Flow Functions, Control Flow Functions, and the Reach Control Problem

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Abstract

The paper studies the reach control problem (RCP) to make trajectories of an affine system defined on a polytopic state space reach and exit a prescribed facet of the polytope in finite time without first leaving the polytope. We introduce the notion of a *flow function*, which provides the analog of a Lyapunov function for the equilibrium stability problem. A flow function comprises a scalar function that decreases along closed-loop trajectories, and its existence is a necessary and sufficient condition for closed-loop trajectories to exit the polytope. It provides an analysis tool for determining if a specific instance of RCP is solved, without the need for calculating the state trajectories of the closed-loop system. Results include a variant of the LaSalle Principle tailored to RCP. An open problem is to identify suitable classes of flow functions. We explore functions of the form $V(x) = \max\{V_i(x)\}$, and we give evidence that these functions arise naturally when RCP is solved using continuous piecewise affine feedbacks. Next we introduce the notion of a *control flow function*. It is shown that the Artstein-Sontag theorem of control Lyapunov functions has direct analogies to RCP via control flow functions.

1 Introduction

We study the reach control problem (RCP) for affine systems on polytopes. The problem is to find a feedback control to make the closed-loop trajectories of an affine system defined on a polytopic state space reach and exit a prespecified facet of the polytope in finite time. The problem arises in the study of piecewise affine hybrid systems consisting of a discrete automaton where each discrete mode is equipped with continuous-time affine dynamics defined on a polytope [19] and has ties to temporal logic specifications [29,18,43]. When the continuous state crosses a facet of a polytope, the system is transferred to a new discrete mode. The reachability problem for piecewise affine hybrid systems at the continuous level reduces to studying RCP for an affine system on a polytope [22]. Interesting applications of RCP can include motion of robots in complex environments [5], aircraft and underwater vehicles [6], anesthesia [17], genetic networks [7], smart buildings, process control [23], among others [19].

The preponderance of literature on RCP regards simplices because their remarkable structure allows to focus on the essence of the reachability problem [21,22,37,10–13,3,4,39]. Moreover, the search for feedback classes to solve RCP on simplices is narrowed due to their natural fit with affine feedback [21]. In contrast with simplices, the status for polytopes is more fragmentary. In [22] a method we call the *simplex-based method* was proposed. In [24,25] the geometric tools of [10] were extended from simplices to polytopes and a variant of RCP called the *monotonic reach control problem* (MRCP) was formulated. The simplex-based method and MRCP are the only known synthesis methods for solving RCP on polytopes [24]. It is unlikely that the geometric tools in [11,13,39] can be extended to polytopes due to the inherent combinatorial complexity of polytopes. One then turns to numerical approaches. Unfortunately, we encounter examples not solvable by either the simplex-based method or MRCP, yet a PWA feedback is numerically obtained and simulations show it solves RCP. This observation sets the stage for this paper.

We require an analysis tool that allows to diagnose rigorously if a candidate PWA (or continuous state) feedback solves RCP, without the need for calculating the state trajectories of the closed-loop system. One immediately recognizes an analogy with Lyapunov analysis for the equilibrium stability problem. But does RCP have an inherent notion of a function that acts like a Lyapunov function? Indeed it does. It was camouflaged as a *flow condition* in [37]. The flow condition is reinterpreted in this paper as a linear scalar function V called a *flow*

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function that strictly decreases along closed-loop trajectories in the polytope $\mathcal{P}$. Our concept of flow functions appears to be related to so-called density functions used to characterize certain reachability problems [34].

The contributions of the paper are as follows. In Section 4 we introduce the notion of a flow function. Flow functions provide a necessary and sufficient condition that all trajectories initiated in $\mathcal{P}$ leave it in finite time. In Section 5 we focus on PWA feedback, which is widely used to solve RCP on polytopes [22,25]. We present results which play the role of converse Lyapunov theorems. The aim is to identify a class of flow functions that naturally emerges when solving RCP by PWA feedback. In Section 6 the analogy with Lyapunov theory is deepened as we explore the Artstein-Sontag theorem for control Lyapunov functions within the context of RCP. Control Lyapunov functions continue to be intensively studied due to important emerging applications in hybrid systems and robotics [20,1]. We are lead to the notion of *control flow functions*, and we propose a "universal formula" for RCP. These results extend what is a verification tool based on flow functions to a synthesis tool based on control flow functions. A preliminary version of this paper appeared in [26].

