structurally stable at λ if and only if it is solvable at λ by a continuous, locally Lipschitz state feedback $u(x)$ satisfying the strict invariance conditions (4).

Proof: ($\Longrightarrow$) Follows from Lemma 4.

($\Longleftarrow$) Suppose RCP is solvable by a continuous, locally Lipschitz state feedback $u(x)$ that satisfies the strict invariance conditions (4). Let $\mathcal{D}$ be an open connected neighborhood of $\mathcal{P}$ and consider the closed-loop system defined on $\mathcal{D}$. Let $\phi_u(t, x_0; \lambda)$ be the unique trajectory of the nominal closed-loop system starting from initial condition $x_0 \in \mathcal{P}$. Since $u(x)$ solves RCP, there exists $\gamma > 0$ and a time $T \geq 0$ such that $\phi_u(t, x_0; \lambda) \in \mathcal{D} \setminus \mathcal{P}$ for all $t \in (T, T + \gamma)$. Let $\overline{x} = \phi_u(T + \gamma/2, x_0; \lambda)$, and select $\epsilon > 0$ such that $\mathbb{B}_\epsilon(\overline{x}) \subset \mathcal{D} \setminus \mathcal{P}$, where $\mathbb{B}_\epsilon(\overline{x})$ denotes the open ball of radius ϵ centered at $\overline{x}$ (such a ball exists because $\mathcal{P}$ is closed).

Let $\tilde{\lambda} = (\tilde{A}, \tilde{B}, \tilde{a})$ denote the perturbed system parameters. By Theorem 3.5 of [10], given $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\tilde{\lambda} - \lambda\| < \delta$, then $\|\phi_u(T + \gamma/2, x_0; \tilde{\lambda}) - \overline{x}\| < \epsilon$. In particular, $\phi_u(T + \gamma/2, x_0; \tilde{\lambda}) \notin \mathcal{P}$. This shows that all trajectories of the perturbed system exit $\mathcal{P}$. Moreover, they must exit $\mathcal{P}$ through $\mathcal{F}_0$ because $u(x)$ satisfies strict invariance conditions for the nominal system. That is, we can select $\delta > 0$ sufficient small such that the invariance conditions, which depend continuously on system parameters, still hold: for all $j \in \{1, \ldots, p\}$ and $x \in \mathcal{F}_j$,

$$h_j \cdot (\tilde{A}x + \tilde{B}u(x) + \tilde{a}) \leq 0\,.$$

We conclude RCP is solved using $u(x)$ for any $\tilde{\lambda}$ in a neighborhood of λ. It follows that RCP by continuous state feedback is well-posed and structurally stable at λ. ∎

We remark that Theorem 5 does not generalize to discontinuous control inputs. In particular, it cannot be shown that the strict invariance conditions are necessary for a well-posed and structurally stable solution of RCP. This issue will be clarified in our future work.

IV. Affine Feedback

A general result for structural stability on polytopes using continuous state feedback was presented in the previous section. The result immediately specializes to simplices and affine feedbacks. However, the result is useful only for analysis, as it does not provide conditions for structurally stable synthesis. On the other hand, using prior results [9], [14], progressively more constructive results can be found which provide necessary and sufficient conditions for a well-posed and structurally stable synthesis by affine feedback for simplices.

Theorem 6: Consider simplex $\mathcal{S}$ and system (1) with system parameters $\lambda = (A, B, a)$. RCP by affine feedback is structurally stable at λ if and only if there exists an affine feedback $u(x) = Kx + g$ such that the strict invariance conditions (4) hold and there is no closed-loop equilibrium in $\mathcal{S}$.

Proof: ($\Longleftarrow$) If RCP by affine feedback is structurally stable at λ, then it is solvable by an affine feedback $u(x)$ at λ. Clearly there can be no closed-loop equilibrium in $\mathcal{S}$. Also by Lemma 4 the strict invariance conditions (4) hold using $u(x)$.

