INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2002; **00**:1–6 *Prepared using* rncauth.cls *[Version: 2002/11/11 v1.00]*

On the Least Restrictive Control for Collision Avoidance of Two Unicycles†

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SUMMARY

We give an explicit analytical characterization of the least restrictive control for collision avoidance of two unicycles. The controller is proved to be least restrictive using viability theory. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: viability theory, invariance, collision avoidance

1. INTRODUCTION

In this paper we study the problem of least restrictive collision avoidance control of two unicycles. A collision avoidance controller is said to be *least restrictive* if it has the following property: if starting at some initial condition there is a collision using the least restrictive controller, then there is a collision using any other measurable control. Our goal is to obtain an explicit analytical characterization of this controller. In order to do so, we apply viability theory in a somewhat new setting.

The theoretical question that arises may be placed in the following context. Given a control system, a subset of the state space is said to be controlled invariant or viable if for all initial conditions in the set, the trajectories of the system remain inside the set by proper choice of control. Controlled invariance has been developed primarily in two contexts. One context is geometric system theory where the invariant set is the zero level set of a smooth function, the control system is typically affine in the control, and there are no constraints on the control values [19, 12]. The second more general context is that of viability theory [2]. Here the invariant set need not be a manifold, the system is described by differential inclusions, and the control typically takes values in a convex set. A comparison of the two contexts can be found in [3].

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†Supported by Natural Sciences and Engineering Research Council of Canada (NSERC)

Received 18 March 2005
Revised

In the present paper, guided by the desire to characterize a least restrictive collision avoidance controller, we consider a control affine system and an invariant set which is a smooth manifold with boundary. The control takes values in a convex set. We propose conditions under which the viability controller is a bang controller; that is, it takes only a single constant control value. A characterization of the viability kernel is given. Recent, relevant work both on theory and numerical approaches to finding viability kernels are [5, 4, 9, 10, 13, 15, 16]. However, the specific class of viability problem treated here has not yet been investigated. In the second half of the paper the theory is applied to the problem of collision avoidance control for two unicycles. The presented theory is also applicable to general nonholonomic systems, with the main increase in complexity compared to unicycles arising in the computations of the minimum distance between two nonholonomic systems as a function of the control value.

Collision avoidance has been studied by many researchers and there are numerous approaches available. See, for instance, [7, 14, 11, 17, 18] for several recent approaches. Ikeda and Kay [11] study collision avoidance of two aircraft maneuvering in 3-space. The problem is formulated as an optimal control problem over a finite horizon where the terminal time occurs when the aircraft are at a minimum separation. The optimal control is a function of the terminal time, which is, in turn, a function of the control. Thus, the optimal control is computed by way of fixed point iterations on a set of implicit equations. In contrast, we obtain an explicit, easily implemented formula for the control as a function of the initial condition.

An alternative theoretical approach to viability theory is presented by Melikyan, Hovakimyan, and Ikeda [14] based on dynamic programming. The optimal control is obtained by solving for the viscosity solution of a Hamilton-Jacobi-Bellman equation. It is proposed to use the method of singular characteristics to solve for the solution of the HJB equation. In contrast we compute explicitly a value function, denoted s^*, only for the critical level set $s^* = c$, essentially the only level set required for collision avoidance. By taking this viability theory approach, the problem is significantly simplified. The theoretical connections between the dynamic programming formulation of the problem and the viability theory formulation is an interesting area of further investigation.

The paper is organized as follows. Section 2 gives the motivating problem of least restrictive collision avoidance control. The theory to solve the problem is developed in Sections 3 and 4. In particular, Section 3 presents the viability problem and preliminary assumptions, Section 4 gives a characterization of the viability controller and viability kernel, and Section 4.1 discusses properties of the viability kernel. The main theoretical results are presented in Section 4.2. We return to the collision avoidance problem in Section 5. The Appendix contains proofs of supporting lemmas.

2. MOTIVATING PROBLEM

Suppose we have two vehicles modelled as unicycles. The vehicles are assumed to travel with unit speed and they each have a minimum turning radius of one. For each vehicle $i = 1, 2$ the kinematic model is

$$\begin{aligned} \dot{x}_i &= \cos\theta_i \\ \dot{y}_i &= \sin\theta_i \\ \dot{\theta}_i &= u_i \,, \end{aligned}$$

where $(x_i, y_i) \in \mathbb{R}^2$ is the position in the plane, $\theta_i \in \mathbb{R}$ is the vehicle's orientation, and the control input $u_i \in \mathbb{R}$ is the angular velocity. The turning radius requirement dictates the control must satisfy $|u_i| \leq 1$. We say that the two vehicles *collide* at time t if the distance between them at t is strictly less than a prespecified positive number c. We define the domain $\mathcal{S}_c$ to be the region of the state space where there is no collision. That is,

$$\mathcal{S}_c = \{(x_1, y_1, \theta_1, x_2, y_2, \theta_2) \mid \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \geq c\}\,.$$

It is assumed that the two vehicles are autonomous, unwilling to form long term plans with each other, but in the face of imminent collision, they execute controllers which harmoniously achieve collision avoidance. We consider the following problem.

Problem 1. *Given two vehicles modelled as unicycles, find a controller u_v with the following property: if starting from an initial condition and using u_v the two vehicles collide, then using any other measurable control input the vehicles also collide.*

In the next two sections we develop a theoretical framework to address this problem. In Section 5 we return to solving the motivating problem.

3. THEORETICAL PROBLEM FORMULATION

Consider a system

$$\dot{x} = f(x) + g(x)u\,, \tag{1}$$

where $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^{n\times m}$ are Lipschitz and the input space is a compact, convex polyhedron $U \subset \mathbb{R}^m$. A control $u : [0, \infty) \to U$ is a measurable function in t which takes values in U. The set of q vertices of U is denoted as

$$V = \{v^1, \ldots, v^q\}\,.$$

Let $\phi(t, x_0)$ be the unique solution of (1) starting at x_0 and using control u. Also, let $s : \mathbb{R}^n \to \mathbb{R}$ be a smooth submersion, i.e. the gradient ∇s is non-vanishing everywhere in $\mathbb{R}^n$.† Suppose we are given $c \in \mathbb{R}$. The domain to be rendered invariant is

$$\mathcal{S}_c = \{x \in \mathbb{R}^n \mid s(x) \geq c\}\,.$$

We make the following assumption on s.

Assumption 1. *The function s has the property that for all $x \in \mathbb{R}^n$, $\dot{s}(x)$, the Lie derivative of s along solutions of (1), is not a function of u. That is, $L_g s(x) = 0$.*

This relative degree-like assumption implies that the Lie derivative of s is $\dot{s} = L_f s$ and it allows us to define the set of states where s is decreasing, namely,

$$\mathcal{W} = \{\ x \in \mathbb{R}^n \mid L_f s(x) < 0\ \}\,.$$

†The condition may be relaxed to say that on a relevant subset of $\mathbb{R}^n$ every point is a regular point of s.

Definition 1 (Aubin, p. 121 [2]) *A subset $\mathcal{S}_c$ is said to be a* viability domain *if for each $x_0 \in \mathcal{S}_c$, there exists a control $u(t)$ such that the solution of (1) starting at x_0 with control u stays in $\mathcal{S}_c$ for all $t \geq 0$. If $\mathcal{S}_c$ is not a viability domain, then there exists a largest closed (possibly empty) viability domain $\mathcal{S}_c^*$ contained in $\mathcal{S}_c$, which is called the* viability kernel *of $\mathcal{S}_c$. A control u_v which renders $\mathcal{S}_c^*$ viable is called a* viability controller.

Our viability problem can be stated as follows.

Problem 2. *Given a control affine system (1) and the set $\mathcal{S}_c$ which is a manifold with boundary, find u_v, a viability controller, and $\mathcal{S}_c^*$, the viability kernel.*

We place a restriction on the type of viability controller that we consider. It is that the viability controller achieves viability in a finite time, rather than asymptotically. This is stated more precisely as follows.

Assumption 2. *For each $x_0 \in \mathcal{S}_c^* \cap \mathcal{W}$ and using u_v, there exists $\bar{t} < \infty$ such that $s(\phi(\bar{t}, x_0)) \geq c$ and $\dot{s}(\phi(\bar{t}, x_0)) \geq 0$.*

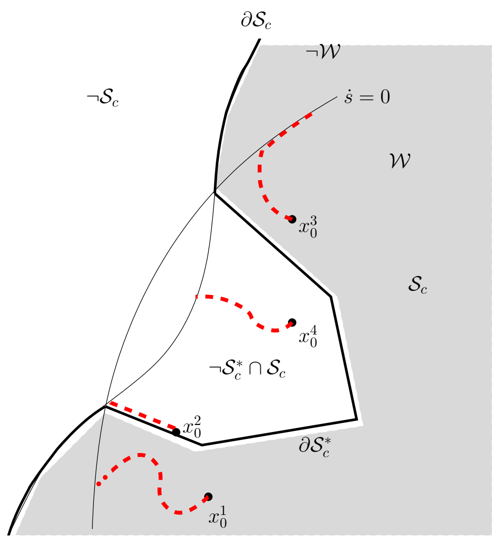

Figure 1. The sets $\mathcal{S}_c$, $\mathcal{W}$, and $\mathcal{S}_c^*$. The set $\mathcal{S}_c^*$ is shaded and has a bold boundary.

