Reach Controllability of Single Input Affine Systems on a Simplex

Elham Semsar-Kazerooni and Mireille E. Broucke

***Abstract*—We study the reach control problem (RCP) for a single input affine system with a simplicial state space. We extend previous results by exploring arbitrary triangulations of the state space; particularly allowing the set of possible equilibria to intersect the interior of simplices. In the studied setting, it is shown that closed-loop equilibria, nevertheless, only arise on the boundary of simplices. This allows to define a notion of reach controllability which quantifies the effect of the control input on boundary equilibria. Using reach controllability we obtain necessary and sufficient conditions for solvability of RCP by affine feedback.**

I. Introduction

This paper studies the *reach control problem* (RCP) on simplices. The problem is for trajectories of an affine system defined on a simplex $\mathcal{S}$ to exit a prespecified exit facet in finite time without first leaving the simplex. The problem has been studied over a series of papers [3], [7], [8], [13] due to its fundamental nature among reachability problems. The reader is referred to [2], [3], [7], [8], [12], [13] for further motivations, including how the problem arises in reachability problems for hybrid systems. Formulating the reach control problem on a simplex is well-founded since the simplex is a canonical object for partitioning space, and it appears in many disciplines ranging from algebraic topology to computational geometry. Since any convex polytope can be triangulated into a set of simplices, solution methods for RCP on a simplex can be extended to polytopes [10].

In [3] RCP was studied under the assumption that the state space was triangulated so that $\mathcal{O}$, the set of possible equilibria of the affine system, intersected with $\mathcal{S}$ was either the empty set or a face of $\mathcal{S}$. In this paper we assume $\mathcal{O}$ intersects the interior of $\mathcal{S}$, and we study only single input systems. Remarkably it emerges that if an equilibrium appears using an affine feedback to solve RCP, then the equilibrium is, nevertheless, on the boundary of $\mathcal{S}$. Using this finding, we propose a notion of *reach controllability* for determining if RCP is solvable by affine feedback. Simply put, an affine system is reach controllable on a simplex if each equilibrium can be "pushed off" the simplex boundary by an admissible affine feedback. Because the feedback is affine, the equilibrium is affected by the control input only through the control values applied at vertices of the face containing the equilibrium. In this sense, reach controllability measures the extent to which the control input can affect the dynamics on faces of the simplex. Since the simplex is a canonical geometric object, this gives rise to an intrinsic notion related to how the control system is actuated; hence, the monicker "reach controllability". Finally, using reach controllability, we obtain new necessary and sufficient conditions for solvability of RCP in the current setting.

The contributions of the paper relative to the literature are as follows. First, we relax the requirement that the state space is triangulated with respect to the set $\mathcal{O}$, departing from our earlier investigations [3]. This requirement had originally been placed for two reasons. First, the choice of triangulation of the state space is under the discretion of the designer, so in principle there is no loss to impose a triangulation that makes the synthesis problem easier. Second, using this triangulation, unequivocal results on the role of affine feedbacks are possible - affine feedbacks and continuous state feedbacks are equivalent with respect to solvability of RCP [3]. By allowing for more general triangulations as in this paper, we provide more flexibility to the designer. At the same time, difficulties immediately assert themselves because it is no longer known if affine feedbacks remain the central object of study for solving RCP. The first and a significant contribution of the paper is the surprising discovery that despite arbitrary triangulations (not enforcing $\mathcal{O}$ to lie on the boundary of $\mathcal{S}$), closed-loop equilibria still only appear on the boundary of $\mathcal{S}$ when using admissible affine feedbacks. The second contribution of the paper is the introduction of a new notion of reach controllability that captures precisely how these boundary equilibria are affected by the input. The third contribution is new necessary and sufficient conditions for solvability of RCP by affine feedback. These conditions improve those in the literature [8], [13] which are stated as properties to be verified for a given candidate controller.

Recent results on RCP include [4], [5], [1], [10], [9], [6]. Because of the choice of triangulation of [3], so-called reach control indices [4] emerged as important structural information about the control system allowing to completely resolve what class of feedbacks to use for RCP [5]. An alternate class of feedbacks was proposed in [1]. The construction of the indices relies on certain $\mathscr{M}$-matrices. Unfortunately, when we go to the more general triangulation used in this paper, this structure disappears. Section IV provides the mathematical machinery that was formerly provided by $\mathscr{M}$-matrices. Second, [10] studies RCP on polytopes in the spirit of [7]. Because of the generality of polytopes, the results are primarily numerical methods to compute feedbacks. When restricted to simplices, they recover the results of [3], [7], [13]. Hence, [10] does not provide new information for the present problem. In conclusion, [4], [5], [1], [10] provide no avenue for solving the problem studied here. Finally, a preliminary version of this paper appeared in [14].

The paper is organized as follows. In Section II we define the reach control problem. In Section III a new necessary condition for single-input systems for solvability of RCP by continuous state feedback is presented, adding to the known necessary conditions [6], [7]. In Section IV preliminary technical results are presented to support Section V where important properties of the set of open-loop equilibria are exposed; particularly, that such equilibria only appear on the boundary $\mathcal{S}$. In Section VI we introduce the notion of reach controllability and the main theoretical result on new necessary and sufficient conditions for solvability of RCP for single-input systems is presented.

Notation. Let $\mathcal{S} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{S}}$, and the interior is $\mathcal{S}^\circ$. The relative interior is denoted $\mathrm{ri}(\mathcal{S})$, the relative boundary of $\mathcal{S}$, denoted $\mathrm{rb}(\mathcal{S})$ is $\overline{\mathcal{S}} \setminus \mathrm{ri}(\mathcal{S})$, and $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. Notation $\mathrm{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$.

II. Problem Statement

Consider an n-dimensional simplex $\mathcal{S} := \mathrm{co}\{v_0, \ldots, v_n\}$, the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Let its vertex set be $V := \{v_0, \ldots, v_n\}$ and its facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$. The facet will be indexed by the vertex it does not contain. Without loss of generality (w.l.o.g.) we assume that $v_0 = 0$. Let h_j, $j \in \{0, \ldots, n\}$ be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $I := \{1, \ldots, n\}$ and define $I(x)$ to be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \mathrm{co}\{v_i \mid i \in I(x)\}$. For $x \in \mathcal{S}$ define the closed, convex cone

$$\mathcal{C}(x) := \{\ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in I \setminus I(x)\ \}.$$

Dept. of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: elham.semsar.kazerooni@utoronto.ca,broucke@control.utoronto.ca). This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and Le Fonds Québécois de la Recherche sur la Nature et les Technologies (FQRNT).

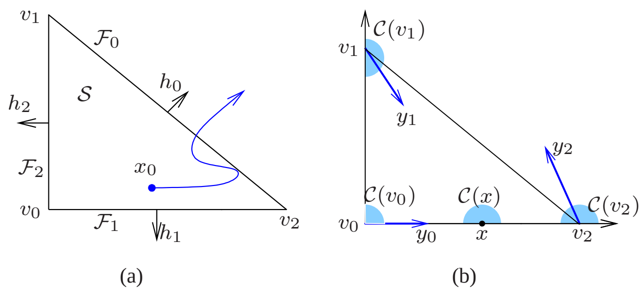


Fig. 1. (a) Notation for the reach control problem. (b) Convex cones and the invariance conditions in a 2D example.