($\Longrightarrow$) Suppose there is $u = Kx + g$ such that the strict invariance conditions hold and there is no closed-loop equilibrium in $\mathcal{S}$. By the results of [9], [14], $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by $u(x)$ for the nominal system. Then by Theorem 5, RCP is structurally stable. ∎

More constructive necessary and sufficient conditions for existence of affine feedbacks solving RCP on a simplex were obtained in [3]. These results depend on choosing a special triangulation of the state space that aligns possible closed-loop equilibria along faces of simplices. This raises a question of whether the requirement of a special triangulation leads to a synthesis that is not well-posed or structurally stable, since a perturbation of system parameters will generally destroy the alignment of possible equilibria with faces. The answer immediately follows from Theorem 5 that a synthesis based on a special triangulation is well-posed and structurally stable so long as the conditions of Theorem 5 are satisfied. We summarize the results below.

Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define $\mathcal{O} := \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$. This set is an affine space corresponding to the possible closed-loop equilibria when using state feedback. Define

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O} \,.$$

Associated with $\mathcal{G}$ is its vertex index set $I_{\mathcal{G}} := \{i \mid v_i \in V \cap \mathcal{G}\}$. The requirement for alignment of the state space triangulation with $\mathcal{G}$ appearing in [3] is as follows.

Assumption 7: Simplex $\mathcal{S}$ and system (1) satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$.

Theorem 8: Consider simplex $\mathcal{S}$ and system (1) with system parameters $\lambda = (A, B, a)$. Suppose $\mathcal{G} = \emptyset$. RCP by affine feedback is structurally stable at λ if and only if the strict invariance conditions (4) are solvable.

Proof: ($\Longrightarrow$) Select the control $u_i \in \mathbb{R}^m$ for each vertex v_i to satisfy the strict invariance conditions (4). Using the method of [8], one can find unique K and g corresponding to the affine feedback $u(x) = Kx + g$ such that $u(v_i) = u_i$, $0 \leq i \leq n$. Since $\mathcal{G} = \emptyset$, the closed-loop system has no equilibria in $\mathcal{S}$. Then, the result follows from Theorem 5.

($\Longleftarrow$) If RCP by affine feedback is structurally stable at $\lambda = (A, B, a)$, then the strict invariance conditions are solvable for $\mathcal{S}$ at λ using Theorem 5. ∎

Theorem 9: Consider simplex $\mathcal{S}$ and system (1) with system parameters $\lambda = (A, B, a)$. Suppose $\mathcal{G} \neq \emptyset$ and Assumption 7 holds. RCP by affine feedback is structurally stable at λ if and only if

(i) The strict invariance conditions (4) are solvable.
(ii) Either $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq \mathbf{0}$ or there exists a linearly independent selection $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$.

Proof: ($\Longrightarrow$) If there exists $u = Kx + g$ s.t. (i) and (ii) hold, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback [3]. Thus, an affine feedback exists such that the strict invariance conditions (4) hold and there is no closed-loop equilibrium in $\mathcal{S}$. Then, the result follows from Theorem 5.

($\Longleftarrow$) If RCP by affine feedback is structurally stable at $\lambda = (A, B, a)$, then the strict invariance conditions are solvable for $\mathcal{S}$ at λ using Theorem 5. For condition (ii), if neither case is true, then by Theorem 7.3 of [3], RCP is not solvable by continuous state feedback. Therefore, RCP by affine feedback is not well-posed, and so not structurally stable. ∎

The main conclusion of [3] is that under the Assumption 7, RCP is solvable by affine feedback if and only if it is solvable by continuous state feedback. In the following we show that this equivalence can be extended to well-posedness and structural stability of RCP.

Theorem 10: Consider simplex $\mathcal{S}$ and system (1) with system parameters $\lambda = (A, B, a)$. Suppose Assumption 7 holds. Then the following statements are equivalent:

(a) RCP by affine feedback is structually stable at λ.
(b) RCP by continuous state feedback is structually stable at λ.