Let $\neg\mathcal{W}$ denote $\mathbb{R}^n \setminus \mathcal{W}$. Essentially the assumption says that starting in the set $\mathcal{S}_c^* \cap \mathcal{W}$, the viability controller u_v drives the system to $\mathcal{S}_c \cap \neg\mathcal{W}$ in finite time. Consider Figure 1. The set $\mathcal{S}_c^*$ is shaded and with a bold boundary, while the other curves depict the boundaries of $\mathcal{W}$ and $\mathcal{S}_c$. Consider the following three cases: the trajectory starting at x_0^1 approaches an equilibrium on the boundary of $\mathcal{W}$ while remaining in $\mathcal{S}_c^*$; the trajectory starting at x_0^2 moves along the boundary of $\mathcal{S}_c^*$ until the boundary of $\mathcal{W}$ is reached in finite time; and the trajectory starting at x_0^3 reaches the boundary of $\mathcal{W}$ in infinite time. The first and third cases is ruled out by Assumption 2. A viability controlller satisfying Assumption 2 is said to be a *finite time viability* *controller*. Finite time viability controllers are desirable from an applications viewpoint. Thus, in this paper we only consider finite time viability controllers.

4. VIABILITY CONTROLLER

Consider $x_0 \in \mathbb{R}^n$ and for each $i = 1, \ldots, q$, define $\phi_i(t, x_0)$ to be the solution of the autonomous system

$$\dot{x} = f(x) + g(x)v^i, \qquad v^i \in V \tag{2}$$

starting from $x(0) = x_0$ and evaluated at time t. We define the following variables:

$$s_i(t, x_0) := s(\phi_i(t, x_0)), \qquad i = 1, \ldots, q\,.$$

For $x_0 \in \mathcal{W}$, let $\overline{t}_i(x_0)$ be the first time when $\phi_i(t, x_0)$ reaches the boundary of $\mathcal{W}$; that is when $\frac{ds_i}{dt}(t, x_0) = 0$. For $x_0 \in \neg\mathcal{W}$, set $\overline{t}_i(x_0) = 0$. (We will write $\overline{t}_i(x_0)$ as $\overline{t}_i$ where the dependence on x_0 is clear.) Since initially for $x_0 \in \mathcal{W}$, $\frac{ds_i}{dt}(t, x_0)\big|_{t=0} < 0$, $\overline{t}_i$ is the time when $s_i(\cdot, x_0)$ (and therefore s) reaches a local minimum along the trajectory starting at x_0 when restricted to the time interval $[0, \overline{t}_i]$. For $x_0 \in \mathbb{R}^n$, we define $\overline{s}_i(x_0)$ to be the value of s_i at $\overline{t}_i$, i.e.,

$$\overline{s}_i(x_0) := s_i(\overline{t}_i, x_0)\,. \tag{3}$$

We observe that by definition $\overline{s}_i$ is constant when evaluated along the trajectory $\phi_i(t, x_0)$ over the interval $[0, \overline{t}_i]$.

We require the following.

Assumption 3. *Each $\overline{t}_i(\cdot)$ is a continuous function of the initial condition $x_0 \in \mathbb{R}^n$, and $\overline{t}_i(x_0) < \infty$ for all $x_0 \in \mathbb{R}^n$ and $i = 1, \ldots, q$. Moreover, $\overline{s}_i(\cdot)$ is a continuously differentiable function on $\mathcal{W}$.*

Continuity of $\overline{t}_i$ can be guaranteed by imposing transversality of the flow $\phi_i(t, x_0)$ with $\partial\mathcal{W}$. See [6] for similar arguments in the context of proving continuity of a minimum time function over a finite horizon. Once we have that $\overline{t}_i$ is continuous and using Lipschitz continuity of the vector fields, it is a standard argument to show that $\overline{s}_i$ is continuous. The differentiability assumption is introduced to be able to compute gradients of $\overline{s}_i$ in $\mathcal{W}$, and, in general, is too restrictive; however, it can be removed using tools of non-smooth analysis [2, 8]. We retain the assumption since it holds in our main application.

For each $v^i \in V$ and each $x_0 \in \mathcal{W}$, there is a finite time $\overline{t}_i$ when the trajectory reaches the boundary of $\mathcal{W}$. The first step of our design is to specify a control which acts in the region $\mathcal{S}_c \cap \neg\mathcal{W}$.

Assumption 4. *There exists a controller $u_p : [0, \infty) \to U$ such that if $x_0 \in \mathcal{S}_c \cap \neg\mathcal{W}$, then using u_p, $\dot{s}(\phi(t, x_0)) > 0$ for all $t > 0$.*

Remark 1. *Several remarks about u_p are in order. First, a viability controller need only act on the boundary of its viability kernel. In $\neg\mathcal{W}$, we will see the viability kernel is simply $\mathcal{S}_c$, so u_p is only used in $\partial\mathcal{S}_c \cap \neg\mathcal{W}$. However, the system naturally remains viable in $\mathcal{S}_c \cap \neg\mathcal{W}$, since $L_f s(x) > 0$ along $\partial\mathcal{S}_c$ in $\neg\mathcal{W}$. Hence, any control will, in fact, do in this region. The control u_p is selected primarily to be able to conveniently refer to a single controller in $\mathcal{S}_c \cap \neg\mathcal{W}$ in the later theoretical development, and therefore presents no loss of generality. In practice, u_p*

is only absolutely required at points in $\partial\mathcal{S}_c \cap \partial\mathcal{W}$. Intuitively, we can understand this idea in terms of our two unicycle system. At the termination of collision avoidance, the unicycles are at a minimum separation from each other. The control u_p takes over and forces the unicycles to head away from each other, thus guaranteeing no further possible collision. In this application, u_p is easy to design and it simplies our discussion of what happens after collision avoidance.

Next we turn to the more challenging task of finding a viability controller for the region $\mathcal{S}_c \cap \mathcal{W}$. We propose a bang controller (a controller that uses only one constant control value for each initial condition), denoted u^*, that we claim is the viability controller in the region $\mathcal{S}_c \cap \mathcal{W}$. The overall viability controller is then u_v, equal to u_p or u^*, depending on the initial condition. Associated with u_v is a viability kernel $\mathcal{S}_c^*$ of $\mathcal{S}_c$. Using u_v, if the state is initialized in $\mathcal{S}_c^*$ then it remains in $\mathcal{S}_c^* \subseteq \mathcal{S}_c$ for all time. The controller u_v is active only on the boundary of $\mathcal{S}_c^*$. In the interior of $\mathcal{S}_c^*$ other controllers may be used. We say that the controller u_v is *least restrictive* in the sense that if viability is violated starting at some initial condition using u_v, then it is violated with any other measurable control.

We give a characterization of u^*. For $x \in \mathcal{S}_c \cap \mathcal{W}$, define the set of indices

$$I^*(x) = \text{argmax}_{i\in\{1,\ldots,q\}}\{\ \overline{s}_i(x)\ \}\,. \tag{4}$$

Notice that the cardinality of this set may vary with x. Define the function $\mu^* : \mathcal{S}_c \cap \mathcal{W} \to V$ by

$$\mu^*(x) := v^j\,, \qquad\qquad j \in I^*(x)\,. \tag{5}$$

Finally, for each initial condition $x_0 \in \mathcal{S}_c \cap \mathcal{W}$ we define

$$u^*(t, x_0) := \mu^*(x_0)\,, \qquad\qquad t \in [0, \overline{t}(x_0)]\,, \tag{6}$$

where $\overline{t}(x_0) := \overline{t}_j(x_0)$ if $\mu^*(x_0) = v^j$. This controller will henceforth be called the "bang controller". Intuitively, this choice of controller maximizes the first local minimum value of s on an interval $[0, \overline{t}]$, by using only a single control value in V. The controller u^* terminates at the time $\overline{t}$ when, by construction, $\dot{s} = 0$; that is, u^* terminates and u_p is initiated when the trajectory exits the set $\mathcal{W}$. (The controller u_p guarantees that the local minimum of s on the interval $[0, \overline{t}]$ is in fact a global mimimum on the interval $[0, \infty)$.)

Remark 2. *Observe that μ^* is in feedback form: at each $x \in \mathcal{S}_c \cap \mathcal{W}$, the set $I^*(x)$ must be evaluated and a control value in V selected. However, u^* is an open loop control. Its value and its duration $\overline{t}$ are computed at $t = 0$ based on the initial condition only. We will see in Lemma 2 that, under a suitable condition on the vector fields* (2)*, u^* is identically equal to μ^* at each point, so that u^* is effectively also a feedback.*

Next we introduce the viability kernel. First, we define

$$s^*(x) = \begin{cases} \max_{i\in\{1,\ldots,q\}}\{\ \overline{s}_i(x)\ \} & x \in \mathcal{W} \\ s(x) & x \in \neg\mathcal{W}\,. \end{cases}$$

It is a straightforward exercise to show that s^* is a continuous function. Define the set

$$\mathcal{D}_c^* = \{x \in \mathbb{R}^n \mid s^*(x) < c\}\,. \tag{7}$$

We claim the viability kernel is

$$\mathcal{S}_c^* := \neg D_c^*\,. \tag{8}$$

It is evident from this definition and the continuity of s^* that $\mathcal{S}_c^*$ is closed. We can further interpret $\mathcal{S}_c^*$ as follows:

$$\mathcal{S}_c^* = (S_c \cap \neg\mathcal{W}) \cup (\neg\mathcal{D}_c^* \cap \mathcal{W}) .$$

It is obviously true for $x \in \mathcal{W}$ that $\mathcal{S}_c^* \cap \mathcal{W} = \neg\mathcal{D}_c^* \cap \mathcal{W}$. For $x \in \neg\mathcal{W}$, we know $s^*(x) = s(x)$, so $\mathcal{S}_c^* \cap \neg\mathcal{W} = \mathcal{S}_c \cap \neg\mathcal{W}$. Thus, the interpretation of $\mathcal{S}_c^*$ is as follows. In the region $\neg\mathcal{W}$ where $L_f s \geq 0$, the viability kernel is simply $\mathcal{S}_c$. In particular, on the boundary of $\mathcal{S}_c$, the control u_p may be used to ensure viability, as already discussed. In the region $\mathcal{W}$ where $L_f s < 0$, we claim the viability kernel is $\neg\mathcal{D}_c^*$, and on the boundary of $\mathcal{D}_c^*$ the bang control u^* is used. To summarize, $u_v : \mathbb{R}^n \to U$ consists of two parts corresponding to the two regions of $\mathcal{S}_c^*$, and it acts only on the boundary of $\mathcal{S}_c^*$. Precisely,

$$u_v(t, x_0) = \begin{cases} u^*(t, x_0) & x_0 \in \partial D_c^* \cap \mathcal{W} \\ u_p(t) & x_0 \in \partial S_c \cap \neg\mathcal{W} . \end{cases}$$

This is illustrated for the initial condition x_0^2 in Figure 1.

