A Viability Problem for Control Affine Systems with Application to Collision Avoidance

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***Abstract*— Given a control affine system and a domain $\mathcal{S}$ which is a smooth manifold with boundary, we present results on explicit construction of a viability controller and viable kernel of $\mathcal{S}$. Under a standard invariance condition on the viability domain, it is shown that there is a viability controller that takes a particularly simple form: it is a bang control. The proposed theory is applied to the problem of collision avoidance control of two vehicles.**

I. INTRODUCTION

In this paper we study the problem of least restrictive collision avoidance control of two unicycles. A collision avoidance controller is said to be *least restrictive* if it has the following property: if starting at some initial condition there is a collision using the least restrictive controller, then there is a collision using any other measurable control. Our goal is to obtain an explicit analytical characterization of this controller. In order to do so, we apply viability theory in a somewhat new setting.

The theoretical question that arises may be placed in the following context. Given a control system, a subset of the state space is said to be controlled invariant or viable if for all initial conditions in the set, the trajectories of the system remain inside the set by proper choice of control. Controlled invariance has been developed primarily in two contexts. One context is geometric system theory where the invariant set is the zero level set of a smooth function, the control system is typically affine in the control, and there are no constraints on the control values [20], [13]. The second more general context is that of viability theory [2]. Here the invariant set need not be a manifold, the system is described by differential inclusions, and the control typically takes values in a convex set. A comparison of the two contexts can be found in [3].

In the present paper, guided by the desire to characterize a least restrictive collision avoidance controller, we consider a control affine system and an invariant set which is a smooth manifold with boundary. The control takes values in a convex set. We propose conditions under which the viability controller is a bang controller; that is, it takes only a single constant control value. A characterization of the viability kernel is given. Recent, relevant work both on theory and numerical approaches to finding viability kernels are [5], [4], [10], [11], [14], [16], [17]. However, the specific class of viability problem treated here has not yet been investigated. In the second half of the paper the theory is applied to the problem of collision avoidance control for two unicycles. The presented theory is also applicable to general nonholonomic systems, with the main increase in complexity compared to unicycles arising in the computations of the minimum distance between two nonholonomic systems as a function of the bang control value.

Collision avoidance has been studied by many researchers and there are numerous approaches available. See, for instance, [7], [15], [12], [18], [19] for several recent approaches. Of particular relevance are the results of Ikeda and Kay [12] and Melikyan, Hovakimyan, and Ikeda [15]. The latter reference shows that an alternate approach to solving the least restrictive collision avoidance problem is to use dynamic programming. This approach leads to the solution of a Hamilton-Jacobi-Bellman equation, and it is proposed to use the method of singular characteristics to solve for the value function. Instead we obtain an explicit solution using direct arguments from viability theory. The theoretical connections between the dynamic programming formulation and the viability theory formulation is an interesting area of further investigation.

II. MOTIVATING PROBLEM

Suppose we have two vehicles modelled as unicycles. The vehicles are assumed to travel with unit speed and they each have a minimum turning radius of one. For each vehicle $i = 1,2$ the kinematic model is

$$\begin{aligned} \dot{x}_i &= \cos\theta_i \\ \dot{y}_i &= \sin\theta_i \\ \dot{\theta}_i &= u_i\,, \end{aligned}$$

where $(x_i, y_i) \in \mathbb{R}^2$ is the position in the plane, $\theta_i \in \mathbb{R}$ is the vehicle's orientation, and the control input $u_i \in \mathbb{R}$ is the angular velocity. The turning radius requirement dictates the control must satisfy $|u_i| \le 1$. We say that the two vehicles *collide* at time t if the distance between them at t is strictly less than a prespecified positive number c. We define the domain $\mathcal{S}_c$ to be the region of the state space where there is no collision. That is,

$$\mathcal{S}_c = \{(x_1, y_1, \theta_1, x_2, y_2, \theta_2) \mid \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ge c\}\,.$$

It is assumed that the two vehicles are autonomous, unwilling to form long term plans with each other, but in the face of imminent collision, they execute controllers which

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harmoniously achieve collision avoidance. We consider the following problem.

Problem 1: Given two vehicles modelled as unicycles, find a controller u_v with the following property: if starting from an initial condition and using u_v the two vehicles collide, then using any other measurable control input the vehicles also collide.

In the next two sections we develop a theoretical framework to address this problem. In Section V we return to solving the motivating problem.

III. THEORETICAL PROBLEM FORMULATION

Consider a system

$$\dot{x} = f(x) + g(x)u\,, \tag{1}$$

where $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^{n\times m}$ are Lipschitz and the input space is a compact, convex polyhedron $U \subset \mathbb{R}^m$. A control $u : [0,\infty) \to U$ is a measurable function in t which takes values in U. The set of q vertices of U is denoted as

$$V = \{v^1, \ldots, v^q\}\,.$$

Let $\phi(t,x_0)$ be the unique solution of (1) starting at x_0 and using control u. Also, let $s : \mathbb{R}^n \to \mathbb{R}$ be a smooth submersion, i.e. the gradient ∇s is non-vanishing everywhere in $\mathbb{R}^n$.[1] Suppose we are given $c \in \mathbb{R}$. The domain to be rendered invariant is

$$\mathcal{S}_c = \{x \in \mathbb{R}^n \mid s(x) \geq c\}\,.$$

We make the following assumption on s.