Proof: (a) $\Longrightarrow$ (b) is obvious.
(b) $\Longrightarrow$ (a) Suppose RCP by continuous state feedback $u(x)$ is well-posed and structurally stable at λ. Then, it is solvable by affine feedback at λ using Theorem 8.1 of [3] and the strict invariance conditions (4) are solvable by the results of Lemma 4 (or 2). If $\mathcal{G} = \emptyset$, then by Theorem 8, RCP by affine feedback is well-posed and structurally stable. Suppose instead $\mathcal{G} \neq \emptyset$ and either $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq \mathbf{0}$ or there exists a linearly independent selection $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$. Then by Theorem 9, RCP by affine feedback is well-posed and structurally stable at λ. Suppose instead $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$ and every selection $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ is linearly dependent. Then, by Theorem 7.3 of [3], RCP is not solvable by continuous state feedback, a contradiction. Hence, RCP by affine feedback is structurally stable at λ. ∎

V. Piecewise Affine Feedback

So far we have obtained a general result on structural stability of RCP on polytopes by continuous state feedback in Theorem 5, and we have specialized this result to simplices and affine feedback in Theorems 8 and 9. We particularly address that the non-generic triangulation of Assumption 7 does not confound the well-posedness and structural stability outcome. Unfortunately, these findings are not sufficient to close the investigation of well-posedness and structural stability of RCP. For it is known that, even under Assumption 7, the class of continuous state feedbacks is not large enough to solve RCP on simplices [3]. In [6], [5] a discontinuous control method based on piecewise affine feedback is developed to address those cases when RCP is solvable by open-loop controls but not by continuous state feedback. Unfortunately, the general well-posedness and structural stability result of Theorem 5 no longer applies because it is only for continuous state feedbacks. By a careful analysis of the method of [6], [5], we show precisely when a structurally stable synthesis is obtained.

Following the results of [3] and under Assumption 7, we study the case when RCP is not solvable by continuous state feedback but it is solvable by open-loop controls.

Assumption 11: Simplex $\mathcal{S}$ and system (1) satisfy the following conditions.

(R1) $\mathcal{G} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$, where $0 \leq \kappa < n$.
(R2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$.
(R3) The maximum number of linearly independent vectors in any set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ (with only

one vector for each $\mathcal{B} \cap \mathcal{C}(v_i), i \in I_{\mathcal{G}}$) is $\widehat{m}$ with $1 \leq \widehat{m} \leq \kappa$.

(R4) $\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}\,, \quad i \in I_{\mathcal{G}}$.

Let $p := \kappa + 1 - \widehat{m} \geq 1$. One of the principal findings of [6], [4] is the existence of so-called reach control indices. To present the result we require the following notation. Let $r_1, \ldots, r_p \geq 0$ be integers and define the numbers

$$\begin{aligned} m_k &:= r_1 + \cdots + r_{k-1} + 1\,, \qquad k = 1, \ldots, p \\ r &:= r_1 + \cdots + r_p\,. \end{aligned}$$

Theorem 12 ([6], [4]): Suppose Assumption 11 holds. Then there exist integers $r_1, \ldots, r_p \geq 2$ and a decomposition of $\mathcal{B}$ into p subsets such that

$$\mathcal{B} \cap \mathcal{C}(v_i) \ \subset\ \mathcal{B}_1 := \mathrm{sp}\{b_{m_1}, \ldots, b_{m_1+r_1-1}\}\,,\ i = m_1, \ldots, m_1 + r_1 - 1\,, \tag{5}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\mathcal{B} \cap \mathcal{C}(v_i) \ \subset\ \mathcal{B}_p := \mathrm{sp}\{b_{m_p}, \ldots, b_{m_p+r_p-1}\}\,,\ i = m_p, \ldots, m_p + r_p - 1\,, \tag{6}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. Each set $\{b_{m_k}, \ldots, b_{m_k+r_k-1}\}$ is linearly independent if any one vector is removed and for each $k = 1, \ldots, p$, there exist coefficients c_i such that

$$b_{m_k+r_k-1} = c_{m_k} b_{m_k} + \cdots + c_{m_k+r_k-2} b_{m_k+r_k-2}\,, \quad c_i < 0\,. \tag{7}$$

The integers $\{r_1, \ldots, r_p\}$ are called the *reach control indices* of system (1) with respect to simplex $\mathcal{S}$. We additionally require the following result.