4.1. Properties of $\mathcal{S}_c^*$

We examine some properties of $\mathcal{D}_c^*$ and $\mathcal{S}_c^*$. First, we know $\partial\mathcal{D}_c^* = \{x \mid s^*(x) = c\}$. Let us define the sets $\mathcal{D}_c^i = \{x \mid \overline{s}_i(x) < c\}$, $i = 1, \ldots, q$. The set $\mathcal{D}_c^*$ can be written in terms of the $\mathcal{D}_c^i$'s as $\mathcal{D}_c^* = \bigcap_{i\in\{1,\ldots,q\}} \mathcal{D}_c^i$. This is useful to understand the differentiability properties of the boundary of $\mathcal{D}_c^*$. The boundary $\partial\mathcal{D}_c^*$ of $\mathcal{D}_c^*$ is obtained as the boundary of the intersection of the sets $\mathcal{D}_c^i$. Therefore, it will not in general be differentiable. Non-differentiable points in $\partial\mathcal{D}_c^*$ can occur in the intersection of boundaries of some of the $\mathcal{D}_c^i$'s.

Set $\mathcal{S}_c^*$ also has a property observed in other examples of viability kernel calculations. The boundary of $\mathcal{S}_c^*$ contains arcs of trajectories of the system (1) using bang controls in V. First, we know that $\overline{s}_i$ is constant when evaluated along a trajectory $\phi_i(t, x_0)$ over the interval $t \in [0, \overline{t}_i]$. Now suppose at $x_0 \in \mathcal{W} \cap \partial\mathcal{S}_c^*$, $\mu(x_0) = v^j$. Then so long as $j \in I^*(\phi_j(t, x_0))$ for all $t \in [0, \overline{t}_j)$, $s^*(\phi_j(t, x_0)) = c$. In other words, under these conditions, the trajectory $\phi_j(t, x_0)$ lies in the boundary of $\mathcal{S}_c^*$. Showing that for all $t \in [0, \overline{t})$, $j \in I^*(\phi_j(t, x_0))$, i.e. the trajectory moves along the boundary of $\mathcal{S}_c^*$ for the duration of application of u^*, will be proved, under suitable conditions, in Lemma 2. At this stage, however, we can already say that a trajectory starting at $x_0 \in \mathcal{S}_c^* \cap \mathcal{W}$ and using the constant control $\mu^*(x_0)$ over the interval $[0, \overline{t}]$ cannot exit $\mathcal{S}_c^*$.

Lemma 1. *Let $x_0 \in \mathcal{S}_c^* \cap \mathcal{W}$ and $i \in I^*(x_0)$. Then $\phi_i(t, x_0) \in \mathcal{S}_c^*$ for all $t \in [0, \overline{t}_i]$.*

Proof. By assumption $\overline{s}_i(x_0) \geq c$ and since $\overline{s}_i$ is constant along $\phi_i(t, x_0)$, we have $\overline{s}_i(\phi_i(t, x_0)) \geq c$ for all $t \in [0, \overline{t}_i]$. By definition of $\overline{t}_i$, $\phi_i(t, x_0) \in \mathcal{W}$ for all $t \in [0, \overline{t}_i)$. Thus, for all $t \in [0, \overline{t}_i]$, $s^*(\phi_i(t, x_0)) = \max_{j\in\{1,\ldots,q\}}\{\overline{s}_j(\phi_i(t, x_0))\} \geq c$, which implies $\phi_i(t, x_0) \in \mathcal{S}_c^*$ for all $t \in [0, \overline{t}_i]$. □

4.2. Main Results

In this section we prove our main theoretical results. We say that a control $u(t)$ is *bang-bang* if it is piecewise constant and it takes values in V, for all $t \geq 0$. Let a *k-switch controller* be a bang-bang control that allows k switches in its value. In particular, u^* is a 0-switch controller. Some remarks are in order.

Remark 3.

1. *The control u_p is trivially least restrictive for initial conditions in the region $\neg\mathcal{W}$ since u_p maintains the state in $\mathcal{S}_c$ for the largest possible set of initial conditions, namely $\mathcal{S}_c \cap \neg\mathcal{W}$. For this reason, the theoretical development that follows focuses only on the question of whether u^* is least restrictive for initial conditions in $\mathcal{W}$. Once this is shown, then it is immediate that u_v is a viability controller associated with $\mathcal{S}_c^*$ and $\mathcal{S}_c^*$ is the viability kernel.*
2. *The control u^* is trivially least restrictive with respect to 0-switch controls in the region $\mathcal{W}$. This is because $\mu^*(x_0)$ selects at each point $x_0 \in \mathcal{W}$ a control $v^j \in V$ that maximizes the first local minimum value of $s(\phi_j(t, x_0))$. Any other 0-switch control will achieve an equal or worse local minimum value of s. Thus, to show u^* is least restrictive with respect to measurable controls, we begin with 1-switch controls.*

In Lemma 3 we give a condition under which u^* is least restrictive with respect to 1-switch controls. The main idea is the following. We consider the set $\mathcal{D}_c^* \cap \mathcal{S}_c$ which comprises the initial conditions x_0 for which u^* cannot maintain the system in $\mathcal{S}_c$, since $s^*(x_0) < c$, but some other control $u(t)$ may be able to. To maintain viability the control $u(t)$ must be able to steer the system to $\neg\mathcal{W}$ without first entering $\neg\mathcal{S}_c$ (recall we only consider viability controls that reach $\neg\mathcal{W}$ in finite time). By imposing an appropriate invariance condition on the vector fields (2) on $\partial\mathcal{S}_c^* \cap \mathcal{W}$, it is shown that no such 1-switch control $u(t)$ exists. In particular, the invariance condition guarantees that trajectories starting in $\mathcal{D}_c^* \cap \mathcal{S}_c$ cannot exit directly to $\mathcal{S}_c^*$, but instead first reach $\neg\mathcal{S}_c$. This property is illustrated in Figure 1 for initial condition x_0^4; we will show that a trajectory such as the one starting at x_0^4 cannot occur. This property is the crucial step to show least restrictiveness of u^* in $\mathcal{W}$. Lemma 4 uses an induction argument to extend this result to bang-bang controls. Finally, in Theorem 1 we prove u^* is least restrictive in $\mathcal{W}$ with respect to measurable controls.

Before proceeding, we present a preliminary result showing, under the same invariance condition as Lemma 3, s^* is constant using u^*. The proof gives a prelude to the proof technique of the main results.

Lemma 2. *Given $c \in \mathbb{R}$, suppose that for all $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$ and for all $j \in I^*(x)$ and $i \in \{1, \dots, q\} \setminus j$, we have that*

$$\nabla \overline{s}_j(x) \cdot (f(x) + g(x)v^i) < 0 \,. \tag{9}$$

Suppose also that for all $i \in \{1, \dots, q\}$ and $x_0 \in \partial\mathcal{S}_c \cap \mathcal{W}$, $I^*(\phi_i(\cdot, x_0))$ is a piecewise constant set-valued map on the interval $[0, \overline{t}_i]$. Then for all $x_0 \in \partial\mathcal{S}_c^* \cap \mathcal{W}$, if $i \in I^*(x_0)$, then for all $t \in [0, \overline{t}_i)$, $i \in I^*(\phi_i(t, x_0))$.

Proof. Suppose by way of contradiction there exists $x_0 \in \partial\mathcal{S}_c^* \cap \mathcal{W}$, $i \in I^*(x_0)$, and $t^2 \in [0, \overline{t}_i)$ such that $i \notin I^*(\phi_i(t^2, x_0))$. Now we know several facts:

1. $\overline{s}_i(x_0) = \overline{s}_i(\phi_i(t, x_0)) = c$ for $t \in [0, \overline{t}_i]$,
2. if $i \in I^*(x_0)$, then $\overline{s}_i(x_0) = s^*(x_0) = c$,
3. $s^*(\phi_i(t, x_0)) = \max_j \overline{s}_j(\phi_i(t, x_0)) \geq \overline{s}_i(\phi_i(t, x_0)) = c$ for $t \in [0, \overline{t}_i]$.