Assumption 1: The function s has the property that for all $x \in \mathbb{R}^n$, $\dot{s}(x)$, the Lie derivative of s along solutions of (1), is not a function of u. That is, $L_g s(x) = 0$.

This relative degree-like assumption implies that the Lie derivative of s is $\dot{s} = L_f s$ and it allows us to define the set of states where s is decreasing, namely,

$$\mathcal{W} = \{\ x \in \mathbb{R}^n \mid L_f s(x) < 0\ \}\,.$$

Definition 1 (Aubin, p. 121 [2]): A subset $\mathcal{S}_c$ is said to be a *viability domain* if for each $x_0 \in \mathcal{S}_c$, there exists a control $u(t)$ such that the solution of (1) starting at x_0 with control u stays in $\mathcal{S}_c$ for all $t \geq 0$. If $\mathcal{S}_c$ is not a viability domain, then there exists a largest closed (possibly empty) viability domain $\mathcal{S}_c^*$ contained in $\mathcal{S}_c$, which is called the *viability kernel* of $\mathcal{S}_c$. A control u_v which renders $\mathcal{S}_c^*$ viable is called a *viability controller*.

Our viability problem can be stated as follows.

Problem 2: Given a control affine system (1) and the set $\mathcal{S}_c$ which is a manifold with boundary, find u_v, a viability controller, and $\mathcal{S}_c^*$, the viability kernel.

We place a restriction on the type of viability controller that we consider. It is that the viability controller achieves viability in a finite time, rather than asymptotically. This is stated more precisely as follows.

Assumption 2: For each $x_0 \in \mathcal{S}_c^* \cap \mathcal{W}$ and using u_v, there exists $\bar{t} < \infty$ such that $s(\phi(\bar{t},x_0)) \geq c$ and $\dot{s}(\phi(\bar{t},x_0)) \geq 0$.

[1]The condition may be relaxed to say that on a relevant subset of $\mathbb{R}^n$ every point is a regular point of s.

Let $\neg\mathcal{W}$ denote $\mathbb{R}^n \setminus \mathcal{W}$. Essentially the assumption says that starting in the set $\mathcal{S}_c^* \cap \mathcal{W}$, the viability controller u_v drives the system to $\mathcal{S}_c \cap \neg\mathcal{W}$ in finite time. A viability controlller satisfying Assumption 2 is said to be a *finite time viability controller*. Finite time viability controllers are desirable from an applications viewpoint, and thus are the only ones considered in this paper.

IV. VIABILITY CONTROLLER

Consider $x_0 \in \mathbb{R}^n$ and for each $i = 1,\ldots,q$, define $\phi_i(t,x_0)$ to be the solution of the autonomous system

$$\dot{x} = f(x) + g(x)v^i, \qquad v^i \in V \tag{2}$$

starting from $x(0) = x_0$ and evaluated at time t. We define the following variables:

$$s_i(t,x_0) := s(\phi_i(t,x_0)), \qquad i = 1,\ldots,q\,.$$

For $x_0 \in \mathcal{W}$, let $\bar{t}_i(x_0)$ be the first time when $\phi_i(t,x_0)$ reaches the boundary of $\mathcal{W}$; that is when $\frac{ds_i}{dt}(t,x_0) = 0$. For $x_0 \in \neg\mathcal{W}$, set $\bar{t}_i(x_0) = 0$. (We will write $\bar{t}_i(x_0)$ as $\bar{t}_i$ where the dependence on x_0 is clear.) Since initially for $x_0 \in \mathcal{W}$, $\frac{ds_i}{dt}(t,x_0)\Big|_{t=0} < 0$, $\bar{t}_i$ is the time when $s_i(\cdot,x_0)$ (and therefore s) reaches a local minimum along the trajectory starting at x_0 when restricted to the time interval $[0,\bar{t}_i]$. For $x_0 \in \mathbb{R}^n$, we define $\bar{s}_i(x_0)$ to be the value of s_i at $\bar{t}_i$, i.e.,

$$\bar{s}_i(x_0) := s_i(\bar{t}_i,x_0)\,. \tag{3}$$

We observe that by definition $\bar{s}_i$ is constant when evaluated along the trajectory $\phi_i(t,x_0)$ over the interval $[0,\bar{t}_i]$.

We require the following.

Assumption 3: Each $\bar{t}_i(\cdot)$ is a continuous function of the initial condition $x_0 \in \mathbb{R}^n$, and $\bar{t}_i(x_0) < \infty$ for all $x_0 \in \mathbb{R}^n$ and $i = 1,\ldots,q$. Moreover, $\bar{s}_i(\cdot)$ is a continuously differentiable function on $\mathcal{W}$.

Continuity of $\bar{t}_i$ can be guaranteed by imposing transversality of the flow $\phi_i(t,x_0)$ with $\partial\mathcal{W}$. See [6] for similar arguments in the context of proving continuity of a minimum time function over a finite horizon. Once we have that $\bar{t}_i$ is continuous and using Lipschitz continuity of the vector fields, it is a standard argument to show that $\bar{s}_i$ is continuous. The differentiability assumption is introduced to be able to compute gradients of $\bar{s}_i$ in $\mathcal{W}$, and, in general, is too restrictive; however, it can be removed using tools of non-smooth analysis [2], [8]. We retain the assumption since it holds in our main application.