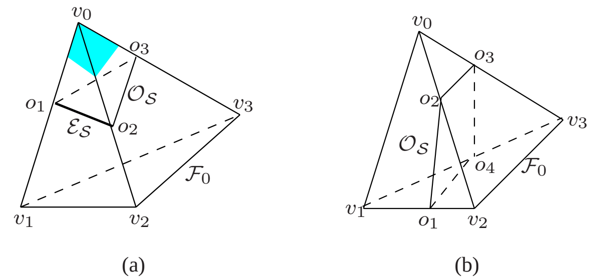


Fig. 2. (a) $\mathcal{O}_\mathcal{S}$ satisfies Assumption 1. (b) $\mathcal{O}_\mathcal{S}$ violates Assumption 1.

Figure 1(a) illustrates the notation for a 2D simplex, and Figure 1(b) illustrates the cones $\mathcal{C}(x)$ for several representative points in $\mathcal{S}$. We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B) = m = 1$. Let $\mathcal{B} = \mathrm{Im}(B)$, the image of B. Define $\mathcal{O} := \{\, x \in \mathbb{R}^n \mid Ax+a \in \mathcal{B}\}$, $\mathcal{E} := \{\, x \in \mathbb{R}^n \mid Ax + a = 0\}$, $\mathcal{O}_\mathcal{S} := \mathcal{S} \cap \mathcal{O}$, and $\mathcal{E}_\mathcal{S} := \mathcal{S} \cap \mathcal{E}$. One can show that either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ is an affine space with $m \le \dim(\mathcal{O}) \le n$. Notice that $\mathcal{E}$ is the set of open-loop equilibria (when $u = 0$); whereas $Ax + Bu + a$ for $x \in \mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of possible equilibrium points of the system. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at x_0 under control input u. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{S}$. Find a feedback $u(x)$ such that: for each $x_0 \in \mathcal{S}$ there exist $T \ge 0$ and $\delta > 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \delta)$.

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{S}$ exit $\mathcal{S}$ through the exit facet $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{S}$. In particular, a trajectory initialized at $x_0 \in \mathcal{S}$ may reach $\mathcal{F}_0$, remain in $\mathcal{S}$ and exit $\mathcal{F}_0$ at some time later as illustrated in Figure 1(a). In the sequel we use the shorthand notation $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ to denote that conditions (i)-(iii) of Problem 1 hold under some control law.

To solve RCP we require conditions that disallow trajectories to exit from the facets $\mathcal{F}_i$, $i \in I$. We say the *invariance conditions are solvable* if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that,

$$Av_i + Bu_i + a \in \mathcal{C}(v_i)\,,\ \ i \in \{0, \ldots, n\}\,. \tag{2}$$

The inequalities (2) are called *invariance conditions*, and they guarantee that trajectories that exit $\mathcal{S}$ only do so through $\mathcal{F}_0$, and they are used to construct affine feedbacks [7]. Consider Figure 1(b). The cones $\mathcal{C}(v_i)$ are depicted as the shaded cones attached at each vertex (of course their apex is at 0). The invariance conditions (2) are depicted in the figure, where velocity vectors $y_i := Av_i + Bu_i + a$ are shown lying inside their respective cones $\mathcal{C}(v_i)$.

III. Necessary Conditions for Solvability by Continuous State Feedback

The goal of this paper is to obtain new necessary and sufficient conditions for solvability of RCP by affine feedback; unlike the conditions given in Theorem 8 of [13] (or Theorem 4.16 of [8]), we seek conditions that lead to synthesis of the controller. To aid in this endeavor, we first seek necessary conditions for solvability by continuous state feedback. One such necessary condition is provided by Proposition 3.1 in [7], where it is shown that solvability of the invariance conditions is necessary for solvability of RCP by continuous state feedback. The goal of this section is to provide a second necessary condition. The result is presented for single-input systems only, as the multi-input result is still unknown. Note that the presented result requires no assumption on the placement of $\mathcal{O}_\mathcal{S}$ with respect to $\mathcal{S}$.

The set $\mathcal{O}_\mathcal{S} = \mathcal{S} \cap \mathcal{O}$, the intersection of a simplex and an affine space, is a polyhedron. Suppose that $\{o_1, \ldots, o_{\kappa+1}\}$ is its vertex set; thus, $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, \ldots, o_{\kappa+1}\}$. Let $I_{\mathcal{O}_\mathcal{S}} := \{1, \ldots, \kappa + 1\}$. Similarly, suppose $\mathcal{E}_\mathcal{S} = \mathrm{co}\{\epsilon_1, \ldots, \epsilon_{\kappa_0+1}\}$ is a polytope with vertex set $\{\epsilon_1, \ldots, \epsilon_{\kappa_0+1}\}$, and let $I_{\mathcal{E}_\mathcal{S}} := \{1, \ldots, \kappa_0 + 1\}$. Define the cone

$$\mathrm{cone}(\mathcal{O}_\mathcal{S}) := \bigcap_{i \in I_{\mathcal{O}_\mathcal{S}}} \mathcal{C}(o_i)\,.$$

Consider Figure 2(a). Here $\mathcal{C}(v_0)$ is depicted as the shaded cone with apex at v_0. The set $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2, o_3\}$ is not only a polyhedron, but also a simplex. It is clear from the figure that $\mathrm{cone}(\mathcal{O}_\mathcal{S}) = \mathcal{C}(o_1) \cap \mathcal{C}(o_2) \cap \mathcal{C}(o_3)$ is precisely $\mathcal{C}(v_0)$. The next result says to solve RCP by continuous state feedback there must be a non-zero vector in $\mathcal{B}$ that lies in $\mathrm{cone}(\mathcal{O}_\mathcal{S})$.

Theorem 1: Suppose $\mathcal{O}_\mathcal{S} \neq \emptyset$. If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by continuous state feedback, then $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.

Proof: Suppose by way of contradiction that $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) = \mathbf{0}$. Since $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$, by Proposition 3.1 of [7], one can find a continuous state feedback $u(x)$ such that $y(x) := Ax + Bu(x) + a \in \mathcal{C}(x)$, $\forall x \in \mathcal{S}$. Let $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, \ldots, o_{\kappa+1}\}$. If $\kappa = 0$, then $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) = \mathbf{0}$ implies that $\mathcal{B} \cap \mathcal{C}(o_1) = \mathbf{0}$. Thus, o_1 is an equilibrium of the closed-loop system. Instead suppose $\kappa > 0$ and w.l.o.g. $0 \neq b_1 := Ao_1 + Bu(o_1) + a \in \mathcal{B} \cap \mathcal{C}(o_1)$. Then the assumption $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) = \mathbf{0}$ implies there exists $k \in \{2, \ldots, \kappa + 1\}$ such that $b_1 \notin \mathcal{C}(o_k)$. Consider the segment $\overline{o_1 o_k}$. Since $\overline{o_1 o_k} \subset \mathcal{O}$, $y(x) \in \mathcal{B}$ for $x \in \overline{o_1 o_k}$. Thus there exists a continuous function $c : \mathbb{R}^n \to \mathbb{R}$ such that $y(x) = c(x) b_1$ for $x \in \overline{o_1 o_k}$, with $c(o_1) > 0$ and $c(o_k) \le 0$. By the Intermediate Value Theorem, there exists $x^* \in \overline{o_1 o_k} \subset \mathcal{S}$ such that $c(x^*) = 0$. Thus, the closed-loop system has an equilibrium in $\mathcal{S}$, a contradiction to $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$. ∎

IV. Preliminary Technical Results

In this section we present preliminary technical results that will enable us to characterize (in Section V) useful geometric properties of $\mathcal{O}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}$. We begin by posing our main assumptions.