The condition $i \notin I^*(\phi_i(t^2, x_0))$ implies $s^*(\phi_i(t^2, x_0)) > c$. Because $s^*(\phi_i(t, x_0))$ is a continuous function of t, there exists a last time $t^1 \in [0, t^2)$ when $s^*(\phi_i(t^1, x_0)) = c$, and for all $t \in (t^1, t^2]$, $s^*(\phi_i(t, x_0)) > c$. Since $I^*(\phi_i(t, x_0))$ is piecewise constant, there exists $\delta > 0$ such that for all $t \in (t^1, t^1+\delta)$, $I^*(\phi_i(t,x_0))$ is constant and $i \notin I^*(\phi_i(t,x_0))$. Let $\{t_k\}$, $t_k \in (t^1, t^1+\delta]$ be a decreasing sequence of times such that $t_k \downarrow t^1$. Define a sequence $\{j(t_k)\}$ such that $j(t_k) = j \in I^*(\phi_i(t_k,x_0))$. That is, $\{j(t_k)\}$ is a constant sequence. We claim $j \in I^*(\phi_i(t^1,x_0))$. This follows because $\overline{s}_j(\phi_i(t_k,x_0)) = s^*(\phi_i(t_k,x_0))$ and $s^*(\phi_i(t^1,x_0)) = c$, so by continuity of $\overline{s}_j$, $\overline{s}_j(\phi_i(t^1,x_0)) = c$. Now we have

$$\dot{\overline{s}}_j(\phi_i(t^1,x_0)) = \lim_{k\to\infty} \frac{\left(\overline{s}_j(\phi_i(t_k,x_0)) - \overline{s}_j(\phi_i(t^1,x_0))\right)}{t_k - t^1} = \lim_{k\to\infty} \frac{\left(\overline{s}_j(\phi_i(t_k,x_0)) - c\right)}{t_k - t^1} \geq 0\,.$$

This contradicts the assumption (9) that $\dot{\overline{s}}_j(\phi_i(t^1,x_0)) < 0$. □

Remark 4. *Lemma 2 shows that under condition* (9) s^* *is constant while applying* u^**.*

Lemma 3. *Given* $c \in \mathbb{R}$, *suppose that for all* $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$ *and for all* $i \notin I^*(x)$ *and* $j \in I^*(x)$, *we have that* (9) *holds. Then the bang controller* u^* *for* $\mathcal{S}_c^* \cap \mathcal{W}$ *is least restrictive with respect to* 1*-switch controllers.*

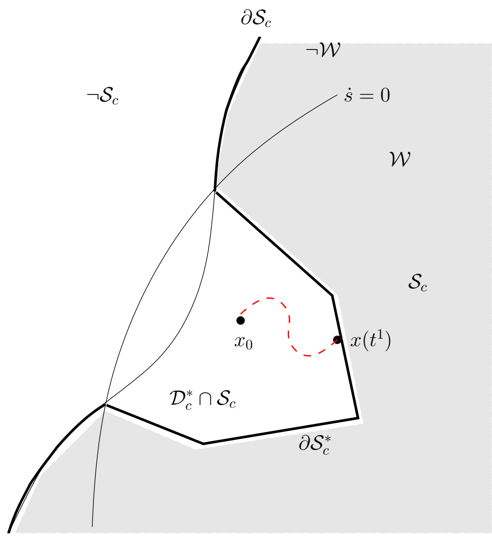

Figure 2. The trajectory starting at $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ reaches $\partial\mathcal{S}_c^* \cap \mathcal{W}$ at the point $x(t^1)$.

Proof. We argue by contradiction. Suppose there exists an initial condition $x_0 \in \mathcal{W}$ and a control $u(t)$ such that viability is violated with $u^*(t,x_0)$ and not with u. Let $x(t)$ be the solution of (1) using control u. Viability is only violated with u^* if $x_0 \in \mathcal{D}_c^*$ but to preserve viability using u it must be that $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. Denote u as u^1u^2, the concatentation of control $u^1 \in V$ followed by $u^2 \in V$. Suppose the control switches value at time $0 < t^2 < \infty$. Since a 1-switch controller becomes a 0-switch controller at the switching time, it must be that $x(t^2) \in \neg(\mathcal{D}_c^* \cap \mathcal{S}_c)$. Hence, there exists $t^1 \leq t^2$, the first time that $x(t^1) \in \partial(\mathcal{D}_c^* \cap \mathcal{S}_c)$. If $x(t^1) \in \mathcal{D}_c^* \cap \partial\mathcal{S}_c$ (where we must have $\dot{s} \geq 0$ for u to be a viable control), then one can apply the control u_p from $x(t^1)$, which means that the 0-switch controller u^1 is a viable control starting from $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$, a contradiction. It must be that $x(t^1) \in \partial\mathcal{D}_c^* \cap \mathcal{S}_c$. See Figure 2. Since $s^*(x(t))$ is a continuous function of t, $t^1 > 0$. We cannot have $i \in I^*(x(t^1))$, where i corresponds to $u^1 = v^i$; otherwise we could continue with u^1, a 0-switch controller, to maintain viability starting from $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. Instead, it must be that $i \notin I^*(x(t^1))$. Let $j \in I^*(x(t^1))$. Let $\{t_k\}$, $t_k \in [0, t^1)$ be an increasing sequence of times such that $t_k \to t^1$. Since $x(t_k) \in \mathcal{D}_c^* \cap \mathcal{S}_c$, $\overline{s}_j(x(t_k)) \leq s^*(x(t_k)) < c$. Then we have

$$\dot{\overline{s}}_j(x(t^1)) = \lim_{k\to\infty} \frac{\left(\overline{s}_j(x(t^1)) - \overline{s}_j(x(t_k))\right)}{t^1 - t_k} = \lim_{k\to\infty} \frac{c - \overline{s}_j(x(t_k))}{t^1 - t_k} \geq 0\,.$$

This contradicts the assumption (9) that $\dot{\overline{s}}_j(x(t^1)) < 0$. As a result there does not exist a 1-switch controller that is less restrictive than u^*. □

Remark 5. *The proof shows that it is only necessary that $\overline{s}_i$ be continuously differentiable at points $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$ where $i \in I^*(x)$; that is, where the level set $\overline{s}_i(x) = c$ forms part of the boundary of $\mathcal{S}_c^*$. This relaxation of the Assumption 3 will be useful for the application to collision avoidance. Note also a slight difference in the use of the condition* (9) *in Lemma 2 and Lemma 3. In Lemma 2, we want to guarantee that trajectories using u^* cannot go up the level sets of s^* starting from the level set $s^*(x) = c$. In Lemma 3, we must guarantee that no control allows trajectories to reach the level set $s^*(x) = c$ from a lower level set. This is obviously true for u^* and must be guaranteed for other control values in V.*

Lemma 4. *Given $c \in \mathbb{R}$, suppose that for all $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$ and for all $i \notin I^*(x)$ and $j \in I^*(x)$, condition* (9) *holds. Then the bang controller $u^*(t, x_0)$ for $\mathcal{S}_c^* \cap \mathcal{W}$ is least restrictive with respect to bang-bang controls.*

Proof. We argue by induction. By Lemma 3, u^* is least restrictive with respect to 1-switch controllers. Now assume it is least restrictive with respect to 1 up to $k-1$ switch controllers. We will show it is least restrictive with respect to 1 to k switch controllers. By way of contradiction, suppose there is a k-switch controller u that is less restrictive than u^*. That is, there exists an initial condition $x_0 \in \mathcal{D}_c^*$ for which the k-switch controller maintains viability. This means that $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. Consider the point x^1 where the $(k-1)$th switch happens. It must be that $x^1 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ since the bang controller is least restrictive with respect to $k-2$ switch controllers. Starting from x^1 we have a 1-switch controller to maintain viability. This contradicts that the bang controller is the least restrictive controller with respect to 1-switch controllers. □

Finally, we must prove that u^* is least restrictive with respect to measurable controls. We require a general result for control affine systems, called the Chattering Lemma, on the reachability of states under measurable controls and bang-bang controls.

Lemma 5 (Chattering Lemma [1]) *Let $x(t)$ be a solution of* (1) *corresponding to some control $u(t)$. There exists a sequence $\{u^k\,,\ k \in \mathbb{N}\}$ of piecewise constant bang-bang controls, such that if $\left\{x^k\right\}$ is the corresponding family of solutions to* (1)*, then for each $T > 0$*

$$\sup_{0\leq t\leq T} \|x^k(t) - x(t)\| \to 0\,, \quad \textit{as } k \to \infty\,. \tag{10}$$

The following is our main result.

Theorem 1. *Suppose that for all* $c \in \mathbb{R}$, $x \in \partial\mathcal{S}_c^* \cap \mathcal{W}$, *and for all* $i \notin I^*(x)$ *and* $j \in I^*(x)$, *condition* (9) *holds. Then the bang controller* u^* *for* $\mathcal{S}_c^* \cap \mathcal{W}$ *is least restrictive with respect to measurable controls. Consequently* u^* *is a viability controller for* $\mathcal{S}_c^* \cap \mathcal{W}$ *and* $\mathcal{S}_c^*$ *is the viability kernel.*

Proof. Fix $c \in \mathbb{R}$. Suppose there exists a measurable control $u(t)$ that is less restrictive than u^*, the bang controller for $\mathcal{S}_c^* \cap \mathcal{W}$. This implies there exists $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ and a time $\bar{t} < \infty$ such that if $x(t)$ is the trajectory starting at x_0 using control $u(t)$, then

1. $s(x(t)) > c$ and $\dot{s}(x(t)) < 0$ for all $t \in [0, \bar{t})$.
2. $s(x(\bar{t})) =: \bar{c} \geq c$ and $\dot{s}(x(\bar{t})) = 0$.
3. After time $\bar{t}$, w.l.o.g. set $u(t) = u_p(t)$, as in Assumption 4.