For each $v^i \in V$ and each $x_0 \in \mathcal{W}$, there is a finite time $\bar{t}_i$ when the trajectory reaches the boundary of $\mathcal{W}$. The first step of our design is to specify a control which acts in the region $\mathcal{S}_c \cap \neg\mathcal{W}$.

Assumption 4: There exists a controller $u_p : [0,\infty) \to U$ such that if $x_0 \in \mathcal{S}_c \cap \neg\mathcal{W}$, then using u_p, $\dot{s}(\phi(t,x_0)) > 0$ for all $t > 0$.

Remark 1: A viability controller need only act on the boundary of its viability kernel. In $\neg\mathcal{W}$, we will see the viability kernel is simply $\mathcal{S}_c$, so u_p is only used in $\partial\mathcal{S}_c \cap \neg\mathcal{W}$. However, the system naturally remains viable in $\mathcal{S}_c \cap \neg\mathcal{W}$, since $L_f s(x) > 0$ along $\partial\mathcal{S}_c$ in $\neg\mathcal{W}$. Hence, any control will, in fact, do in this region. The control u_p is selected primarily to be able to conveniently refer to a single controller in $\mathcal{S}_c \cap \neg\mathcal{W}$ in the later theoretical development, and therefore presents no loss of generality.

Next we turn to the more challenging task of finding a viability controller for the region $\mathcal{S}_c \cap \mathcal{W}$. We propose a bang controller (a controller that uses only one constant control value for each initial condition), denoted u^*, that we claim is the viability controller in the region $\mathcal{S}_c \cap \mathcal{W}$. The overall viability controller is then u_v, equal to u_p or u^*, depending on the initial condition. Associated with u_v is a viability kernel $\mathcal{S}_c^*$ of $\mathcal{S}_c$. Using u_v, if the state is initialized in $\mathcal{S}_c^*$ then it remains in $\mathcal{S}_c^* \subseteq \mathcal{S}_c$ for all time. The controller u_v is active only on the boundary of $\mathcal{S}_c^*$. In the interior of $\mathcal{S}_c^*$ other controllers may be used. We say that the controller u_v is *least restrictive* in the sense that if viability is violated starting at some initial condition using u_v, then it is violated with any other measurable control.

We give a characterization of u^*. For $x \in \mathcal{S}_c \cap \mathcal{W}$, define the set of indices

$$I^*(x) = \mathrm{argmax}_{i \in \{1,\dots,q\}} \{\ \bar{s}_i(x)\ \}. \qquad (4)$$

Notice that the cardinality of this set may vary with x. Define the function $\mu^* : \mathcal{S}_c \cap \mathcal{W} \to V$ by

$$\mu^*(x) := v^j, \qquad j \in I^*(x). \qquad (5)$$

Finally, for each initial condition $x_0 \in \mathcal{S}_c \cap \mathcal{W}$ we define

$$u^*(t, x_0) := \mu^*(x_0), \qquad t \in [0, \bar{t}(x_0)], \qquad (6)$$

where $\bar{t}(x_0) := \bar{t}_j(x_0)$ if $\mu^*(x_0) = v^j$. This controller will henceforth be called the “bang controller”. First, notice it is an open-loop control. Intuitively, this choice of controller maximizes the first local minimum value of s on an interval $[0, \bar{t}]$, by using only a single control value in V. The controller u^* terminates at the time $\bar{t}$ when, by construction, $\dot{s} = 0$; that is, u^* terminates and u_p is initiated when the trajectory exits the set $\mathcal{W}$. (The controller u_p guarantees that the local minimum of s on the interval $[0, \bar{t}]$ is in fact a global mimimum on the interval $[0, \infty)$.)

Next we introduce the viability kernel. First, we define

$$s^*(x) = \begin{cases} \max_{i \in \{1,\dots,q\}} \{\ \bar{s}_i(x)\ \} & x \in \mathcal{W} \\ s(x) & x \in \neg\mathcal{W}. \end{cases}$$

It is a straightforward exercise to show that s^* is a continuous function. Define the set

$$\mathcal{D}_c^* = \{x \in \mathbb{R}^n \mid s^*(x) < c\}. \qquad (7)$$

We claim the viability kernel is

$$\mathcal{S}_c^* := \neg D_c^*. \qquad (8)$$

It is evident from this definition and the continuity of s^* that $\mathcal{S}_c^*$ is closed. We can further interpret $\mathcal{S}_c^*$ as follows:

$$\mathcal{S}_c^* = (S_c \cap \neg\mathcal{W}) \cup (\neg\mathcal{D}_c^* \cap \mathcal{W}).$$