Assumption 1: The system (1) satisfies:

(A1) $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, \ldots, o_{\kappa+1}\}$, a κ-dimensional simplex with $m \le \kappa < n$.

(A2) If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} = \mathrm{co}\{\epsilon_1, \ldots, \epsilon_{\kappa_0+1}\}$, a κ_0-dimensional simplex with $0 \le \kappa_0 \le \kappa$.

(A3) $\mathcal{O}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$.
(A4) $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$.

Remark 1: If $\mathcal{O}_\mathcal{S} = \emptyset$, then the solution of RCP is completely understood [8], [13]. Here we only focus on the case when $\mathcal{O}_\mathcal{S} \neq \emptyset$. In [3] we assumed that if $\mathcal{O}_\mathcal{S} \neq \emptyset$, then $\mathcal{O}_\mathcal{S}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \le \kappa \le n$. More generally, if the intersection is arbitrary, then $\mathcal{O}_\mathcal{S}$ is a convex polytope. In the present paper we assume $\mathcal{O}_\mathcal{S}$ is a simplex that intersects the interior of $\mathcal{S}$. Also, we restrict $\mathcal{O}_\mathcal{S}$ so that it does not touch $\mathcal{F}_0$. The motivation for these restrictions is to posit a generic situation distinct from the one studied in [3]. First, (A3) is clearly generic. Second, generically (A, B) is controllable. Then $\dim(\mathcal{O}) = 1$, so (A1) and (A2) are satisfied. Thus, if we restrict to single-input systems, then Assumption (A1)-(A3) include the generic case. Finally, (A4) is a simplifying assumption and is the only notable loss of generality. However, it is important to note that how $\mathcal{O}$ intersects $\mathcal{S}$ is determined by the choice of triangulation. If the designer chooses to disregard (A4), then either a trial and error style of synthesis must be used [8], [13] or other triangulations must be adopted [3].

Example 1: Assumption 1 is illustrated in Figure 2(a). We observe that $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2, o_3\}$ is a simplex intersecting the interior of $\mathcal{S}$, but it does not intersect the facet $\mathcal{F}_0$. Therefore (A1) and (A3) hold. Also $\mathcal{E}_\mathcal{S} = \mathrm{co}\{o_1, o_2\}$ is a simplex so (A2) holds. Figure 2(b) illustrates a situation when Assumption 1 fails. In this case $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2, o_3, o_4\}$ is a polytope, not a simplex. Moreover, it intersects $\mathcal{F}_0$ along the segment $\overline{o_1 o_4}$. Thus, both (A1) and (A4) fail. ◁

The following basic properties of $\mathcal{O}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}$ are derived from the fact that they are formed as intersections of affine spaces and a simplex.

Lemma 1: If $\dim(\mathcal{O}_\mathcal{S}) \ge 1$, then $\mathrm{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$. If $\dim(\mathcal{E}_\mathcal{S}) \ge 1$, then $\mathrm{rb}(\mathcal{E}_\mathcal{S}) \subset \mathrm{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$.

When conditions (A3)-(A4) hold for a simplex $\mathcal{O}_\mathcal{S}$ inside $\mathcal{S}$, certain restrictions on the index sets $I(o_i)$ arise. The next result identifies those restrictions for a general simplex $\mathcal{P} \subset \mathcal{S}$.

Lemma 2: Let $\mathcal{P} = \mathrm{co}\{w_1, \dots, w_{p+1}\}$ be a p-dimensional simplex with vertex set $\{w_1, \dots, w_{p+1}\}$. Suppose $\mathcal{P} \subset \mathcal{S}$, $\mathrm{rb}(\mathcal{P}) \subset \partial\mathcal{S}$, $\mathcal{P} \cap \mathcal{S}^\circ \neq \emptyset$, and $\mathcal{P} \cap \mathcal{F}_0 = \emptyset$. Then each index set $I(w_k)$, $k \in \{1, \dots, p+1\}$, has a nonzero exclusive member. That is, there exists $i_k \in I(w_k)$, $i_k \neq 0$, such that $i_k \notin I(w_j)$ for all $j \in \{1, \dots, p+1\} \setminus \{k\}$.

Proof: First note that the exclusive member of $I(w_k)$ cannot be zero because $\mathcal{P} \cap \mathcal{F}_0 = \emptyset$ implies $0 \in I(w_j)$ for all $j \in \{1, \dots, p+1\}$. If $p = 0$ we are done. Instead suppose w.l.o.g. $I(w_1) \subset \cup_{j=2}^{p+1} I(w_j)$. Since $\mathcal{P} \cap \mathcal{S}^\circ \neq \emptyset$, $\cup_{j=1}^{p+1} I(w_j) = \{0, \dots, n\}$. Thus, $\cup_{j=2}^{p+1} I(w_j) = \{0, \dots, n\}$. Define $\mathcal{P}' = \mathrm{co}\{w_2, \dots, w_{p+1}\}$. Since $\mathcal{P}$ is a simplex, $\mathcal{P}'$ is a $(p-1)$-dimensional facet of $\mathcal{P}$ so $\mathcal{P}' \subset \mathrm{rb}(\mathcal{P})$. However, $\cup_{j=2}^{p+1} I(w_j) = \{0, \dots, n\}$ implies $\mathcal{P}' \cap \mathcal{S}^\circ \neq \emptyset$. This contradicts that $\mathrm{rb}(\mathcal{P}) \subset \partial\mathcal{S}$. ■

We now turn to an algebraic characterization of points in $\mathcal{O}$. Let $x = \sum_{i=0}^n \alpha_i^x v_i$, where $\sum_i \alpha_i^x = 1$. Since $\{v_0, \dots, v_n\}$ are affinely independent, $\alpha^x := (\alpha_0^x, \dots, \alpha_n^x)$ are the unique barycentric coordinates of x. If $x \in \mathcal{O}$ then there exists $u^x \in \mathbb{R}^m$ such that $Ax + Bu^x + a = 0$. This yields

$$
\begin{aligned}
Ax + Bu^x + a = 0 &\iff h_j \cdot (Ax + Bu^x + a) = 0\,, j \in I \\
&\iff \sum_{i=0}^n \alpha_i^x (h_j \cdot Av_i) + h_j \cdot (Bu^x + a) = 0\,, j \in I\,. \quad (3)
\end{aligned}
$$

Similarly, for $x \in \mathcal{E}$ we have $u^x = 0$, so we get

$$
Ax + a = 0 \iff \sum_{i=0}^n \alpha_i^x (h_j \cdot Av_i) + h_j \cdot a = 0\,, j \in I\,. \quad (4)
$$

Using (3) and (4) we can relate geometric properties of $\mathcal{O}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}$ to the feasibility of certain algebraic constraints. The following are the main results of this section.