Lemma 13 ([6], [4]): Suppose Assumption 11 holds. Then for $k = 1, \ldots, p$, $i = m_k, \ldots, m_k + r_k - 1$, and $j \in I \setminus \{m_k, \ldots, m_k + r_k - 1\}$,

$$h_j \cdot b_i = 0\,, \tag{8}$$

We have seen in Theorem 5 that the strict invariance conditions (4) are necessary for a well-posed and structurally stable synthesis by continuous state feedback. Lemma 13 shows that strict invariance conditions are not achievable when reach control indices are defined. Essentially, reach control indices are defined when the system has insufficient inputs. Therefore, to rule out the possibility of the equality constraints (8) which are not robust to perturbation of system parameters, we require that the system have sufficient inputs.

Lemma 14: Suppose Assumption 11 holds. If the strict invariance conditions (4) are solvable, then $m = \kappa = n - 1$.

Proof: By Theorem 12, $\mathcal{B} \cap \mathcal{C}(v_i) \subset \mathcal{B}_k$, $i = m_k, \ldots, m_k + r_k - 1$. Thus, the equality constraints (8) hold for any $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, $i = m_k, \ldots, m_k + r_k - 1$. This contradicts that the strict invariance conditions are solvable. It must be that for every $k = 1, \ldots, p$,

$$I \setminus \{m_k, \ldots, m_k + r_k - 1\} = \emptyset\,.$$

This can only happen if $p = 1$ and $r_1 = n$. In turn, this implies $\kappa = n - 1$ and $\widehat{m} = \kappa + 1 - p = n - 1$. Since $m \neq n$, this implies $m = n - 1$. ■

We note that the outcome of Lemma 14 echoes similar results for the robust regulator problem. See, for instance, Corollary 8.2 of [15]. Lemma 14 illustrates the conservatism of well-posedness and structural stability in the form of a stringent constraint on the number of inputs. This conservativism arises from two sources. First, well-posedness is an inherently conservative notion; for instance, by allowing arbitrary perturbations of parameters, the physics of a system can be violated. Second, the requirement that the invariance conditions must hold strictly is conservative. We have seen in our study of polytopes in Section III that it may be tolerable to allow invariance conditions on internal simplices to hold with equality constraints, even if these are apparently non-robust, so long as global behavior on the polytope achieves the requirements of RCP.

In [6], [5] a subdivision algorithm for synthesizing piecewise affine feedbacks consisting of p steps is proposed. Lemma 14 shows that only one step of the algorithm is required. We now investigate the extent to which the single step subdivision algorithm can provide a well-posed and structurally stable synthesis for RCP.

We are focused on the case when $\mathcal{G} = \mathcal{F}_0 = \mathrm{co}\{v_1, \ldots, v_n\}$ and $\mathcal{B}_1 = \mathrm{sp}\{b_1, \ldots, b_n \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$. One can see that $\mathcal{B}_1 \not\subset \mathcal{H}_0 := \{y \in \mathbb{R}^n \mid h_0 \cdot y = 0\}$ for otherwise all velocity vectors of (1) on $\mathcal{F}_0$ are tangent to $\mathcal{F}_0$, and assuming uniqueness of solutions, by a standard argument trajectories starting inside $\mathcal{S}$ will not be able to reach $\mathcal{F}_0$ in finite time. It is then easy to show that, without loss of generality (by reordering indices $\{1, \ldots, n\}$), there is $b_1 \in \mathcal{B} \cap \mathcal{C}(v_1)$ "pointing out" of $\mathcal{F}_0$. That is,