It is obviously true for $x \in \mathcal{W}$ that $\mathcal{S}_c^* \cap \mathcal{W} = \neg\mathcal{D}_c^* \cap \mathcal{W}$. For $x \in \neg\mathcal{W}$, we know $s^*(x) = s(x)$, so $\mathcal{S}_c^* \cap \neg\mathcal{W} = \mathcal{S}_c \cap \neg\mathcal{W}$. Thus, the interpretation of $\mathcal{S}_c^*$ is as follows. In the region $\neg\mathcal{W}$ where $L_f s \geq 0$, the viability kernel is simply $\mathcal{S}_c$. In particular, on the boundary of $\mathcal{S}_c$, the control u_p may be used to ensure viability, as already discussed. In the region $\mathcal{W}$ where $L_f s < 0$, we claim the viability kernel is $\neg\mathcal{D}_c^*$, and on the boundary of $\mathcal{D}_c^*$ the bang control u^* is used. To summarize, $u_v : \mathbb{R}^n \to U$ consists of two parts corresponding to the two regions of $\mathcal{S}_c^*$, and it acts only on the boundary of $\mathcal{S}_c^*$. Precisely,

$$u_v(t, x_0) = \begin{cases} u^*(t, x_0) & x_0 \in \partial D_c^* \cap \mathcal{W} \\ u_p(t) & x_0 \in \partial S_c \cap \neg\mathcal{W}. \end{cases}$$

A. Main Results

In this section we prove our main theoretical results. We say that a control $u(t)$ is *bang-bang* if it is piecewise constant and it takes values in V, for all $t \geq 0$. Let a *k-switch controller* be a bang-bang control that allows k switches in its value. In particular, u^* is a 0-switch controller.

In Lemma 1 we give a condition under which u^* is least restrictive with respect to 1-switch controls. The main idea is the following. We consider the set $\mathcal{D}_c^* \cap \mathcal{S}_c$ which comprises the initial conditions x_0 for which u^* cannot maintain the system in $\mathcal{S}_c$, since $s^*(x_0) < c$, but some other control $u(t)$ may be able to. To maintain viability the control $u(t)$ must be able to steer the system to $\neg\mathcal{W}$ without first entering $\neg\mathcal{S}_c$ (recall we only consider viability controls that reach $\neg\mathcal{W}$ in finite time). By imposing an appropriate invariance condition on the vector fields (2) on $\partial\mathcal{S}_c^* \cap \mathcal{W}$, it is shown that no such 1-switch control $u(t)$ exists. In particular, the invariance condition guarantees that trajectories starting in $\mathcal{D}_c^* \cap \mathcal{S}_c$ cannot exit directly to $\mathcal{S}_c^*$, but instead first reach $\neg\mathcal{S}_c$. Lemma 2 uses an induction argument to extend this result to bang-bang controls. Finally, in Theorem 1 we prove u^* is least restrictive in $\mathcal{W}$ with respect to measurable controls.

Lemma 1: Given $c \in \mathbb{R}$, suppose that for all $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$ and for all $i \notin I^*(x)$ and $j \in I^*(x)$, we have that

$$\nabla\bar{s}_j(x) \cdot (f(x) + g(x)v^i) < 0. \qquad (9)$$

Then the bang controller u^* for $\mathcal{S}_c^* \cap \mathcal{W}$ is least restrictive with respect to 1-switch controllers.

Proof: We argue by contradiction. Suppose there exists an initial condition $x_0 \in \mathcal{W}$ and a control $u(t)$ such that viability is violated with $u^*(t, x_0)$ and not with u. Let $x(t)$ be the solution of (1) using control u. Viability is only violated with u^* if $x_0 \in \mathcal{D}_c^*$ but to preserve viability using u it must be that $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. Denote u as $u^1 u^2$, the concatentation of control $u^1 \in V$ followed by $u^2 \in V$. Suppose the control switches value at time $0 < t^2 < \infty$. Since a 1-switch controller becomes a 0-switch controller at the switching time, it must be that $x(t^2) \in \neg(\mathcal{D}_c^* \cap \mathcal{S}_c)$. Hence, there exists $t^1 \leq t^2$, the first time that $x(t^1) \in \partial(\mathcal{D}_c^* \cap \mathcal{S}_c)$. If $x(t^1) \in \mathcal{D}_c^* \cap \partial\mathcal{S}_c$ (where we must have $\dot{s} \geq 0$ for u to be a viable control), then one can apply the control u_p from $x(t^1)$, which means that the 0-switch controller u^1 is a viable control starting from $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$, a contradiction. It must be that $x(t^1) \in \partial\mathcal{D}_c^* \cap \mathcal{S}_c$. Since $s^*(x(t))$ is a continuous function of t, $t^1 > 0$. We cannot have $i \in I^*(x(t^1))$, where i corresponds to $u^1 = v^i$;

otherwise we could continue with u^1, a 0-switch controller, to maintain viability starting from $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. Instead, it must be that $i \notin I^*(x(t^1))$. Let $j \in I^*(x(t^1))$. Let $\{t_k\}$, $t_k \in [0,t^1)$ be an increasing sequence of times such that $t_k \to t^1$. Since $x(t_k) \in \mathcal{D}_c^* \cap \mathcal{S}_c$, $\bar{s}_j(x(t_k)) \le s^*(x(t_k)) < c$. Then we have

$$\dot{\bar{s}}_j(x(t^1)) = \lim_{k\to\infty} \frac{\left(\bar{s}_j(x(t^1)) - \bar{s}_j(x(t_k))\right)}{t^1 - t_k} = \lim_{k\to\infty} \frac{c - \bar{s}_j(x(t_k))}{t^1 - t_k} \ge 0\,.$$

This contradicts the assumption (9) that $\dot{\bar{s}}_j(x(t^1)) < 0$. As a result there does not exist a 1-switch controller that is less restrictive than u^*. ■

Lemma 2: Given $c \in \mathbb{R}$, suppose that for all $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$ and for all $i \notin I^*(x)$ and $j \in I^*(x)$, condition (9) holds. Then the bang controller $u^*(t,x_0)$ for $\mathcal{S}_c^* \cap \mathcal{W}$ is least restrictive with respect to bang-bang controls.