Proposition 1: Suppose Assumptions (A1), (A3), and (A4) hold. Then the following cannot hold:

$$
h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, h_j \cdot B = 0\,, k \in I_{\mathcal{O}_\mathcal{S}}, i \in I(o_k), j \in I \setminus I(o_k)\,. \quad (5)
$$

Proof: By Assumptions (A1), (A3), and (A4), Lemma 2 applies with $\mathcal{P} = \mathcal{O}_\mathcal{S}$. Thus, w.l.o.g. the vertices of $\mathcal{S}$ can be ordered according to exclusive members of $\{I(o_k)\}$, $k \in I_{\mathcal{O}_\mathcal{S}}$. That is, the indices $\{0, \dots, n\}$ are ordered as $\{0, m_0, \dots, m_0 + r_0 - 1, m_1, \dots, m_1 + r_1 - 1, \dots, m_{\kappa+1}, \dots, m_{\kappa+1} + r_{\kappa+1} - 1\}$. Here $\{0, m_0, \dots, m_0 + r_0 - 1\}$ are the indices appearing in more than one index set $I(o_k)$, $k \in I_{\mathcal{O}_\mathcal{S}}$. Indices $\{m_k, \dots, m_k + r_k - 1\}$ only appear in $I(o_k)$. By Lemma 2, $r_k \ge 1$, $k \in I_{\mathcal{O}_\mathcal{S}}$. Arguing by contradiction, (5) implies $h_j \cdot B = 0$, $h_j \cdot a = 0$, for all $j \in I \setminus I(o_1) \cup \cdots \cup I \setminus I(o_{\kappa+1})$. Applying the new index ordering we have $I \setminus I(o_1) \cup \cdots \cup I \setminus I(o_{\kappa+1}) = I \setminus \big[I(o_1) \cap \cdots \cap I(o_{\kappa+1})\big] \supset \{m_1, \dots, m_1 + r_1 - 1, \dots, m_{\kappa+1}, \dots, m_{\kappa+1} + r_{\kappa+1} - 1\}$. We conclude that for $j = m_1, \dots, m_1 + r_1 - 1, \dots, m_{\kappa+1}, \dots, m_{\kappa+1} + r_{\kappa+1} - 1$,

$$
h_j \cdot (Bu^x + a) = 0\,. \quad (6)
$$

Now consider any $x \in \mathcal{O}$. Combining (3), (5), and (6), and invoking $v_0 = 0$, we have

$$
\sum_{i=0}^n \alpha_i^x = 1 \quad (7a)
$$

$$
\sum_{i=1}^n \alpha_i^x (h_j \cdot Av_i) + h_j \cdot (Bu^x + a) = 0\,, j = m_0, \dots, m_0 + r_0 - 1 \quad (7b)
$$

$$
\sum_{i=m_1}^{m_1+r_1-1} \alpha_i^x (h_j \cdot Av_i) = 0\,, j = m_1, \dots, m_1 + r_1 - 1 \quad (7c)
$$

$$
\vdots
$$

$$
\sum_{i=m_{\kappa+1}}^{m_{\kappa+1}+r_{\kappa+1}-1} \alpha_i^x (h_j \cdot Av_i) = 0\,, j = m_{\kappa+1}, \dots, m_{\kappa+1} + r_{\kappa+1} - 1\,. \quad (7d)
$$

For $k = 0, \dots, \kappa+1$, let $M_k = [h_{m_k} \cdots h_{m_k + r_k - 1}]^T [Av_{m_k} \ \dots \ Av_{m_k + r_k - 1}]$. Then in matrix form, (7a)-(7d) become

$$
M\alpha^x := \begin{bmatrix} 1 & \star & \star & \dots & \star \\ & M_0 & \star & \dots & \star \\ & & M_1 & & \\ & & & \ddots & \\ & & & & M_{\kappa+1} \end{bmatrix} \alpha^x = \begin{bmatrix} 1 \\ \star \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (8)
$$

where $M \in \mathbb{R}^{(n+1)\times(n+1)}$ and empty entries are zero. Let $k \in \{1, \dots, \kappa+1\}$ and consider M_k. Suppose $\mathrm{rank}(M_k) = r_k$. Then for all $x \in \mathcal{O}$, $\alpha_{m_k}^x = \cdots = \alpha_{m_k + r_k - 1}^x = 0$. In particular, $\mathcal{O}_\mathcal{S} \subset \mathrm{co}\{v_0, \dots, v_{m_k - 1}, v_{m_k + r_k}, \dots, v_n\} \subset \partial\mathcal{S}$, a contradiction. We conclude $\mathrm{rank}(M_k) < r_k$ for $k = 1, \dots, \kappa+1$. Consequently we can find $\beta^1, \dots, \beta^{\kappa+1} \in \mathrm{Ker}(M) \subset \mathbb{R}^{n+1}$ such that $\beta_0^k = 0$ for $k = 1, \dots, \kappa+1$ and $\{\beta^1, \dots, \beta^{\kappa+1}\}$ are linearly independent. Define $\xi^k := \sum_{i=0}^n (\beta_i^k + \alpha_i^x) v_i$ for $k = 1, \dots, \kappa+1$. Observe that $\xi^k \in \mathcal{O}$ because $(\beta^k + \alpha^x)$ is a solution of (8). Now we claim $\{x, \xi^1, \dots, \xi^{\kappa+1}\}$ are affinely independent. (Proof of claim: Define $\beta_{1,n}^k = (\beta_1^k, \dots, \beta_n^k)$, $V := [v_1 \cdots v_n] \in \mathbb{R}^{n\times n}$ and $\Gamma := [\beta_{1,n}^1 \cdots \beta_{1,n}^{\kappa+1}] \in \mathbb{R}^{n\times(\kappa+1)}$. Then $[(\xi^1 - x) \cdots (\xi^{\kappa+1} - x)] = V\Gamma$. Since $\mathrm{rank}(V) = n$ and $\mathrm{rank}(\Gamma) = \kappa+1$, $\mathrm{rank}(V\Gamma) = \kappa+1$.) We conclude $\dim(\mathcal{O}) > \kappa$, a contradiction. ■

Proposition 2: Suppose Assumptions (A1)-(A4) hold.

(a) If $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$ and $\dim(\mathcal{E}) = 0$, then the following cannot hold:

$$(\exists k \in I_{\mathcal{O}_\mathcal{S}})\;\; h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, \quad i \in I(o_k), j \in I \backslash I(o_k)\,. \tag{9}$$

(b) If $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$ and $\dim(\mathcal{E}) = \kappa_0 > 0$, then the following cannot hold:

$$h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, \quad k \in I_{\mathcal{E}_\mathcal{S}}, i \in I(\epsilon_k), j \in I \setminus I(\epsilon_k)\,. \tag{10}$$

Proof: (a) W.l.o.g. suppose $k = 1$ and $I(o_1) = \{0, 1, \ldots, q\}$ for some $1 \leq q \leq n-1$. The form of this index set is dictated by the following facts: (a1) $0 \in I(o_1)$ by (A4); (a2) $q < n$ because (A3) implies $\dim(\mathcal{O}_\mathcal{S}) \geq 1$, so Lemma 1 gives $o_1 \in \partial\mathcal{S}$; (a3) $q \geq 1$, otherwise $o_1 = v_0$ and $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$ together imply $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 \neq \emptyset$, a contradiction to (A4). Then (9) becomes:

$$h_j \cdot Av_i = 0\,, h_j \cdot a = 0\,, \quad i \in \{0, \ldots, q\}, j \in \{q+1, \ldots, n\}\,. \tag{11}$$

Consider any $x \in \mathcal{E}$. Combining (4) with (11) and invoking $v_0 = 0$, we have

$$\sum_{i=0}^{n} \alpha_i^x = 1 \tag{12a}$$

$$\sum_{i=1}^{n} \alpha_i^x (h_j \cdot Av_i) + h_j \cdot a = 0\,, \qquad j = 1, \ldots, q \tag{12b}$$

$$\sum_{i=q+1}^{n} \alpha_i^x (h_j \cdot Av_i) = 0\,, \qquad j = q+1, \ldots, n\,. \tag{12c}$$

Let $M_1 := [h_1 \cdots h_q]^T \; [Av_1 \;\ldots\; Av_q]$ and $M_{q+1} := [h_{q+1} \cdots h_n]^T \; [Av_{q+1} \;\ldots\; Av_n]$. Then (12a)-(12c) become

$$M\alpha^x := \begin{bmatrix} 1 & \star & \star \\ 0 & M_1 & \star \\ 0 & 0 & M_{q+1} \end{bmatrix} \alpha^x = \begin{bmatrix} 1 \\ \star \\ 0 \end{bmatrix},$$

where $M \in \mathbb{R}^{(n+1)\times(n+1)}$. By the same argument as in the proof of Proposition 1 we deduce that $\text{rank}(M_{q+1}) < n-q$, $\text{rank}(M) < n+1$, and $\dim(\mathcal{E}) \geq 1$, a contradiction.