The first and second statements arise as follows. First, $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$ implies that $x_0 \in \mathcal{W}$. Also, we cannot have $x_0 \in \partial\mathcal{S}_c \cap \mathcal{W}$ for then viability would be immediately violated at x_0. Instead, $x_0 \in int(\mathcal{S}_c) \cap \mathcal{W}$ and $\bar{t}$ is the first time when $x(t) \in \mathcal{S}_c \cap \partial\mathcal{W}$.

Let $T = \bar{t}+1$. We note that $\inf_{t\in[0,\infty)} s(x(t)) = \min_{t\in[0,T]} s(x(t)) = \bar{c}$ by the definition of u_p. Let $\{u^k\}$ be a sequence of bang-bang controls defined on $[0, \infty)$ as in the Chattering Lemma and $\{x^k(t)\}$ the associated trajectories such that (10) holds. By continuity of s we have that $\sup_{0\leq t\leq T} \|s(x^k(t)) - s(x(t))\| \to 0$ as $k \to \infty$. It follows that

$$\min_{t\in[0,T]} s(x^k(t)) \to \bar{c}\,, \quad \text{as } k \to \infty\,, \tag{11}$$

and since $\dot{s}(x(t)) < 0$ for $t \in [0, \bar{t})$ and $\dot{s}(x(t)) > 0$ for $t \in (\bar{t}, \infty)$, one can also show $\mathrm{arginf}_{t\in[0,\infty)} s(x^k(t)) \to \bar{t}$ as $k \to \infty$. Therefore, there exists $\kappa > 0$ such that for all $k > \kappa$, there exists $\bar{t}^k \in [0, T]$ such that

$$\inf_{t\in[0,\infty)} s(x^k(t)) = \min_{t\in[0,T]} s(x^k(t)) = s(x^k(\bar{t}^k))\,. \tag{12}$$

Let $c^* := s^*(x_0)$. We know $c^* < c$ since $x_0 \in \mathcal{D}_c^* \cap \mathcal{S}_c$. By Lemma 4, u^* is less restrictive than any bang-bang control for the domain $\mathcal{S}_{c^*}$. This means for all $k > \kappa$, $\min_{t\in[0,T]} s(x^k(t)) \leq c^*$. From (11), it follows that $\bar{c} \leq c^* < c$, a contradiction. □

5. COLLISION AVOIDANCE

Suppose we have two vehicles $i = 1, 2$ modelled as unicycles. The vehicles are assumed to travel with unit speed and they each have a minimum turning radius of one. The pair $(x_i, y_i) \in \mathbb{R}^2$ is the position in the plane, $\theta_i \in \mathbb{R}$ is the vehicle's orientation, and the control input $u_i \in \mathbb{R}$ is the angular velocity. Also, $V = \{-1, 1\} \times \{-1, 1\}$. We say that two vehicles *collide* at time t if the distance between them at t is strictly less than a prespecified positive number c. Let r be the distance between the vehicles, $\phi \in (-\pi, \pi]$ the heading of the first vehicle that would take it directly towards the second vehicle, and $\theta \in (-\pi, \pi]$ the difference between the two vehicle's headings, taking vehicle 1's heading as the reference. The two unicycle system in relative polar

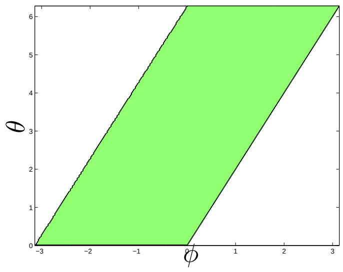

Figure 3. The set $\mathcal{W}$ in the $\phi - \theta$ plane.

coordinates, valid for $r > 0$, is

$$\dot{r} = \cos(\phi - \theta) - \cos(\phi) \tag{13}$$

$$\dot{\phi} = \frac{1}{r}\sin(\theta - \phi) + \frac{1}{r}\sin(\phi) - u_1 \tag{14}$$

$$\dot{\theta} = u_2 - u_1 \,. \tag{15}$$

We use the notation $\xi = (r, \phi, \theta)$ to refer to the state of the system in relative coordinates. In terms of these coordinates the viability domain is

$$\mathcal{S}_c = \{\xi \in \mathbb{R}^3 \mid r \geq c\}\,.$$

It is clear from (13) that Assumption 1 holds, so we can characterize the set $\mathcal{W}$. From Equation (13) we have $\dot{r}(t) = 0$ when $\cos(\phi - \theta) - \cos(\phi) = 0$. The roots are $\theta = 0$ and $\theta = 2\phi$. It follows that

$$\mathcal{W} = \Big\{\xi \mid \phi \in [0, \pi), \theta \in (2\phi, 2\pi)\Big\} \bigcup \Big\{\xi \mid \phi \in [-\pi, 0), \theta \in (0, 2\pi + 2\phi)\Big\}.$$

See Figure 3. Next, we must address Assumption 4. The following is easily proved.

Lemma 6. *If $\xi \in \neg\mathcal{W}$ at $t = 0$, then there exists a controller u_p such that $\dot{r}(t) > 0$ for all $t > 0$.*

Next we compute u^* and s^*. The main step of the computation is computing the $\overline{s}_i$'s. The following convention on subscripts is used. Subscript $i = 1$ when $v^1 := (u_1, u_2) = (1, 1)$; $i = 2$ when $v^2 := (u_1, u_2) = (1, -1)$; $i = 3$ when $v^3 := (u_1, u_2) = (-1, 1)$; and $i = 4$ when $v^4 := (u_1, u_2) = (-1, -1)$. Using this notation, $\overline{s}_i(\xi_0)$ is the value of r at the first time $\overline{t}_i$ when $\dot{r}(\overline{t}_i) = 0$ starting from the initial condition ξ_0 and using the constant control $v^i \in V$.

Each vehicle follows one of two circles depending on its own control input. See Figure 4. A circle is identified with its center. Circle o_1 (o_2) is the circle followed by vehicle 1 when $u_1 = 1$ ($u_1 = -1$). Similarly, o_3 (o_4) is the circle followed by vehicle 2 when $u_2 = -1$ ($u_2 = 1$). Let $D_1(t)$ be the length of the line connecting o_1 and o_4; $D_2(t)$ is the length of the line connecting o_1 and o_3; $D_3(t)$ is the length of the line connecting o_2 and o_4; and $D_4(t)$ is the length of the line connecting o_2 and o_3. Also define D_{i0} to be $D_i(t)$ at time $t = 0$. Let m_i, $i = 1, 2$, be the position of ith vehicle. To obtain the parameters D_i we use the positions of

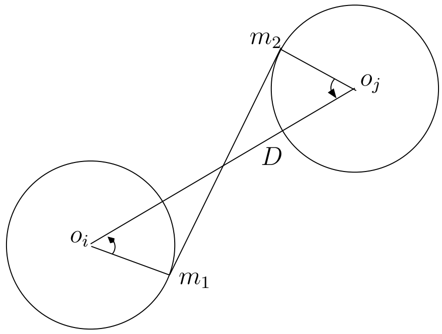

Figure 4. Geometry for computing $\overline{s}_i$.

the centers of the circles: $o_1 = (0, 1)$, $o_2 = (0, -1)$, $o_3 = (r\cos\phi + \sin\theta, r\sin\phi - \cos\theta)$ and $o_4 = (r\cos\phi - \sin\theta, r\sin\phi + \cos\theta)$. It follows immediately that

$$D_1^2 = r^2 - 4r\cos(\phi - \frac{\theta}{2})\sin(\frac{\theta}{2}) + 4\sin^2(\frac{\theta}{2}) \quad (16)$$
$$D_2^2 = r^2 - 4r\sin(\phi - \frac{\theta}{2})\cos(\frac{\theta}{2}) + 4\cos^2(\frac{\theta}{2}) \quad (17)$$
$$D_3^2 = r^2 + 4r\sin(\phi - \frac{\theta}{2})\cos(\frac{\theta}{2}) + 4\cos^2(\frac{\theta}{2}) \quad (18)$$
$$D_4^2 = r^2 + 4r\cos(\phi - \frac{\theta}{2})\sin(\frac{\theta}{2}) + 4\sin^2(\frac{\theta}{2})\,. \quad (19)$$

One can now analyze the geometry of the four circles and determine the minimum distance between the two points m_1 and m_2 moving on their respective circles, as a function of the initial condition and the D_i's. This is an elementary but lengthy computation of geometry, so we only present the final result. Let

$$\sigma^-(\xi_0) := \left|2\cos(\frac{\theta_0}{2}) - r_0\sin(\phi_0 - \frac{\theta_0}{2})\right| \quad (20)$$
$$\sigma^+(\xi_0) := \left|2\cos(\frac{\theta_0}{2}) + r_0\sin(\phi_0 - \frac{\theta_0}{2})\right|\,. \quad (21)$$

Then given $\xi_0 \in \mathcal{W} \cap \mathcal{S}_c$ we have for $i = 1, 4$,

$$\bar{s}_i(\xi_0) = \left|D_{i0} - 2|\sin(\frac{\theta_0}{2})|\right| \quad (22)$$

Also,

$$\bar{s}_2(\xi_0) = \begin{cases} |r_0\cos(\phi_0 - \frac{\theta_0}{2})| & \sigma^-(\xi_0) \leq 2 \\ \sqrt{D_{20}^2 + 4 - 4\sigma^-(\xi_0)} & \sigma^-(\xi_0) > 2\,. \end{cases} \quad (23)$$

and

$$\bar{s}_3(\xi_0) = \begin{cases} |r_0\cos(\phi_0 - \frac{\theta_0}{2})| & \sigma^+(\xi_0) \leq 2 \\ \sqrt{D_{30}^2 + 4 - 4\sigma^+(\xi_0)} & \sigma^+(\xi_0) > 2\,. \end{cases} \quad (24)$$

We must determine the largest among the $\bar{s}_i$'s in order to obtain the viability kernel. First, we obtain a concise expression for $\max\{\bar{s}_1, \bar{s}_4\}$. Second, we show that the first cases of $\bar{s}_2$ in (23) and $\bar{s}_3$ in (24) do not appear in the final expression for s^*. Third, we obtain a concise expression for the maximum of the second two cases of $\bar{s}_2$ and $\bar{s}_3$. With the aid of these three elements, we are able to write s^*.