Proof: We argue by induction. By Lemma 1, u^* is least restrictive with respect to 1-switch controllers. Now assume it is least restrictive with respect to 1 up to $k-1$ switch controllers. We will show it is least restrictive with respect to 1 to k switch controllers. By way of contradiction, suppose there is a k-switch controller u that is less restrictive than u^*. That is, there exists an initial condition $x_0 \in \mathcal{D}_c^*$ for which the k-switch controller maintains viability. This means that $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. Consider the point x^1 where the $(k-1)$th switch happens. It must be that $x^1 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ since the bang controller is least restrictive with respect to $k-2$ switch controllers. Starting from x^1 we have a 1-switch controller to maintain viability. This contradicts that the bang controller is the least restrictive controller with respect to 1-switch controllers. ■

Finally, we must prove that u^* is least restrictive with respect to measurable controls. We require a general result for control affine systems, called the Chattering Lemma, on the reachability of states under measurable controls and bang-bang controls.

Lemma 3 (Chattering Lemma [1]): Let $x(t)$ be a solution of (1) corresponding to some control $u(t)$. There exists a sequence $\{u^k,\, k \in \mathbb{N}\}$ of piecewise constant bang-bang controls, such that if $\left\{x^k\right\}$ is the corresponding family of solutions to (1), then for each $T > 0$

$$\sup_{0\le t\le T} \|x^k(t) - x(t)\| \to 0\,, \quad \text{as } k \to \infty\,. \tag{10}$$

The following is our main theoretical result.

Theorem 1: Suppose that for all $c \in \mathbb{R}$, $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$, and for all $i \notin I^*(x)$ and $j \in I^*(x)$, condition (9) holds. Then the bang controller u^* for $\mathcal{S}_c^* \cap \mathcal{W}$ is least restrictive with respect to measurable controls. Consequently u^* is a viability controller for $\mathcal{S}_c^* \cap \mathcal{W}$ and $\mathcal{S}_c^*$ is the viability kernel.

Proof: Fix $c \in \mathbb{R}$. Suppose there exists a measurable control $u(t)$ that is less restrictive than u^*, the bang controller for $\mathcal{S}_c^* \cap \mathcal{W}$. This implies there exists $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ and a time $\bar{t} < \infty$ such that if $x(t)$ is the trajectory starting at x_0 using control $u(t)$, then

1) $s(x(t)) > c$ and $\dot{s}(x(t)) < 0$ for all $t \in [0,\bar{t})$.
2) $s(x(\bar{t})) =: \bar{c} \ge c$ and $\dot{s}(x(\bar{t})) = 0$.
3) After time $\bar{t}$, w.l.o.g. set $u(t) = u_p(t)$, as in Assumption 4.

The first and second statements arise as follows. First, $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ implies that $x_0 \in \mathcal{W}$. Also, we cannot have $x_0 \in \partial\mathcal{S}_c \cap \mathcal{W}$ for then viability would be immediately violated at x_0. Instead, $x_0 \in int(\mathcal{S}_c) \cap \mathcal{W}$ and $\bar{t}$ is the first time when $x(t) \in \mathcal{S}_c \cap \partial\mathcal{W}$.

Let $T = \bar{t} + 1$. We note that $\inf_{t\in[0,\infty)} s(x(t)) = \min_{t\in[0,T]} s(x(t)) = \bar{c}$ by the definition of u_p. Let $\{u^k\}$ be a sequence of bang-bang controls defined on $[0,\infty)$ as in the Chattering Lemma and $\{x^k(t)\}$ the associated trajectories such that (10) holds. By continuity of s we have that $\sup_{0\le t\le T} \|s(x^k(t)) - s(x(t))\| \to 0$ as $k \to \infty$. It follows that

$$\min_{t\in[0,T]} s(x^k(t)) \to \bar{c}\,, \quad \text{as } k \to \infty\,, \tag{11}$$

and since $\dot{s}(x(t)) < 0$ for $t \in [0,\bar{t})$ and $\dot{s}(x(t)) > 0$ for $t \in (\bar{t},\infty)$, one can also show $\operatorname{arginf}_{t\in[0,\infty)} s(x^k(t)) \to \bar{t}$ as $k \to \infty$. Therefore, there exists $\kappa > 0$ such that for all $k > \kappa$, there exists $\bar{t}^k \in [0,T]$ such that

$$\inf_{t\in[0,\infty)} s(x^k(t)) = \min_{t\in[0,T]} s(x^k(t)) = s(x^k(\bar{t}^k))\,. \tag{12}$$