(b) By assumption $\mathcal{E}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$ and by (A4), $\mathcal{E}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$. By (A2), $\mathcal{E}_\mathcal{S}$ is a simplex. Therefore Lemma 2 applies with $\mathcal{P} = \mathcal{E}_\mathcal{S}$. Suppose w.l.o.g. the vertices of $\mathcal{S}$ are ordered according to exclusive members of $\{I(\epsilon_k)\}$, $k \in I_{\mathcal{E}_\mathcal{S}}$, as per Lemma 2. That is, the indices are ordered as $\{0, m_0, \ldots, m_0 + r_0 - 1, m_1, \ldots, m_1 + r_1 - 1, \ldots, m_{\kappa_0+1}, \ldots, m_{\kappa_0+1} + r_{\kappa_0+1} - 1\}$. Here $\{0, m_0, \ldots, m_0 + r_0 - 1\}$ are the indices appearing in more than one index set $I(\epsilon_k)$, $k \in I_{\mathcal{E}_\mathcal{S}}$. Indices $\{m_k, \ldots, m_k + r_k - 1\}$ only appear in $I(\epsilon_k)$. By Lemma 2, $r_k \geq 1$, $k \in I_{\mathcal{E}_\mathcal{S}}$. From (10) we know $h_j \cdot a = 0$ for all $j \in I \setminus I(\epsilon_1) \cup \cdots \cup I \setminus I(\epsilon_{\kappa_0+1})$. Applying the new index ordering we have $I \setminus I(\epsilon_1) \cup \cdots \cup I \setminus I(\epsilon_{\kappa_0+1}) = I \setminus \big[I(\epsilon_1) \cap \cdots \cap I(\epsilon_{\kappa_0+1})\big] \supset \{m_1, \ldots, m_1 + r_1 - 1, \ldots, m_{\kappa_0+1}, \ldots, m_{\kappa_0+1} + r_{\kappa_0+1} - 1\}$. We conclude that for $j = m_1, \ldots, m_1 + r_1 - 1, \ldots, m_{\kappa_0+1}, \ldots, m_{\kappa_0+1} + r_{\kappa_0+1} - 1$,

$$h_j \cdot a = 0\,. \tag{13}$$

Consider any $x \in \mathcal{E}$. Combining (4), (10), and (13), and invoking $v_0 = 0$, we obtain (8) with $u^x = 0$. By the same argument as in the proof of Proposition 1, we deduce that $\text{rank}(M) < n + 1 - \kappa_0$ and $\dim(\mathcal{E}) > \kappa_0$, a contradiction. ∎

V. Properties of Equilibrium Set

In this section we exploit the algebraic properties discovered in the previous section, and particularly we examine their geometric consequences. The most important result is that equilibria cannot appear in the interior of $\mathcal{S}$ when the necessary conditions for solvability of RCP by continuous state feedback (in Assumption 2) are satisfied.

Assumption 2: The system (1) satisfies:

(A5) $Av_i + a \in \mathcal{C}(v_i)$, $i \in \{0, \ldots, n\}$.
(A6) $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.

First we present a technical lemma that links the appearance of an equilibrium with algebraic constraints of the type studied in the previous section.

Lemma 3 ([14]): Suppose (A5) holds. Suppose there exists $x \in \mathcal{O}_\mathcal{S}$ and $j \in I \setminus I(x)$ such that $0 \in I(x)$ and $h_j \cdot (Ax + a) = 0$. Then

$$h_j \cdot Av_i = 0\,, \quad h_j \cdot a = 0\,, \quad i \in I(x)\,.$$

The previous algebraic results lead to a remarkable property on the placement of equilibria in $\mathcal{S}$: under the assumption that the necessary conditions of Proposition 3.1 of [7] and Theorem 1 hold, open-loop equilibria can only appear on the boundary of $\mathcal{S}$.

Theorem 2: Suppose Assumptions 1 and 2 hold. If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} \subset \text{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$.

Proof: Suppose by way of contradiction there is $\overline{x} \in \mathcal{S}^\circ$ such that $A\overline{x} + a = 0$. By (A3), $\mathcal{O}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$, so $\dim(\mathcal{O}_\mathcal{S}) = \dim(\mathcal{O}) \geq 1$ .Then by Lemma 1, $\text{rb}(\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$. Therefore, $\overline{x} \in \text{ri}(\mathcal{O}_\mathcal{S})$.

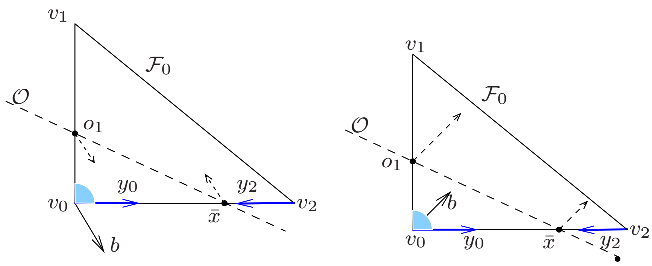
First, suppose $\dim(\mathcal{E}) = 0$ and let $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$. Since $\dim(\mathcal{O}) \geq 1$, $\mathcal{E} = \{\overline{x}\}$, and $\overline{x} \neq o_i$, $i \in I_{\mathcal{O}_\mathcal{S}}$, w.l.o.g. at least one pair of vertices of $\mathcal{O}_\mathcal{S}$, say (o_1, o_2), satisfy $Ao_1 + a = \eta_1 b$ and $Ao_2 + a = \eta_2 b$ with $\eta_1 < 0$ and $\eta_2 > 0$. Since $b \in \mathcal{C}(o_1)$, $h_j \cdot b \leq 0$, $j \in I \setminus I(o_1)$. By (A5) and convexity $Ao_1 + a \in \mathcal{C}(o_1)$. Thus, $h_j \cdot (Ao_1 + a) = h_j \cdot (\eta_1 b) \leq 0$, $j \in I \setminus I(o_1)$. Since $\eta_1 < 0$, the previous two inequalities imply $h_j \cdot b = 0$, $j \in I \setminus I(o_1)$. Equivalently we get $h_j \cdot (Ao_1 + a) = 0$, $j \in I \setminus I(o_1)$. Then by Lemma 3 we get $h_j \cdot Av_i = 0$, $h_j \cdot a = 0$ for $i \in I(o_1)$, $j \in I \setminus I(o_1)$. By Proposition 2(a), we reach a contradiction.