2 Background

We use the following notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, the interior is $\mathcal{K}^\circ$, and the boundary is $\partial\mathcal{K} := \overline{\mathcal{K}} \setminus \mathcal{K}^\circ$, where the notation $\mathcal{K}_1 \setminus \mathcal{K}_2$ denotes elements of the set $\mathcal{K}_1$ not contained in the set $\mathcal{K}_2$. The notation $T_{\mathcal{K}}(x)$ denotes the Bouligand tangent cone to the set $\mathcal{K}$ at point x [15]. For $x \in \mathbb{R}^n$, $\mathscr{B}_\delta(x)$ denotes the open ball in $\mathbb{R}^n$ centered at x with radius δ. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\mathbb{R}_+$ denotes the set of non-negative real numbers. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be of class $\mathscr{C}^k$ if all its partial derivatives up to order k exist and are continuous. The notation $L_f V(x) = \frac{\partial V}{\partial x} f(x)$ denotes the Lie derivative of $\mathscr{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to function $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions, and let $\phi(t, x_0)$ denote the unique solution of $\dot{x} = f(x)$ starting at x_0. The *upper right Dini derivative* of $V(\phi(t, x_0))$ with respect to t is $D^+V(\phi(t, x_0)) := \limsup_{\tau \to 0^+} \frac{V(\phi(t+\tau, x_0)) - V(\phi(t, x_0))}{\tau}$. The *upper Dini derivative of* V *with respect to* f is given by $D_f^+ V(x) := \limsup_{\tau \to 0^+} \frac{V(x+\tau f(x)) - V(x)}{\tau}$.

We use some notions from algebraic topology [31]. An n-dimensional simplex $\mathcal{S} :=$ co $\{v_0, \ldots, v_n\}$ is the convex hull of $(n+1)$ affinely independent points $\{v_0, \ldots, v_n\}$ in $\mathbb{R}^n$. A *face* of $\mathcal{S}$ is any simplex spanned by a subset of $\{v_0, \ldots, v_n\}$. A *proper face* of $\mathcal{S}$ is any face of $\mathcal{S}$ different from $\mathcal{S}$ itself. A *facet* of $\mathcal{S}$ is an $(n-1)$-dimensional face. The union of the proper faces of $\mathcal{S}$ is called the boundary of $\mathcal{S}$, denoted bd$(\mathcal{S})$. The interior of $\mathcal{S}$ is int$(\mathcal{S}) = \mathcal{S} \setminus$ bd$(\mathcal{S})$. An n-dimensional polytope $\mathcal{P} :=$ co $\{v_1, \ldots, v_p\}$ is the convex hull of p points $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ whose affine hull has dimension n. A triangulation $\mathbb{T}$ of an n-dimensional polytope $\mathcal{P}$ is a finite collection of n-dimensional simplices $\mathcal{S}_1, \cdots, \mathcal{S}_L$ such that (i) $\mathcal{P} = \bigcup_{i=1}^L \mathcal{S}_i$, (ii) For all $i,\ j \in \{1, \cdots, L\}$ with $i \neq j$, the intersection $\mathcal{S}_i \cap \mathcal{S}_j$ is either empty or a common face of $\mathcal{S}_i$ and $\mathcal{S}_j$. Let $\mathbb{T}$ be a triangulation of $\mathcal{P}$. A point $x \in \mathcal{P}$ lies in the interior of precisely one simplex $\mathcal{S}_x$ in $\mathbb{T}$ whose vertices are, say, $v_1, \ldots, v_k$ (note that $\mathcal{S}_x$ is not necessarily an n-dimensional simplex). Then $x = \sum_{i=1}^k \beta_i v_i$, where $\beta_i > 0$ and $\sum_i \beta_i = 1$. Coefficients $\beta_1, \ldots, \beta_k$ are called the *barycentric coordinates* of x. If w is a vertex of $\mathbb{T}$, the *star* of w in $\mathbb{T}$, denoted by st(w), is the union of the interiors of those simplices in $\mathbb{T}$ that have w as a vertex. It is an open set in $\mathbb{R}^n$. The closure of st(w), denoted $\overline{\text{st}}(w)$, is called the *closed star* of w in $\mathbb{T}$.