Let $c^* := s^*(x_0)$. We know $c^* < c$ since $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. By Lemma 2, u^* is less restrictive than any bang-bang control for the domain $\mathcal{S}_{c^*}$. This means for all $k > \kappa$, $\min_{t\in[0,T]} s(x^k(t)) \le c^*$. From (11), it follows that $\bar{c} \le c^* < c$, a contradiction. ■

V. COLLISION AVOIDANCE

Suppose we have two unit-speed vehicles $i = 1,2$ modelled as unicycles. We define $V = \{-1,1\} \times \{-1,1\}$. Let r be the distance between the vehicles, $\phi \in (-\pi,\pi]$ the heading of the first vehicle that would take it directly towards the second vehicle, and $\theta \in (-\pi,\pi]$ the difference between the two vehicle's headings, taking vehicle 1's heading as the reference. The two unicycle system in relative polar coordinates, valid for $r > 0$, is

$$\dot{r} = \cos(\phi - \theta) - \cos(\phi) \tag{13}$$

$$\dot{\phi} = \frac{1}{r}\sin(\theta - \phi) + \frac{1}{r}\sin(\phi) - u_1 \tag{14}$$

$$\dot{\theta} = u_2 - u_1\,. \tag{15}$$

We use the notation $\xi = (r,\phi,\theta)$ to refer to the state of the system in relative coordinates. In terms of these coordinates the viability domain is

$$\mathcal{S}_c = \{\xi \in \mathbb{R}^3 \mid r \ge c\}\,.$$

It is clear from (13) that Assumption 1 holds, so we can characterize the set $\mathcal{W}$. From Equation (13) we have $\dot{r}(t) = 0$ when $\cos(\phi - \theta) - \cos(\phi) = 0$. The roots are $\theta = 0$ and $\theta = 2\phi$. It follows that

$$\mathcal{W} = \left\{\xi \mid \phi \in [0,\pi), \theta \in (2\phi, 2\pi)\right\} \bigcup \left\{\xi \mid \phi \in [-\pi,0), \theta \in (0, 2\pi + 2\phi)\right\}$$

Next, we must address Assumption 4. The following is easily proved.

Lemma 4: If $\xi \in \neg\mathcal{W}$ at $t = 0$, then there exists a controller u_p such that $\dot{r}(t) > 0$ for all $t > 0$.

Next we compute u^* and s^*. The main step of the computation is computing the $\bar{s}_i$'s. The following convention on subscripts is used. Subscript $i=1$ when $v^1:=(u_1,u_2)=(1,1)$; $i=2$ when $v^2:=(u_1,u_2)=(1,-1)$; $i=3$ when $v^3:=(u_1,u_2)=(-1,1)$; and $i=4$ when $v^4:=(u_1,u_2)=(-1,-1)$. Using this notation, $\bar{s}_i(\xi_0)$ is the value of r at the first time $\bar{t}_i$ when $\dot{r}(\bar{t}_i)=0$ starting from the initial condition ξ_0 and using the constant control $v^i\in V$.

Each vehicle follows one of two circles depending on its own control input. A circle is identified with its center. Circle o_1 (o_2) is the circle followed by vehicle 1 when $u_1=1$ ($u_1=-1$). Similarly, o_3 (o_4) is the circle followed by vehicle 2 when $u_2=-1$ ($u_2=1$). Let $D_1(t)$ be the length of the line connecting o_1 and o_4; $D_2(t)$ is the length of the line connecting o_1 and o_3; $D_3(t)$ is the length of the line connecting o_2 and o_4; and $D_4(t)$ is the length of the line connecting o_2 and o_3. Also define D_{i0} to be $D_i(t)$ at time $t=0$. Let m_i, $i=1,2$, be the position of ith vehicle. To obtain the parameters D_i we use the positions of the centers of the circles: $o_1=(0,1)$, $o_2=(0,-1)$, $o_3=(r\cos\phi+\sin\theta, r\sin\phi-\cos\theta)$ and $o_4=(r\cos\phi-\sin\theta, r\sin\phi+\cos\theta)$. It follows immediately that

$$D_1^2 = r^2-4r\cos(\phi-\frac{\theta}{2})\sin(\frac{\theta}{2})+4\sin^2(\frac{\theta}{2}) \quad (16)$$

$$D_2^2 = r^2-4r\sin(\phi-\frac{\theta}{2})\cos(\frac{\theta}{2})+4\cos^2(\frac{\theta}{2}) \quad (17)$$

$$D_3^2 = r^2+4r\sin(\phi-\frac{\theta}{2})\cos(\frac{\theta}{2})+4\cos^2(\frac{\theta}{2}) \quad (18)$$

$$D_4^2 = r^2+4r\cos(\phi-\frac{\theta}{2})\sin(\frac{\theta}{2})+4\sin^2(\frac{\theta}{2}). \quad (19)$$