$$h_0 \cdot b_1 > 0\,. \tag{9}$$

We consider any point v' in the open segment (v_0, v_1). That is, let $\lambda \in (0, 1)$ and define

$$v' = \lambda v_1 + (1 - \lambda) v_0\,. \tag{10}$$

Now define the following simplices in $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}^1 &= \mathrm{co}\{v', v_1, v_2, \ldots, v_n\} \\ \mathcal{S}^2 &= \mathrm{co}\{v_0, v', v_2, \ldots, v_n\}\,. \end{aligned}$$

Also define the new exit facet for $\mathcal{S}^2$ by $\mathcal{F}_0' := \mathrm{co}\{v', v_2, \ldots, v_n\}$. See Figure 2. The following lemma provides a formula for the normal vector h' of $\mathcal{F}_0'$.

Lemma 15 ([6], [5]): Let $h_0 = -\gamma_1 h_1 - \ldots - \gamma_n h_n$ with $\gamma_i > 0$, and let $\lambda \in (0, 1)$. Then the normal vector to $\mathcal{F}_0'$ pointing out of $\mathcal{S}^1$ is

$$h' = \gamma_1 h_1 + \lambda \sum_{j=2}^{n} \gamma_j h_j = \gamma_1 (1 - \lambda) h_1 - \lambda h_0\,. \tag{11}$$

Lemma 16 ([6], [5]): Suppose Assumption 11 holds. There exists $v' \in (v_0, v_1)$, such that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1) \neq \mathbf{0}$. Moreover, $b_1 \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1)$ with $h' \cdot b_1 < 0$.

Now we show that solvability of the strict invariance conditions for $\mathcal{S}^1$ and for $\mathcal{S}^2$ is inherited from solvability of the strict invariance conditions for $\mathcal{S}$.

Lemma 17: Suppose Assumption 11 holds. If the strict invariance conditions (4) for $\mathcal{S}$ are solvable, then the strict invariance conditions for $\mathcal{S}^1$ are solvable.

Lemma 18: Suppose Assumption 11 holds. If the strict invariance conditions for $\mathcal{S}$ are solvable, then the strict

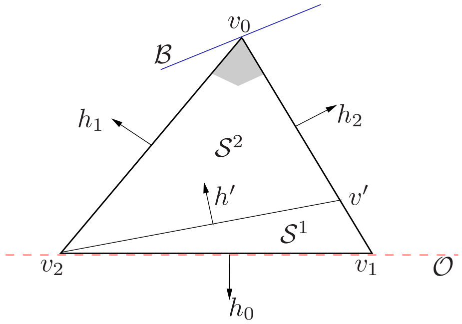

Fig. 2. Subdivision into two sub-simplices $\mathcal{S}^1$ and $\mathcal{S}^2$.

invariance conditions for $\mathcal{S}^2$ are solvable.

Proof: By assumption the strict invariance conditions are solvable for $\mathcal{S}$, and since the strict invariance conditions for $\mathcal{S}^2$ are identical (the only facet which changed for $\mathcal{S}^2$ is $\mathcal{F}_0$, which plays no role in invariance conditions), they are also solvable for $\mathcal{S}^2$. ■

Theorem 19: Consider simplex $\mathcal{S}$ and system (1) with system parameters $\lambda = (A, B, a)$. Suppose Assumption 11 holds. Then RCP by piecewise affine feedback is structurally stable at λ if and only if the strict invariance conditions are solvable for $\mathcal{S}$.

Proof: ($\Longrightarrow$) Follows from Lemma 3.