The next lemma enables a comparision of $\bar{s}_4$ and $\bar{s}_1$.

Lemma 7. *If* $\bar{s}_4(\xi_0) \geq \bar{s}_1(\xi_0)$*, then* $\bar{s}_4(\xi_0) = D_{40} - 2\left|\sin(\frac{\theta_0}{2})\right|$*. If* $\bar{s}_1(\xi_0) \geq \bar{s}_4(\xi_0)$*, then* $\bar{s}_1(\xi_0) = D_{10} - 2\left|\sin(\frac{\theta_0}{2})\right|$*.*

In light of this, we have

$$\begin{aligned} \rho(\xi) &:= \max\{\bar{s}_1(\xi), \bar{s}_4(\xi)\} = \max\{D_1, D_4\} - 2|\sin(\frac{\theta}{2})| \\ &= \sqrt{r^2 + 4r\left|\cos(\phi - \frac{\theta}{2})\sin(\frac{\theta}{2})\right| + 4\sin^2(\frac{\theta}{2})} - 2\left|\sin(\frac{\theta}{2})\right| \end{aligned} \quad (25)$$

Next we require two preliminary lemmas. The first lemma is used to compare the two cases for $\bar{s}_2$ and for $\bar{s}_3$. The second provides a comparison of $\max\{\bar{s}_1, \bar{s}_4\}$ and $\max\{\bar{s}_3, \bar{s}_2\}$.

Lemma 8. *The following inequalities hold:*

$$\left|r\cos(\phi - \frac{\theta}{2})\right| \leq \sqrt{D_3^2 + 4 - 4\left|2\cos(\frac{\theta}{2}) + r\sin(\phi - \frac{\theta}{2})\right|} \quad (26)$$

$$\left|r\cos(\phi - \frac{\theta}{2})\right| \leq \sqrt{D_2^2 + 4 - 4\left|2\cos(\frac{\theta}{2}) - r\sin(\phi - \frac{\theta}{2})\right|}. \quad (27)$$

Let

$$\sigma(\xi) := 2|\cos(\frac{\theta}{2})| + r|\sin(\phi - \frac{\theta}{2})|. \quad (28)$$

Note that $\sigma(\xi) \leq 2$ if and only if $\sigma^+(\xi) \leq 2$ and $\sigma^-(\xi) \leq 2$.

Lemma 9. *If* $\sigma(\xi_0) \leq 2$ *then* $\max\{\bar{s}_1, \bar{s}_4\} \geq \max\{\bar{s}_3, \bar{s}_2\}$*. If, in addition,* $\xi_0 \in \mathcal{W}$*, then the inequality is strict.*

Lemma 9 shows that if $\sigma^+(\xi) \leq 2$ and $\sigma^-(\xi) \leq 2$ then ρ is used to define $\partial\mathcal{D}_c^*$. Lemma 8 shows that if either $\sigma^+(\xi) > 2$ and $\sigma^-(\xi) \leq 2$, or $\sigma^+(\xi) \leq 2$ and $\sigma^-(\xi) > 2$, then one of the second cases of (23) and (24) is used for $\partial\mathcal{D}_c^*$. Consequently the first cases of (23) and (24) do not contribute to forming $\partial\mathcal{D}_c^*$. The only remaining cases to compare are the second two cases of (23) and (24) when both $\sigma^+(\xi) > 2$ and $\sigma^-(\xi) > 2$. Let $\rho'(\xi)$ be the maximum of the second two cases of (23) and (24). Then we have

Lemma 10. $\rho'(\xi) :=$

$$\sqrt{r^2 + 4r\left|\sin(\phi - \frac{\theta}{2})\cos(\frac{\theta}{2})\right| + 4\cos^2(\frac{\theta}{2}) + 4 - 8\left|\cos(\frac{\theta}{2})\right| - 4r\left|\sin(\phi - \frac{\theta}{2})\right|}. \quad (29)$$

With these results, we obtain the final form of $s^*(\xi_0)$ and thus an analytical characterization of the set of initial conditions ξ_0 where $\max\{\overline{s}_i(\xi_0)\} = c$. We have

$$s^*(\xi_0) = \begin{cases} \rho(\xi_0)\,, & \sigma(\xi_0) \leq 2 \\ \max\{\rho(\xi_0), \rho'(\xi_0)\}\,, & \sigma(\xi_0) > 2 \end{cases} \tag{30}$$

Remark 6. *At this stage it is appropriate to address Assumption 3. First, Lemma 7 shows that $\overline{s}_4$ and $\overline{s}_1$ are continuously differentiable functions in $\mathcal{W}$ whereever they form the boundary of $\mathcal{D}_c^*$. This is because $\partial\mathcal{D}_c^*$ is formed by the level set $\overline{s}_4 = c$ only when $\overline{s}_4 \geq \overline{s}_1$ and by the level set $\overline{s}_1 = c$ only when $\overline{s}_1 \geq \overline{s}_4$. Thus, non-differentiability of* (22) *occurs only when* $\sin(\frac{\theta}{2}) = 0$ *or $\theta = 0$, which corresponds to a point ξ outside $\mathcal{W}$. Thus, Assumption 3 holds for $\overline{s}_4$ and $\overline{s}_1$ in the region of interest. Second, the functions $\overline{s}_2$ and $\overline{s}_3$ are continuous by comparing the two cases in* (23) *and* (24)*. In particular, it is easily verified that when $\sigma^-(\xi) = 2$, $|r\cos(\phi - \frac{\theta}{2})| = \sqrt{D_2^2 - 4}$. Also, when $\sigma^+(\xi) = 2$, $|r\cos(\phi - \frac{\theta}{2})| = \sqrt{D_3^2 - 4}$. Finally, we can verify that the second cases of* (23) *and* (24) *are continuously differentiable in $\mathcal{W}$ whereever they form the boundary of $\mathcal{D}_c^*$. Consider the second case of* (23)*. It is differentiable everywhere except when $\sigma^-(\xi) = 2$. However, by Lemma 9, when $\sigma^-(\xi) = 2$ and $\xi \in \mathcal{W}$, then $\sqrt{D_2^2 - 4} < \rho(\xi)$. Similar arguments apply for the second case of* (24)*. Thus, non-differentiable points of $\overline{s}_2$ and $\overline{s}_3$ do not appear in $\partial\mathcal{D}_c^*$. This concludes our verification of Assumption 3.*

Having obtained a characterization of s^*, we turn to the sets $\mathcal{D}_c^*$ and $\mathcal{S}_c^*$. First, from (30) it is evident that surface $\partial\mathcal{D}_c^*$ is formed by the c level sets of the functions ρ and ρ'. The set $\mathcal{D}_c^*$ also has the following property, useful in representing it graphically.

Lemma 11. *Given a pair (ϕ, θ), there is a unique finite value r such that $(r, \phi, \theta) \in \partial\mathcal{D}_c^*$. Moreover, in spherical coordinates $\mathcal{D}_c^*$ is star convex, i.e. for all $\alpha \in [0, 1)$, if $(r, \phi, \theta) \in \partial\mathcal{D}_c^*$, then $(\alpha r, \phi, \theta) \in \mathcal{D}_c^*$.*

This fact justifies a visualization of $\mathcal{S}_c^*$ by projecting its boundary colorcoded with the appropriate u^* value to the set $\mathcal{W}$ in the $\phi - \theta$ plane (refer to Figure 3 for a visualization of $\mathcal{W}$). Figure 5(a), (b), (c), and (d) show u^* on the boundary of $\mathcal{S}_c^*$ for $c = .1, 1, 5, 100$ after projecting to $\mathcal{W}$ in the $\phi - \theta$ plane. There are four regions corresponding to the four choices of control. The boundary curves are the points where certain of the $\overline{s}_i$'s are equal.

5.1. Verification of Condition (9)

The final step of the design is to verify that condition (9) is satisfied. We have already verified that gradients of $\overline{s}_i$ are defined on $\partial\mathcal{D}_c^*$. Here we compute the gradient vectors $\nabla\rho$ and $\nabla\rho'$ of the level surfaces $\rho(\xi_0) = c$ and $\rho'(\xi_0) = c$, and take the dot product of these gradient vectors with the appropriate vector fields. These expressions are compute symbolically using Maple. The notation $\nabla\rho$ and $\nabla\rho'$ is used as a shorthand for the gradients of the $\overline{s}_i$'s and not to connote these functions are themselves differentiable.