One can now analyze the geometry of the four circles and determine the minimum distance between the two points m_1 and m_2 moving on their respective circles, as a function of the initial condition and the D_i's. This is an elementary but lengthy computation of geometry, so we only present the final result. Let

$$\sigma^-(\xi_0) := \left|2\cos(\frac{\theta_0}{2})-r_0\sin(\phi_0-\frac{\theta_0}{2})\right| \quad (20)$$

$$\sigma^+(\xi_0) := \left|2\cos(\frac{\theta_0}{2})+r_0\sin(\phi_0-\frac{\theta_0}{2})\right|. \quad (21)$$

Then given $\xi_0\in\mathcal{W}\cap\mathcal{S}_c$ we have for $i=1,4$,

$$\bar{s}_i(\xi_0)=\left|D_{i0}-2|\sin(\frac{\theta_0}{2})|\right|. \quad (22)$$

Also,

$$\bar{s}_2(\xi_0)=\begin{cases} |r_0\cos(\phi_0-\frac{\theta_0}{2})| & \sigma^-(\xi_0)\le 2 \\ \sqrt{D_{20}^2+4-4\sigma^-(\xi_0)} & \sigma^-(\xi_0)>2. \end{cases} \quad (23)$$

and

$$\bar{s}_3(\xi_0)=\begin{cases} |r_0\cos(\phi_0-\frac{\theta_0}{2})| & \sigma^+(\xi_0)\le 2 \\ \sqrt{D_{30}^2+4-4\sigma^+(\xi_0)} & \sigma^+(\xi_0)>2. \end{cases} \quad (24)$$

We must determine the largest among the $\bar{s}$'s in order to obtain the viability kernel. Here we present only the final result. See [9] for the details of the computations. First we have

$$\rho(\xi) := \max\{\bar{s}_1(\xi),\bar{s}_4(\xi)\}=\max\{D_1,D_4\}-2|\sin(\frac{\theta}{2})|$$
$$= \sqrt{r^2+4r\left|\cos(\phi-\frac{\theta}{2})\sin(\frac{\theta}{2})\right|+4\sin^2(\frac{\theta}{2})}-2\left|\sin(\frac{\theta}{2})\right|.$$

Next, let

$$\sigma(\xi):=2|\cos(\frac{\theta}{2})|+r|\sin(\phi-\frac{\theta}{2})|.$$

Note that $\sigma(\xi)\le 2$ if and only if $\sigma^+(\xi)\le 2$ and $\sigma^-(\xi)\le 2$.

Lemma 5: If $\sigma(\xi_0)\le 2$ then $\max\{\bar{s}_1,\bar{s}_4\}\ge\max\{\bar{s}_3,\bar{s}_2\}$. If, in addition, $\xi_0\in\mathcal{W}$, then the inequality is strict.

Let $\rho'(\xi)$ be the maximum of the second two cases of (23) and (24). Then we have $\rho'(\xi):=$

$$\sqrt{r^2+4r\left|\sin(\phi-\frac{\theta}{2})\cos(\frac{\theta}{2})\right|+4\cos^2(\frac{\theta}{2})+4-4\sigma(\xi)}.$$

With these results, we obtain the final form of $s^*(\xi_0)$ and thus an analytical characterization of the set of initial conditions ξ_0 where $\max\{\bar{s}_i(\xi_0)\}=c$. We have

$$s^*(\xi_0)=\begin{cases} \rho(\xi_0), & \sigma(\xi_0)\le 2 \\ \max\{\rho(\xi_0),\rho'(\xi_0)\}, & \sigma(\xi_0)>2 \end{cases} \quad (25)$$

Having obtained a characterization of s^*, we turn to the sets $\mathcal{D}_c^*$ and $\mathcal{S}_c^*$. First, from (25) it is evident that surface $\partial\mathcal{D}_c^*$ is formed by the c level sets of the functions ρ and ρ'. The set $\mathcal{D}_c^*$ also has the following property, useful in representing it graphically.

Lemma 6: Given a pair (ϕ,θ), there is a unique finite value r such that $(r,\phi,\theta)\in\partial\mathcal{D}_c^*$. Moreover, in spherical coordinates $\mathcal{D}_c^*$ is star convex, i.e. for all $\alpha\in[0,1)$, if $(r,\phi,\theta)\in\partial\mathcal{D}_c^*$, then $(\alpha r,\phi,\theta)\in\mathcal{D}_c^*$.

This fact justifies a visualization of $\mathcal{S}_c^*$ by projecting its boundary colorcoded with the appropriate u^* value to the set $\mathcal{W}$ in the $\phi-\theta$ plane. Figure 1 shows u^* on the boundary of $\mathcal{S}_c^*$ for $c=1$ after projecting to $\mathcal{W}$ in the $\phi-\theta$ plane. There are four regions corresponding to the four choices of control. The boundary curves are the points where certain of the $\bar{s}_i$'s are equal.

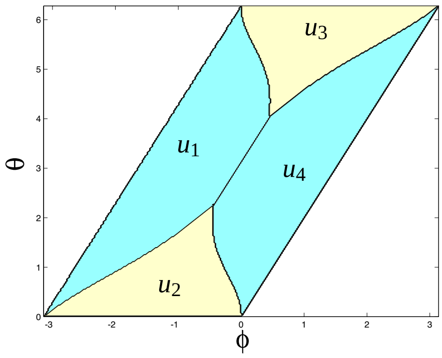

Fig. 1. The set $\mathcal{S}_c^*\cap\mathcal{W}$ projected to the $\phi-\theta$ plane when $c=1$.