($\Longleftarrow$) Suppose the strict invariance conditions are solvable. By Lemma 14, $p = 1$, $r_1 = n$, and $m = \kappa = n - 1$. Consider the subdivision of $\mathcal{S}$ into $\mathcal{S}^1$ and $\mathcal{S}^2$ as above. Lemmas 17 and 18 guarantee that the strict invariance conditions (4) are solvable for $\mathcal{S}^1$ and $\mathcal{S}^2$, as well. Similar to the proof of Theorem 24 of [6], it can be verified that $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ and $\mathcal{S}^2 \xrightarrow{\mathcal{S}^2} \mathcal{F}_0^1$ by affine feedbacks. Hence, no closed-loop equilibria appear in $\mathcal{S}^1$ and $\mathcal{S}^2$. Using Theorem 6, RCP by affine feedback is structurally stable on $\mathcal{S}^1$ and $\mathcal{S}^2$.

Now, for RCP to be structurally stable on $\mathcal{S}$, we should further verify that for the nominal system $\lambda = (A, B, a)$ as well as for the perturbed system $\tilde{\lambda} = (\tilde{A}, \tilde{B}, \tilde{a})$ trajectories only progress from $\mathcal{S}_2$ to $\mathcal{S}_1$. Consider the boundary between $\mathcal{S}^1$ and $\mathcal{S}^2$ given by $\mathcal{F}_0'$. We must show that for any $x_0 \in \mathcal{S}^1 \setminus \mathcal{F}_0'$, closed-loop trajectories do not reach $\mathcal{F}_0'^1$. This can be deduced from the proof of Lemma 17 where it is shown that the controls $\{u', u_2, \ldots, u_n\}$ can be selected so that

$$\begin{aligned} h' \cdot (Av' + Bu' + a) &< 0 \\ h' \cdot (Av_i + Bu_i + a) &< 0, \qquad i = 2, \ldots, n. \end{aligned}$$

If $u = K_1 x + g_1$ is the affine feedback obtained for $\mathcal{S}^1$ using the above control values, then by convexity, $h' \cdot (Ax + B(K_1 x + g_1) + a) < 0$ for all $x \in \mathcal{F}_0'$, from which the result easily follows. Since these inequalities are strict, the result is applicable to the perturbed system, as well. ■

Example 20: Consider a simplex $\mathcal{S}$ defined by vertices $v_0 = (-1, 1)$, $v_1 = (0, 0)$ and $v_2 = (1, 1)$, and consider the affine system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have

$$\mathcal{O} = \{x \mid x_2 = x_1\}.$$

Hence $\mathcal{S} \cap \mathcal{O} = \mathcal{G} = \mathrm{co}\{v_1, v_2\}$, $\kappa = 1$, $m = 1$, and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = \mathbf{0}$. Therefore, we cannot solve the problem by a continuous state feedback according to [3]. We use the method of [6], [5] to triangulate $\mathcal{S}$ where one step of triangulation is enough. We choose $v' = (0.5, 1)$ so that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}^1) \neq 0$. Then $\mathcal{S}^2 := \mathrm{co}\{v_0, v', v_1\}$, $\mathcal{S}^1 := \mathrm{co}\{v', v_1, v_2\}$, $\mathcal{F}_0' = \mathrm{co}\{v', v_1\}$, and $h' = (-1, 0.5)$. To satisfy the strict invariance conditions for $\mathcal{S}^2$ we choose control inputs at the vertices to be $u_0 = -1$, $u' = -.5$, and $u_{21} = 1$. To satisfy the strict invariance conditions for $\mathcal{S}^1$ we choose control inputs at the vertices to be $u' = -.5$, $u_2 = -1$, and $u_{11} = -1$. The associated piecewise affine feedback is

$$u = \begin{cases} \begin{bmatrix} 0.3333 & -1.6667 \end{bmatrix} x + 1, & x \in \mathcal{S}^2 \\ \begin{bmatrix} -1 & 1 \end{bmatrix} x - 1, & x \in \mathcal{S}^1. \end{cases}$$

Now we may verify that the conditions of Assumption 11 are satisfied and moreover, the feedback control for $\mathcal{S}_2$ solves the strict invariance conditions for $\mathcal{S}$. Hence, by Theorem 19, RCP is structurally stable by the piecewise affine feedback (20).

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