Second, suppose $\dim(\mathcal{E}) = \kappa_0$ with $\kappa_0 > 0$. Then $h_j \cdot (A\epsilon_k + a) = h_j \cdot 0 = 0$ for $k \in I_{\mathcal{E}_\mathcal{S}}$, $j \in I$. By Lemma 3 we get $h_j \cdot Av_i = 0$, $h_j \cdot a = 0$ for $k \in I_{\mathcal{E}_\mathcal{S}}$, $i \in I(\epsilon_k)$, and $j \in I \setminus I(\epsilon_k)$. By Proposition 2(b) this is a contradiction. ∎

In Theorem 2 we showed that the set of equilibria $\mathcal{E}_\mathcal{S}$ lies in the relative boundary of $\mathcal{O}_\mathcal{S}$. In the following we show further that $\mathcal{E}_\mathcal{S}$ is indeed a face of $\mathcal{O}_\mathcal{S}$.

Theorem 3: Suppose Assumptions 1 and 2 hold. If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\mathcal{E}_\mathcal{S} = \text{co}\{o_1, \ldots, o_{\kappa_0+1}\}$, a κ_0-dimensional face of $\mathcal{O}_\mathcal{S}$, where $0 \leq \kappa_0 < \kappa$.

Proof: Suppose $\mathcal{E}_\mathcal{S} \neq \emptyset$ but it is not a face of $\mathcal{O}_\mathcal{S}$. By Theorem 2, $\mathcal{E}_\mathcal{S} \subset \text{rb}(\mathcal{O}_\mathcal{S})$. Hence w.l.o.g. $\mathcal{E}_\mathcal{S} = \text{co}\{\epsilon_1, \ldots, \epsilon_{\kappa_0+1}\} \subset \text{co}\{o_1, \ldots, o_p\}$, where $2 \leq p < \kappa + 1$ and $\widehat{I}_{\mathcal{E}_\mathcal{S}} := \{1, \ldots, p\}$ is the minimal index set such that for all $x \in \mathcal{E}_\mathcal{S}$, $x \in \text{co}\{o_i \mid i \in \widehat{I}_{\mathcal{E}_\mathcal{S}}\}$. Since $\mathcal{E}_\mathcal{S}$ is on a face but not an entire face of $\mathcal{O}_\mathcal{S}$ and since the faces of $\mathcal{O}_\mathcal{S}$ are simplices, at least one of the vertices of $\mathcal{E}_\mathcal{S}$, say ϵ_1, is not a vertex of $\mathcal{O}_\mathcal{S}$. Hence, there exist $2 \leq q \leq p$ and $\alpha_i \in (0, 1)$ with $\sum_{i=1}^q \alpha_i = 1$ such that $\epsilon_1 = \sum_{i=1}^q \alpha_i o_i$. Let $y(o_i) := Ao_i + a = \lambda_i B$ with $\lambda_i \in \mathbb{R}$, $i \in I_{\mathcal{O}_\mathcal{S}}$. Then, $0 = \sum_{i=1}^q \alpha_i y(o_i) = \left(\sum_i \alpha_i \lambda_i\right) B$. Thus $\sum_{i=1}^q \alpha_i \lambda_i = 0$. Since $\alpha_i > 0$, either $\lambda_i = 0$ for all $i \in \{1, \ldots, q\}$, or there exists at least one pair of vertices of $\mathcal{O}_\mathcal{S}$, say (o_1, o_2), such that $\lambda_1 < 0$ and $\lambda_2 > 0$. For the first case, $\text{co}\{o_1, \ldots, o_q\} \subset \mathcal{E}_\mathcal{S}$. This means ϵ_1, a vertex of $\mathcal{E}_\mathcal{S}$, is expressible as a convex combination of points in $\mathcal{E}_\mathcal{S}$, a contradiction. For the second case, we have $\lambda_1 < 0$ and $\lambda_2 > 0$. If $\lambda_{p+1} = 0$, then $o_{p+1} \in \mathcal{E}_\mathcal{S}$, and $p + 1 \in \widehat{I}_{\mathcal{E}_\mathcal{S}}$, a contradiction. Therefore, assume w.l.o.g. that $\lambda_{p+1} > 0$. Then by convexity there exists $x \in \text{co}\{o_1, o_{p+1}\}$ such that $Ax + a = 0$. Hence, $p + 1 \in \widehat{I}_{\mathcal{E}_\mathcal{S}}$, a contradiction. ∎



(a) (b)

Fig. 3. Reach controllability in two 2D examples.

VI. Reach Controllability

In this section we define the notion of reach controllability. This notion describes the condition when a velocity vector $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$ can be injected into the system at vertices of $\mathcal{S}$ that contribute to the generation of equilibria on $\mathcal{O}_{\mathcal{S}}$.

Definition 1: Suppose Assumption 2 holds and $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$. We say the triple (A, B, a) is *reach controllable* if either $\mathcal{E}_{\mathcal{S}} = \emptyset$; or $\mathcal{E}_{\mathcal{S}} = \text{co}\{o_1, \ldots, o_{\kappa_0+1}\}$ with $0 \leq \kappa_0 < \kappa$, and for each $k \in I_{\mathcal{E}_{\mathcal{S}}}$, there exists $i \in I(o_k)$ and $u_i > 0$ such that $Av_i + bu_i + a \in \mathcal{C}(v_i)$.

The question of the choice of vector in $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$ to use is settled by the following result.

Lemma 4: Suppose Assumptions (A1) and (A3)-(A6) hold. If $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$, then $-b \notin \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$.

Proof: Suppose not. Then for all $k \in I_{\mathcal{O}_{\mathcal{S}}}$ and $j \in I \setminus I(o_k)$, $h_j \cdot b \leq 0$ and $h_j \cdot (-b) \leq 0$. This implies $h_j \cdot b = h_j \cdot (Ao_k + a) = 0$ for all $k \in I_{\mathcal{O}_{\mathcal{S}}}$, $j \in I \setminus I(o_k)$. By Lemma 3 we obtain $h_j \cdot Av_i = 0$, $h_j \cdot a = 0$, $h_j \cdot B = 0$ for all $k \in I_{\mathcal{O}_{\mathcal{S}}}$, $i \in I(o_k)$, and $j \in I \setminus I(o_k)$. By Proposition 1 this is a contradiction. ∎

Example 2: We illustrate the concept of reach controllability with a 2D example. However, it must be noted that a true example can only be exhibited in dimension 4 and higher, since in dimensions 2 and 3, no system is not reach controllable while also satisfying the two necessary conditions, Proposition 3.1 of [7] and Theorem 1. This aspect will be further explored elsewhere. Here we illustrate a case when a 2D example simultaneously fails reach controllability and Theorem 1. Consider Figure 3(a). The velocity vectors $y_i = Av_i + a$ at v_i, $i = 0, 2$ produce an equilibrium $\bar{x}$. Adding a positive b component to y_0 or y_2 results in a violation of the invariance conditions. The only option is to add $-b$ to y_0 or y_2. This in turn results in velocity vectors at $\bar{x}$ and o_1 as depicted by dashed arrows. Clearly, the zero vector is in the convex hull of these two vectors so there will be an equilibrium in $\mathcal{S}$ along segment $\overline{o_1\bar{x}}$. Therefore, RCP is not solvable. Notice in this example an equilibrium can appear in the interior of $\mathcal{S}$, apparently violating Theorem 2. This is because $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$, so Theorem 2 actually does not apply. On the other hand, Figure 3(b) shows an example where the system is reach controllable. Here $b \in \mathcal{B} \cap \mathcal{C}(v_0)$ and so it can be added to both y_0 and y_2. This results in new velocity vectors at $\bar{x}$ and o_1 depicted as dashed arrows. Clearly, the equilibrium is pushed out of the convex hull of these two points - it now lies below the simplex. ◁

In the next result we relate reach controllability to the existence of a coordinate transformation that decomposes the dynamics into those that contribute to open-loop equilibria and quotient dynamics. It is noted that a geometric characterization of reach controllability has not yet been obtained, but the following result gives a first evidence that one may exist.