Let f_i denote the vector field (13)-(15) with control v^i. We obtain that

$$\nabla\rho \cdot f_1 = \begin{cases} \frac{2r\dot{r}}{c + 2|\sin(\frac{\theta}{2})|} & \overline{s}_4 > \overline{s}_1 \\ 0 & \overline{s}_4 < \overline{s}_1 \end{cases} \tag{31}$$

$$\nabla\rho \cdot f_4 = \begin{cases} \frac{2r\dot{r}}{c + 2|\sin(\frac{\theta}{2})|} & \overline{s}_4 < \overline{s}_1 \\ 0 & \overline{s}_4 > \overline{s}_1 \end{cases} \tag{32}$$

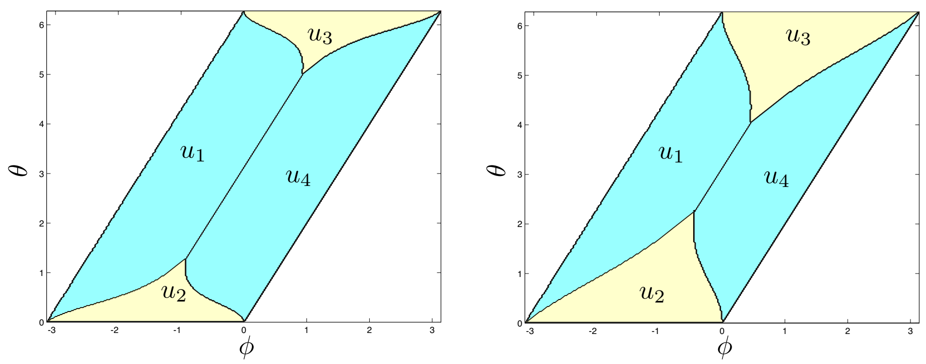

Figure 5. The set $\mathcal{S}_c^* \cap \mathcal{W}$ projected to the $\phi - \theta$ plane when $c = 0.1$ and $c = 1$.

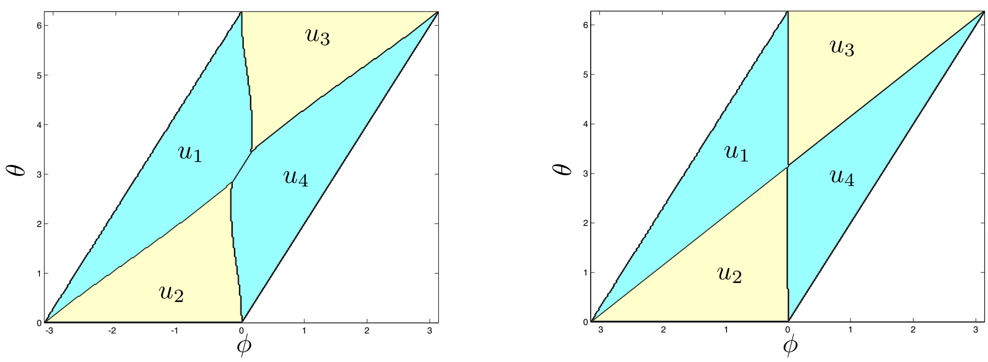

Figure 6. The set $\mathcal{S}_c^* \cap \mathcal{W}$ projected to the $\phi - \theta$ plane when $c = 5$ and $c = 100$.

Using the fact that $\dot{r} < 0$, we see that $\nabla\rho \cdot f_1 < 0$ if $\overline{s}_1 < \overline{s}_4$ and $\nabla\rho \cdot f_4 < 0$ if $\overline{s}_4 < \overline{s}_1$. This dot product is only used when $\rho(\xi) > \rho'(\xi)$ and in that case, if $\overline{s}_1 > \overline{s}_4$ then $u^* = v^1$ and $\nabla\rho \cdot f_1 = 0$, as one would expect of the viability controller. Similarly, if $\rho(\xi) > \rho'(\xi)$ and $\overline{s}_4 > \overline{s}_1$ then $u^* = u_4$ and $\nabla\rho \cdot f_4 = 0$. Hence, these results verify condition (9). Note we do not need to verify (9) when $\overline{s}_4 = \overline{s}_1$ since then $1, 4 \in I^*$.

We also have

$$\nabla\rho' \cdot f_2 = \begin{cases} 2\frac{\dot{r}}{|\sin(\phi-\frac{\theta}{2})|}(r|\sin(\phi-\frac{\theta}{2})|+2|\cos(\frac{\theta}{2})|-2) & \overline{s}_3 > \overline{s}_2 \\ 0 & \overline{s}_3 < \overline{s}_2 \end{cases}$$

$$\nabla\rho' \cdot f_3 = \begin{cases} 2\frac{\dot{r}}{|\sin(\phi-\frac{\theta}{2})|}(r|\sin(\phi-\frac{\theta}{2})|+2|\cos(\frac{\theta}{2})|-2) & \overline{s}_3 < \overline{s}_2 \\ 0 & \overline{s}_3 > \overline{s}_2 \end{cases}$$

These formulas only apply when $\sigma(\xi) > 2$ since that is the only case when ρ' is used. Also, since $\dot{r} < 0$ we cannot have $\sin(\phi - \frac{\theta}{2}) = 0$. The first and third formulas are thus well defined and satisfy (9), by using $\sigma(\xi) > 2$. Finally, the zero dot products appear as discussed above, and also coincide with condition (9).

Finally, we are able to obtain some further information about the remaining dot products. After some algebra,

$$\nabla\rho \cdot (f_3 + f_2) = \frac{2r\dot{r}}{c + 2|\sin(\frac{\theta}{2})|}$$

$$\nabla\rho' \cdot (f_1 + f_4) = \frac{2\dot{r}(r|\sin(\phi-\frac{\theta}{2})|+2|\cos(\frac{\theta}{2})|-2)}{c}.$$

Both expressions are always negative as we have $\dot{r} < 0$. Therefore, the first equation implies either $\nabla\rho \cdot f_3$ is negative or $\nabla\rho \cdot f_2$ is negative. Similarly, the second equation implies either $\nabla\rho' \cdot f_1$ is negative or $\nabla\rho' \cdot f_4$ is negative.

So far we have shown that at any point $\xi \in \partial\mathcal{D}_c^* \cap \mathcal{W} \cap \mathcal{S}_c$, at least three out of the four vector fields satisfy condition (9). The remaining cases are complicated to verify analytically. Therefore, we have verified those cases numerically using Matlab for a range of values of c from 0.01 to 10,000. It was found that the remaining dot products satisfy condition (9).

6. CONCLUSION

We have presented theory for explicit construction of viability kernels and viability controllers for control affine systems when the invariant domain is given as a smooth manifold with boundary. The results are shown to apply to the problem of least restrictive collision avoidance control of two vehicles. Several fruitful research directions are apparent: foremost is the generalization of these results to cases where the set $\mathcal{D}_c^*$ is less regular, by introducing tools of non-smooth analysis. Also, the results may be generalized to the case when the viability domain is defined as the conjunction of k submersions, i.e. $s : \mathbb{R}^n \to \mathbb{R}^k$. This is the main extension needed to address collision avoidance for more than two vehicles.

Acknowledgements: The authors would like to thank Ari Arapostathis for providing the presented version of the Chattering Lemma and its proof, Manfredi Maggiore for helpful comments which improved the clarity of the presentation, and Jacek Nawrot for many helpful discussions and for generating Figures 5 and 6.

REFERENCES

1. L.D. Berkovitz. *Optimal Control Theory.* Springer-Verlag, 1974.
2. J.P. Aubin. *Viability Theory.* Birkhauser, 1991.
3. J.P. Aubin. A survey of viability theory. *SIAM J. Control and Optimization*, vol, 28, no. 4, pp 749-788, 1990.
4. J.P. Aubin, J. Lygeros, M. Quincampoix, and S. Sastry. Impulse differential inclusions: a viability approach to hybrid systems. *IEEE Trans. Automatic Control*, vol. 47, no. 1, pp. 2-20, January 2002.
5. J.P. Aubin. Viability kernels and capture basins of sets under differential inclusions. *SIAM J. Control and Optim.*, vol. 40, no. 3, pp. 853-881, 2001.
6. M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Birkhäuser, Boston, 1997.
7. D. Chang, S. Shadden, J. Marsden, and R. Olfati-Saber. Collision avoidance for multiple agent systems. *IEEE Conference on Decision and Control* (CDC '03), Maui Hawaii, December 2003.
8. F. Clarke, Y. Ledyaev, R. Stern, and P. Wolenski. *Nonsmooth Analysis and Control Theory.* Springer-Verlag, New-York, 1998.
9. H. Frankowska and M. Quincampoix. Viability kernels of differential inclusions with constraints: algorithm and applications. *Journal of Mathematical Systems, Estimation, and Control*, 1(3), pp. 371-388, 1991.
10. M. Heymann, F. Lin, and G. Meyer. Synthesis and viability of minimally interventive legal controllers for hybrid systems. *Discrete Event Dynamic Systems: Theory and Applications*, No 8, pp. 105-135, 1998.
11. Y. Ikeda and J. Kay. An optimal control problem for automatic air collision avoidance. *IEEE Conference on Decision and Control* (CDC '03), Maui Hawaii, December 2003.
12. A. Isidori. *Nonlinear Control Systems.* 3rd Ed. Springer-Verlag, 1995.
13. G. Labinaz and M. Guay. Viability, the solution set and fixed point approximation of hybrid systems. *IEEE Conference on Decision and Control* (CDC '03), Maui Hawaii, December 2003.
14. A. Melikyan, N. Hovakimyan, and Y. Ikeda. Dynamic programming approach to a minimum distance optimal control problem. *IEEE Conference on Decision and Control* (CDC '03), Maui Hawaii, December 2003.
15. M. Quincampoix and P. Saint-Pierre. An algorithm for viability kernels in the Holderian case: approximation by discrete viability kernels. *J. Math. Syst. Estim. Control*, 5, pp. 115-118, 1995.
16. P. Saint-Pierre. Approximation of the viability kernel. *Appl. Math. Optim.*, 29, pp. 187-209, 1994.
17. K. Sigurd and J. How. UAV trajectory design using total field collision avoidance. *AIAA Guidance, Navigation and Control Conference*, August 2003.
18. C. Tomlin, G. Pappas, and S. Sastry. Conflict resolution for air-traffic management: a study in multiagent hybrid systems. *IEEE Trans. Automatic Control*, 43(4):509-521, April 1998.
19. W.M. Wonham. *Linear Multivariable Control, 3rd ed.* Springer-Verlag, 1985.