A. Verification of Condition (9)

The final step of the design is to verify that condition (9) is satisfied. First, it can be verified that the gradients of $\bar{s}_i$ are defined on $\partial\mathcal{D}_c^*$ [9]. Here we compute the gradient vectors $\nabla\rho$ and $\nabla\rho'$ of the level surfaces $\rho(\xi_0) = c$ and $\rho'(\xi_0) = c$, and take the dot product of these gradient vectors with the appropriate vector fields. These expressions are computed symbolically using Maple. The notation $\nabla\rho$ and $\nabla\rho'$ is used as a shorthand for the gradients of the $\bar{s}_i$'s and not to connote these functions are themselves differentiable.

Let f_i denote the vector field (13)-(15) with control v^i. We obtain that

$$\nabla\rho \cdot f_1 = \begin{cases} \frac{2r\dot{r}}{c+2|\sin(\frac{\theta}{2})|} & \bar{s}_4 > \bar{s}_1 \\ 0 & \bar{s}_4 < \bar{s}_1 \end{cases}$$

$$\nabla\rho \cdot f_4 = \begin{cases} \frac{2r\dot{r}}{c+2|\sin(\frac{\theta}{2})|} & \bar{s}_4 < \bar{s}_1 \\ 0 & \bar{s}_4 > \bar{s}_1 \end{cases}$$

Using the fact that $\dot{r} < 0$, we see that $\nabla\rho \cdot f_1 < 0$ if $\bar{s}_1 < \bar{s}_4$ and $\nabla\rho \cdot f_4 < 0$ if $\bar{s}_4 < \bar{s}_1$. This dot product is only used when $\rho(\xi) > \rho'(\xi)$ and in that case, if $\bar{s}_1 > \bar{s}_4$ then $u^* = v^1$ and $\nabla\rho \cdot f_1 = 0$, as one would expect of the viability controller. Similarly, if $\rho(\xi) > \rho'(\xi)$ and $\bar{s}_4 > \bar{s}_1$ then $u^* = u_4$ and $\nabla\rho \cdot f_4 = 0$. Hence, these results verify condition (9). Note we do not need to verify (9) when $\bar{s}_4 = \bar{s}_1$ since then $1, 4 \in I^*$.

We also have

$$\nabla\rho' \cdot f_2 = 2\frac{\dot{r}}{|\sin(\phi - \frac{\theta}{2})|}(r|\sin(\phi - \frac{\theta}{2})| + 2|\cos(\frac{\theta}{2})| - 2)$$

when $\bar{s}_3 > \bar{s}_2$ and $\nabla\rho' \cdot f_2 = 0$ when $\bar{s}_3 < \bar{s}_2$. Also

$$\nabla\rho' \cdot f_3 = 2\frac{\dot{r}}{|\sin(\phi - \frac{\theta}{2})|}(r|\sin(\phi - \frac{\theta}{2})| + 2|\cos(\frac{\theta}{2})| - 2)$$

when $\bar{s}_3 < \bar{s}_2$ and $\nabla\rho' \cdot f_3 = 0$ when $\bar{s}_3 > \bar{s}_2$. These formulas only apply when $\sigma(\xi) > 2$ since that is the only case when ρ' is used. Also, since $\dot{r} < 0$ we cannot have $\sin(\phi - \frac{\theta}{2}) = 0$. The first and third formulas are thus well defined and satisfy (9), by using $\sigma(\xi) > 2$. Finally, the zero dot products appear as discussed above, and also coincide with condition (9).

Finally, we are able to obtain some further information about the remaining dot products. After some algebra,

$$\nabla\rho \cdot (f_3 + f_2) = \frac{2r\dot{r}}{c + 2|\sin(\frac{\theta}{2})|}$$

$$\nabla\rho' \cdot (f_1 + f_4) = \frac{2\dot{r}(r|\sin(\phi - \frac{\theta}{2})| + 2|\cos(\frac{\theta}{2})| - 2)}{c}.$$

Both expressions are always negative as we have $\dot{r} < 0$. Therefore, the first equation implies either $\nabla\rho \cdot f_3$ is negative or $\nabla\rho \cdot f_2$ is negative. Similarly, the second equation implies either $\nabla\rho' \cdot f_1$ is negative or $\nabla\rho' \cdot f_4$ is negative.

So far we have shown that at any point $\xi \in \partial\mathcal{D}_c^* \cap \mathcal{W} \cap \mathcal{S}_c$, at least three out of the four vector fields satisfy condition (9). The remaining cases are complicated to verify analytically. Therefore, we have verified those cases numerically using Matlab for a range of values of c from 0.01 to 10,000. It was found that the remaining dot products satisfy condition (9).

VI. CONCLUSION

We have presented theory for explicit construction of viability kernels and viability controllers for control affine systems when the invariant domain is given as a smooth manifold with boundary. The results are shown to apply to the problem of least restrictive collision avoidance control of two vehicles. Our future work involves improving the proposed theory by casting it in the setting of nonsmooth analysis and by relaxing the assumptions of Theorem 1.

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