Lemma 5: Suppose $Av_i + a \in \mathcal{C}(v_i)$ for $i \in \{0, \ldots, n\}$. Also suppose there exists $\overline{x} \in \mathcal{E}_{\mathcal{S}}$ with $I(\overline{x}) = \{0, \ldots, q\}$ for some $1 \leq q < n$. Then there exists a coordinate transformation $z = T^{-1}x$ that transforms system (1) into

$$\dot{z} = \begin{bmatrix} A_1 & \star \\ 0 & A_2 \end{bmatrix} z + \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u \,, \qquad (14)$$

where $A_1 \in \mathbb{R}^{q \times q}$, $a_1 \in \mathbb{R}^q$, $b_1 \in \mathbb{R}^q$, $A_2 \in \mathbb{R}^{(n-q)\times(n-q)}$, and $b_2 \in \mathbb{R}^{n-q}$.

Proof: Since $\overline{x} \in \mathcal{E}_{\mathcal{S}}$, $A\overline{x} + a = 0$. Thus $h_j \cdot (A\overline{x} + a) = 0$, $j \in I \setminus I(\overline{x})$. By Lemma 3 we have

$$h_j \cdot Av_i = 0 \,, \;\; h_j \cdot a = 0 \,, i = 1, \ldots, q, \;\; j = q+1, \ldots, n \,. \qquad (15)$$

Now consider the coordinate transformation $z = T^{-1}x$, where $T = [v_1 \;\cdots\; v_n]$. Note that T is nonsingular because the vertices of $\mathcal{S}$ are affinely independent and $v_0 = 0$. The transformed vertices are $e_i = T^{-1}v_i$ for $i = 0, \ldots, n$ (where e_i is the ith Euclidean coordinate vector and $e_0 = 0$). Also, the transformed unit normal vectors are $-e_j = \frac{T^T h_j}{\|T^T h_j\|}$, $j \in I$. By a standard argument, (15) implies that the transformed system is (14). ∎

The following result provides constructive necessary and sufficient conditions for solvability of RCP by affine feedback in the studied setting.

Theorem 4: Consider the system (1). Suppose Assumptions 1 and 2 hold and $b_{\kappa+1} := Ao_{\kappa+1} + a \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$. We have $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback if and only if (A, B, a) is reach controllable.

Proof: ($\Longleftarrow$) By (A5), the invariance conditions (2) hold with $u = 0$. If $\mathcal{E}_{\mathcal{S}} = \emptyset$, by Theorem 8 of [13] (or Theorem 4.16 of [8]), $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback $u(x) = 0$. Instead, if $\mathcal{E}_{\mathcal{S}} \neq \emptyset$, then by Theorem 3, $\mathcal{E}_{\mathcal{S}} = \text{co}\{o_1, \ldots, o_{\kappa_0+1}\}$ with $0 \leq \kappa_0 < \kappa$. Let $b_{\kappa+1} := Ao_{\kappa+1} + a \neq 0$. We begin by showing that for all $k \in I_{\mathcal{O}_{\mathcal{S}}}$, $Ao_k + a = \lambda_k b_{\kappa+1}$ with $\lambda_k \geq 0$. The result is obviously true for vertices of $\mathcal{O}_{\mathcal{S}}$ in $\mathcal{E}_{\mathcal{S}}$ since $Ao_k + a = 0$, $k \in I_{\mathcal{E}_{\mathcal{S}}}$. Second, consider vertices of $\mathcal{O}_{\mathcal{S}}$ not in $\mathcal{E}_{\mathcal{S}}$. For these vertices the coefficients λ_k, $k \in I_{\mathcal{O}_{\mathcal{S}}} \setminus I_{\mathcal{E}_{\mathcal{S}}}$, must all have the same sign; otherwise, by convexity there is $x \in \text{co}\{o_k \mid k \in I_{\mathcal{O}_{\mathcal{S}}} \setminus I_{\mathcal{E}_{\mathcal{S}}}\}$ such that $Ax + a = 0$, which implies $x \in \mathcal{E}_{\mathcal{S}}$, a contradiction. Since by assumption $\lambda_{\kappa+1} = 1$, we get $\lambda_k > 0$, $k \in I_{\mathcal{O}_{\mathcal{S}}} \setminus I_{\mathcal{E}_{\mathcal{S}}}$.

Since by assumption $b_{\kappa+1} \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$, we can take $b = b_{\kappa+1}$ in the definition of reach controllability. W.l.o.g. let $B = b$. By reach controllability, for each $k \in I_{\mathcal{E}_{\mathcal{S}}}$, there exist $i_k \in I(o_k)$ and $u_{i_k} > 0$ such that $Av_{i_k} + Bu_{i_k} + a \in \mathcal{C}(v_{i_k})$. Select $i_k \in I(o_k)$ and $u_{i_k} > 0$ as above. Set $u_i = 0$ for the remaining vertices of $\mathcal{S}$. Using the method of [7], form the associated affine feedback $u(x) = Kx + g$ and let $y(x) := Ax + Bu(x) + a$. For $k \in I_{\mathcal{O}_{\mathcal{S}}}$, consider $o_k = \sum_{i \in I(o_k)} \alpha_i^{o_k} v_i$ with $\alpha_i^{o_k} > 0$. For $k \in I_{\mathcal{E}_{\mathcal{S}}}$, $u(o_k) = \sum_{i \in I(o_k)} \alpha_i^{o_k} u_i > 0$ since $u_{i_k} > 0$ and $i_k \in I(o_k)$. For $k \in I_{\mathcal{O}_{\mathcal{S}}} \setminus I_{\mathcal{E}_{\mathcal{S}}}$, $u(o_k) \geq 0$. We obtain that for $k \in I_{\mathcal{E}_{\mathcal{S}}}$, $y(o_k) = Ao_k + Bu(o_k) + a = u(o_k)b$, with $u(o_k) > 0$. For $k \in I_{\mathcal{O}_{\mathcal{S}}} \setminus I_{\mathcal{E}_{\mathcal{S}}}$, $y(o_k) = Ao_k + Bu(o_k) + a = (\lambda_k + u(o_k))b$, with $\lambda_k + u(o_k) > 0$. By convexity for $x \in \mathcal{O}_{\mathcal{S}}$, $y(x) = \lambda(x)b$ with $\lambda(x) > 0$. Thus, $\mathcal{E}_{\mathcal{S}} = \emptyset$ for the closed-loop system. By Theorem 8 of [13], $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback $u(x)$.