3 Reach Control Problem

Consider an n-dimensional polytope in $\mathbb{R}^n$, $\mathcal{P} :=$ co $\{v_1, \ldots, v_p\}$ with vertex set $V := \{v_1, \ldots, v_p \mid v_i \in \mathbb{R}^n\}$ and facets $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$. The *exit facet* is designated to be the facet $\mathcal{F}_0$ of $\mathcal{P}$. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J := \{1, \ldots, r\}$, and $J(x) := \{j \in J \mid x \in \mathcal{F}_j\}$. For each $x \in \mathcal{P}$, define the closed, convex cone $\mathcal{C}(x) := \left\{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in J(x) \right\}$. We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{P}\,, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and rank$(B) = m$. Let $\mathcal{B} =$ Im B, the image of B. Also, let $\phi_u(t, x_0)$ be the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 3.1 (Reach Control Problem (RCP)) *Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that: for each $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T+\gamma)$.*

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{P}$ reach and exit the facet $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. Notice that condition (i) assumes that the dynamics (1) can be extended to a neighborhood of $\mathcal{P}$. A useful shorthand notation is to write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by control $u(x)$ if RCP is solved using $u(x)$.

The class of continuous piecewise affine feedbacks is widely used to solve RCP on polytopes [21,22,25]. Let

$\mathbb{T}$ be a triangulation of $\mathcal{P}$. Given a state feedback $u(x)$ on $\mathcal{P}$, we say u is a piecewise affine (PWA) feedback associated with $\mathbb{T}$ if for any $x \in \mathcal{P}$, $x = \sum_i \beta_i v_i$ implies $u(x) = \sum_i \beta_i u(v_i)$, where $\{v_i\}$ are the vertices of $\mathcal{S}_x$ and the β_i's are the corresponding barycentric coordinates of x. If $u(x)$ is a PWA feedback associated with $\mathbb{T}$, then for each n-dimensional simplex $\mathcal{S}^k \in \mathbb{T}$, there exist $K_k \in \mathbb{R}^{m \times n}$ and $g_k \in \mathbb{R}^m$ such that u takes the form $u(x) = K_k x + g_k$, $x \in \mathcal{S}^k$. In the literature necessary conditions for a PWA feedback to solve RCP have been identified; they guarantee that closed-loop trajectories only exit $\mathcal{P}$ through $\mathcal{F}_0$ [21]. We say the *invariance conditions are solvable* if for each $x \in \mathcal{P}$ there exists $u \in \mathbb{R}^m$ such that

$$Ax + Bu + a \in \mathcal{C}(x)\,. \tag{2}$$

Solvability of (2) can be checked by solving an LP program at each vertex of $\mathcal{P}$. Once control inputs satisfying (2) are obtained at the vertices, one can apply a straightforward procedure presented in [21] to construct a continuous PWA feedback on $\mathcal{P}$ satisfying (2) at all $x \in \mathcal{P}$.

4 Flow Functions

Suppose we are presented with an instance of RCP on a polytope and we have in hand a continuous feedback $u(x)$ as a candidate feedback solution such that starting at each initial condition in $\mathcal{P}$, there is a unique solution of the closed-loop vector field (for information on how to construct feedbacks on polytopes, see [21,25]). Since the invariance conditions (2) are necessary for solvability of RCP by continuous feedback [21], we assume that $u(x)$ already achieves (2). We conclude that trajectories can only exit $\mathcal{P}$ through $\mathcal{F}_0$. Then to verify if $u(x)$ actually solves RCP on $\mathcal{P}$, we only have to verify whether all trajectories initiated in $\mathcal{P}$ leave it in finite time. Of course exact verification through simulation is impossible since there are an infinite number of initial conditions. Like Lyapunov theory, we hope to make this verification without the need for calculating the state trajectories of the closed-loop system.