APPENDIX

Proof. [Proof of Lemma 7] Suppose $\overline{s}_4(\xi_0) \geq \overline{s}_1(\xi_0)$, and assume, by way of contradiction that $D_{40} < 2|\sin(\frac{\theta_0}{2})|$. There are two cases.

1. Suppose $D_{10} \geq 2|\sin(\frac{\theta_0}{2})|$. In this case we have

$$\overline{s}_1 = D_{10} - 2|\sin(\frac{\theta_0}{2})|, \quad \overline{s}_4 = 2|\sin(\frac{\theta_0}{2})| - D_{40} .$$

Squaring and summing these two equations we obtain

$$D_{10}^2 + D_{40}^2 = \bar{s}_1^2 + \bar{s}_4^2 + 8\sin^2(\frac{\theta_0}{2}) + 4\bar{s}_1|\sin(\frac{\theta_0}{2})| - 4\bar{s}_4|\sin(\frac{\theta_0}{2})| . \tag{33}$$

Also from Equations (16) and (19), we have that

$$D_{10}^2 + D_{40}^2 = 2r_0^2 + 8\sin^2(\frac{\theta_0}{2}) . \tag{34}$$

Combining Equations (33) and (34), we find

$$2r_0^2 = \bar{s}_1^2 + \bar{s}_4^2 + 4\bar{s}_1|\sin(\frac{\theta_0}{2})| - 4\bar{s}_4|\sin(\frac{\theta}{2})| \leq \bar{s}_1^2 + \bar{s}_4^2 \leq 2\bar{s}_4^2 ,$$

where we use $\overline{s}_4 \geq \overline{s}_1$ in the last two steps. This leads to a contradiction because $\dot{r}(t) < 0$ on $t \in [0, \overline{t})$, so that $\overline{s}_4 < r_0$.

2. Suppose $D_{10} < 2|\sin(\frac{\theta_0}{2})|$. Then we have:

$$D_{10}^2 + D_{40}^2 < 8\sin^2(\frac{\theta_0}{2})\,.$$

Combining with Equation (34) and simplifying, we obtain $r_0^2 < 0$ which is impossible.

□

Proof. [Proof of Lemma 9] Because $\sigma(\xi_0) \leq 2$, $\overline{s}_3 = \overline{s}_2 = r_0|\cos(\phi_0 - \frac{\theta_0}{2})|$. Also, $\max\{\overline{s}_1, \overline{s}_4\} = \rho(\xi_0)$. Hence, we must show

$$\sqrt{r_0^2 + 4r_0|\cos(\phi_0 - \frac{\theta_0}{2})\sin(\frac{\theta_0}{2})| + 4\sin^2(\frac{\theta_0}{2})} - 2|\sin(\frac{\theta_0}{2})| \geq r_0|\cos(\phi_0 - \frac{\theta_0}{2})|\,.$$

After clearing the squareroot from the l.h.s. we obtain $r_0^2 \geq r_0^2\cos^2(\phi_0 - \frac{\theta_0}{2})$ which is immediately true. Moreover this inequality is strict if $\xi_0 \in \mathcal{W}$. □

Proof. [Proof of Lemma 10] We claim that when $\sigma^-(\xi) > 2$ and $\sigma^+(\xi) > 2$ then

(i) $D_2 = D_3$ implies $\overline{s}_2 = \overline{s}_3$.
(ii) $D_2 > D_3$ implies $\overline{s}_2 > \overline{s}_3$.
(iii) $D_3 > D_2$ implies $\overline{s}_3 > \overline{s}_2$.

Assuming that (i)-(iii) hold, we observe that if $D_3 \geq D_2$, then from (18) and (17) we have $\sin(\phi - \frac{\theta}{2})\cos(\frac{\theta}{2}) \geq 0$ so $\sin(\phi - \frac{\theta}{2})$ and $\cos(\frac{\theta}{2})$ have the same sign. Consequently, $\sigma^+(r, \phi, \theta) = \left|2\cos(\frac{\theta}{2})\right| + \left|r\sin(\phi - \frac{\theta}{2})\right|$. Substituting this in the expression for $\overline{s}_3$ and using (i) and (iii), the result is obtained.

Conversely, if $D_3 < D_2$, then $\sin(\phi - \frac{\theta}{2})$ and $\cos(\frac{\theta}{2})$ have opposite signs, so that $\sigma^-(r, \phi, \theta) = \left|2\cos(\frac{\theta}{2})\right| + \left|r\sin(\phi - \frac{\theta}{2})\right|$. Substituting this in the expression for $\overline{s}_2$ and using (ii), the result is obtained. Thus, it remains to prove the claim.

(i) From (17) and (18), $D_2 = D_3$ implies $\sin(\phi - \frac{\theta}{2})\cos\frac{\theta}{2} = 0$. If either $\sin(\phi - \frac{\theta}{2}) = 0$ or $\cos\frac{\theta}{2} = 0$, it follows immediately that $\overline{s}_2 = \overline{s}_3$.
(ii) Suppose $D_2 > D_3$. By (18) and (17) we have $\sin(\phi - \frac{\theta}{2})\cos\frac{\theta}{2} < 0$. Using the fact that $\cos(\frac{\theta}{2})$, and $\sin(\phi - \frac{\theta}{2})$ have opposite signs and letting $a = 2\cos(\frac{\theta}{2})$ and $b = r\sin(\phi - \frac{\theta}{2})$ we have

$$\overline{s}_2^2 - \overline{s}_3^2 = -4ab - 4|a| - 4|b| + 4|a+b|\,.$$

Hence we must show

$$|ab| + |a+b| > |a| + |b|\,.$$

By squaring both sides, this is equivalent to showing

$$\begin{aligned} a^2b^2 + a^2 + b^2 - 2|ab| + 2|ab||a+b| &> a^2 + b^2 + 2|ab| \\ \iff \quad a^2b^2 - 4|ab| + 2|ab||a+b| &> 0 \\ \iff \quad |ab| - 4 + 2|a+b| &> 0\,. \end{aligned}$$

However, we know $|a+b| = \sigma^+(\xi) > 2$ so the last inequality is true.

(iii) Suppose $D_3 > D_2$. By (18) and (17) we have $\sin(\phi - \frac{\theta}{2})\cos\frac{\theta}{2} > 0$. Using the fact that $\cos(\frac{\theta}{2})$, and $\sin(\phi - \frac{\theta}{2})$ have the same sign and letting $a = 2\cos(\frac{\theta}{2})$ and $b = r\sin(\phi - \frac{\theta}{2})$ we have

$$\overline{s}_3^2 - \overline{s}_3^2 = 4|ab| - 4|a| - 4|b| + 4|a-b|\,.$$

Hence we must show

$$|ab| + |a-b| > |a| + |b|\,.$$

By squaring both sides, this is equivalent to showing

$$\begin{aligned} a^2b^2 + a^2 + b^2 + 2|ab| + 2|ab||a-b| &> a^2 + b^2 + 2|ab| \\ \iff \quad a^2b^2 + 2|ab||a-b| &> 0\,. \end{aligned}$$

However, we know $|a-b| = \sigma^-(\xi) > 2$ so the last inequality is true.

□

Proof. [Proof of Lemma 11] To compute r for a fixed pair (ϕ, θ) we either solve the quadratic $\rho = c$ or the quadratic $\rho' = c$. From (25), the first quadratic is

$$r^2 + 4r|\cos(\phi - \frac{\theta}{2})\sin(\frac{\theta}{2})| - c^2 - 4c|\sin(\frac{\theta}{2})| = 0 \tag{35}$$

while (29) gives the second quadratic

$$r^2 + 4r|\sin(\phi - \frac{\theta}{2})\cos(\frac{\theta}{2})| + 4\cos^2(\frac{\theta}{2}) + 4 - 8|\cos(\frac{\theta}{2})| - 4r|\sin(\phi - \frac{\theta}{2})| - c^2 = 0\,. \tag{36}$$

Since the coefficients of these two quadratic equations are finite for all values of ϕ and θ, the roots are finite. Now considering the first quadratic equation, since the zeroth order term is negative, one of the roots for r must be negative; hence, there is only one valid root. The second quadratic equation only applies when $\sigma(\xi_0) > 2$, so again we have that there is only one valid root for r. We conclude there is only one value for r that satisfies $(r, \phi, \theta) \in \partial\mathcal{D}_c^*$ for a fixed pair (ϕ, θ).

To see that if $(r, \phi, \theta) \in \partial\mathcal{D}_c^*$ then $(\alpha r, \phi, \theta) \in \mathcal{D}_c^*$ for all $\alpha \in [0, 1)$, we observe that if $(r, \phi, \theta) \in \partial\mathcal{D}_c^*$, then r is the positive root of either Equation (35) or (36). In either case, the quadratic equation evaluates to a negative number for $(\alpha r, \phi, \theta)$ (since the other root is negative), which implies $r^* < c$. □