($\Longrightarrow$) Suppose $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback $u(x) = Kx + g$. If $\mathcal{E}_{\mathcal{S}} = \emptyset$ (for the open-loop system), then (A, B, a) is reach controllable. Alternatively, if $\mathcal{E}_{\mathcal{S}} \neq \emptyset$, then by Theorem 3, $\mathcal{E}_{\mathcal{S}} = \text{co}\{o_1, \ldots, o_{\kappa_0+1}\}$ where $0 \leq \kappa_0 < \kappa$. W.l.o.g. suppose $B = b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$. Suppose (A, B, a) is not reach controllable. Then there exists $k \in I_{\mathcal{E}_{\mathcal{S}}}$ such that for all $i \in I(o_k)$, $Av_i + Bu(v_i) + a \in \mathcal{C}(v_i)$ implies $u(v_i) \leq 0$. Since $u(o_k) = \sum_{i \in I(o_k)} \alpha_i^{o_k} u(v_i)$, $\alpha_i^{o_k} > 0$, we obtain $u(o_k) \leq 0$. Either $u(o_k) = 0$, so $Ao_k + Bu(o_k) + a = 0$, which contradicts $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$. Alternatively, $u(o_k) < 0$ so $A + Bu(o_k) + a = \lambda_k b$ with $\lambda_k < 0$. This means $Ao_i + Bu(o_i) + a = \lambda_i b$ with $\lambda_i < 0$ for all $i \in I_{\mathcal{O}_{\mathcal{S}}}$ (for otherwise by convexity there is $\overline{x} \in \mathcal{O}_{\mathcal{S}}$ such that $A\overline{x} + Bu(\overline{x}) + a = 0$, a contradiction). We conclude $-b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$, a contradiction to Lemma 4. ■

VII. Examples

We present two examples of Theorem 4. In the first example reach controllability fails while in the second example it holds.

Example 3: Consider a simplex $\mathcal{S} = \text{co}\{v_0, \ldots, v_4\} \subset \mathbb{R}^4$, where $v_0 = (0,0,0,0)$ and for $i \in I = \{1, \ldots, 4\}$, $v_i = e_i$, the ith Euclidean basis vectors. Note that $h_i = -e_i,\ i \in I$. Consider the following affine system

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -3 & -6 & -3 & -2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ -5 \\ 8 \\ 4 \end{bmatrix} u + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad (16)$$

Let $b := (-3, -5, 8, 4)$. It can be verified that

$$\mathcal{O} \quad := \quad \left\{ x \in \mathbb{R}^n \mid x_1 = x_2 = x_4 + \frac{1}{4}, x_3 = -2x_4 + \frac{1}{4} \right\}.$$

Setting $x_4 = 0$ in the defining equations for $\mathcal{O}$, we get $o_1 := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$. Setting $x_3 = 0$, we get $o_2 := \left(\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{8}\right)$. Thus, $\mathcal{O}_{\mathcal{S}} = \text{co}\{o_1, o_2\}$, where

$$\begin{aligned} o_1 &= \frac{1}{4}v_0 + \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 \in \mathcal{F}_4 \\ o_2 &= \frac{1}{8}v_0 + \frac{3}{8}v_1 + \frac{3}{8}v_2 + \frac{1}{8}v_4 \in \mathcal{F}_3 . \end{aligned}$$

Also we have that $Ao_1 + a = 0$ and $Ao_2 + a \neq 0$, so $\mathcal{E}_{\mathcal{S}} = \{o_1\}$. We observe that $\dim(\mathcal{O}) = 1$, $\dim(\mathcal{E}) = 0$, $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$, and $\mathcal{O}_{\mathcal{S}} \cap \mathcal{S}^{\circ} \neq \emptyset$. Because $o_1 \in \mathcal{F}_4$ and $o_2 \in \mathcal{F}_3$, we have

$$\text{cone}(\mathcal{O}_{\mathcal{S}}) = \left\{ y \in \mathbb{R}^n \mid h_3 \cdot y \leq 0,\ h_4 \cdot y \leq 0 \right\}.$$

Clearly $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$, so solvability of RCP by continuous state feedback cannot be ruled out by Theorem 1. Also it can be verified that $Av_i + a \in \mathcal{C}(v_i),\ i \in \{0, \ldots, n\}$, so solvability of RCP by continuous state feedback cannot be ruled out by Proposition 3.1 of [7]. Nevertheless, for the given simplex $\mathcal{S}$ and system (16), RCP is not solvable by affine feedback. This is due to the fact that (A, B, a) is not reach controllable according to Definition 1. Indeed $Av_i + Bu_i + a \in \mathcal{C}(v_i)$ results in $u_i = 0$ for $\forall\ i \in I(o_1) = \{0, 1, 2, 3\}$. ◁

Example 4: Consider a simplex $\mathcal{S} = \text{co}\{v_0, \ldots, v_4\} \subset \mathbb{R}^4$, where $v_0 = (0,0,0,0)$ and for $i = 1, \ldots, 4$, $v_i = e_i$, the ith Euclidean basis vector. Consider the following affine system

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad (17)$$

Let $b := (-1, 0, 1, 2)$. It can be verified that

$$\mathcal{O} \quad := \quad \left\{ x \in \mathbb{R}^n \mid x_1 = \frac{1 + 5x_4}{4}, x_2 = \frac{1 - x_4}{4}, x_3 = \frac{1 - 5x_4}{4} \right\}.$$

Setting $x_4 = 0$ in the defining equations for $\mathcal{O}$, we get $o_1 := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$. Setting $x_3 = 0$, we get $o_2 := \left(\frac{5}{10}, \frac{2}{10}, 0, \frac{2}{10}\right)$. Thus, $\mathcal{O}_{\mathcal{S}} = \text{co}\{o_1, o_2\}$, where

$$\begin{aligned} o_1 &= \frac{1}{4}v_0 + \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 \in \mathcal{F}_4 \\ o_2 &= \frac{1}{10}v_0 + \frac{5}{10}v_1 + \frac{2}{10}v_2 + \frac{2}{10}v_4 \in \mathcal{F}_3 . \end{aligned}$$

Also we have that $Ao_1 + a = 0$ and $Ao_2 + a = \frac{1}{10}b \neq 0$, so $\mathcal{E}_{\mathcal{S}} = \{o_1\}$. We observe that $\dim(\mathcal{O}) = 1$, $\dim(\mathcal{E}) = 0$, $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$, and $\mathcal{O}_{\mathcal{S}} \cap \mathcal{S}^{\circ} \neq \emptyset$. Because $o_1 \in \mathcal{F}_4$ and $o_2 \in \mathcal{F}_3$, we have

$$\text{cone}(\mathcal{O}_{\mathcal{S}}) = \left\{ y \in \mathbb{R}^n \mid h_3 \cdot y \leq 0,\ h_4 \cdot y \leq 0 \right\}.$$

Clearly $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$. Also, it can be verified that $Av_i + a \in \mathcal{C}(v_i),\ i \in \{0, \ldots, n\}$. Now we show that (A, B, a) is reach controllable. Indeed, for $i = 1, 2 \in I(o_1)$, we can find $u_i > 0$ such that $Av_i + Bu_i + a \in \mathcal{C}(v_i)$. Based on the constructive procedure given in the proof of Theorem 4, to remove o_1 we can inject vector b into the vector field at v_1 or v_2. One possibility is to select $u_1 = 1$ and $u_0 = u_2 = u_3 = u_4 = 0$. Using the method of [7], the resulting affine feedback is $u(x) = [1\ 0\ 0\ 0]x$. It is easy to verify that by using $u(x) = [1\ 0\ 0\ 0]x$, no equilibrium appears in $\mathcal{S}$. Since $Av_i + Bu_i + a \in \mathcal{C}(v_i),\ i \in \{0, \ldots, n\}$, by Theorem 8 of [13], $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using $u(x)$. ◁

VIII. Conclusion

The paper studies the reach control problem for single input affine systems on simplices. We relaxed the assumption that the state space is triangulated with respect to the affine set $\mathcal{O}$, the set of possible equilibria of the affine system. New necessary and sufficient conditions for solvability of RCP are provided in terms of the notion of reach controllability. As yet, a geometric characterization of reach controllability has not been found; our future work will explore this possibility.

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