In the literature on RCP for simplices and affine feedbacks this verification is performed using a *flow condition* comprising a linear scalar function of the form $V(x) := \xi \cdot x$ that strictly decreases along closed-loop trajectories [22,37]. Such a linear function *always exists* if RCP is solved on an n-dimensional simplex by a given affine feedback [22,37]. Thus, for simplices we have specific forms of flow functions (linear flow functions) matching specific forms of the system and feedback (affine systems and feedbacks), in the same way that quadratic Lyapunov functions fit with linear systems and feedbacks. Unfortunately, linear functions are too restrictive as a class when verifying feedback solutions on polytopes [25]. Indeed, we have many examples where simulation results indicate that a continuous feedback $u(x)$ solves RCP, but no linear function exists. We are led to the following definition.

Definition 4.1 *Let $\mathcal{P}$ be an n-dimensional polytope and $\dot{x} = f(x)$ a dynamical system defined on $\mathcal{P}$. A* flow function *$V : \mathbb{R}^n \to \mathbb{R}$ is a scalar function bounded from below on $\mathcal{P}$ and strictly decreasing in $\mathcal{P}$ along solutions of the system.*

An open problem is to identify the most useful classes of flow functions for RCP. We begin with the most general context. Suppose we have a feedback $u(x)$ such that the closed-loop vector field $f(x) := Ax + Bu(x) + a$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Suppose we have a scalar function $V(x)$ bounded from below on $\mathcal{P}$ and satisfying

$$V(\phi(t, x_0)) \leq V(x_0) - t \tag{3}$$

for all $x_0 \in \mathcal{P}$ and $t \geq 0$ such that $\phi(\tau, x_0) \in \mathcal{P}, \tau \in [0, t]$. It is obvious from (3) that trajectories must exit $\mathcal{P}$ in finite time. Conversely, suppose that using $u(x)$, all trajectories initiated in $\mathcal{P}$ leave it in finite time. Then for each $x_0 \in \mathcal{P}$, there exist $T_{x_0} \geq 0$ and $\gamma_{x_0} > 0$ such that $\phi(t, x_0) \in \mathcal{P}$ for all $t \in [0, T_{x_0}]$, and $\phi(t, x_0) \notin \mathcal{P}$ for all $t \in (T_{x_0}, T_{x_0} + \gamma_{x_0})$. Define the map $T : \mathcal{P} \mapsto \mathbb{R}_+$ by $T(x) := T_x$, $x \in \mathcal{P}$. By uniqueness of solutions, T is a well-defined (single-valued) function. Also $T(x) \geq 0$ on $\mathcal{P}$. By the semi-group property, $T(\phi(t, x_0)) = T(x_0) - t$, $t \in [0, T(x_0)]$. Thus, we have proved the following straightforward but fundamental result showing that existence of a function bounded from below on $\mathcal{P}$ and satisfying (3) is a necessary and sufficient condition for leaving $\mathcal{P}$ in finite time.

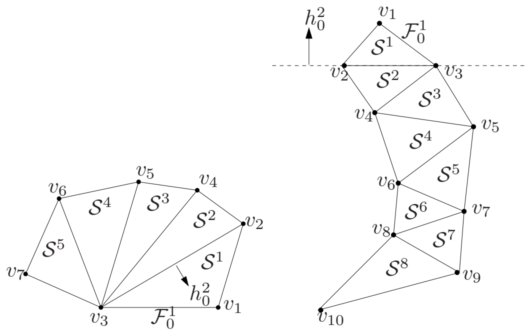
Theorem 4.1 *Consider the system* (1) *defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. All closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time if and only if there exists $V : \mathcal{P} \to \mathbb{R}$ such that $V(x)$ is bounded from below on $\mathcal{P}$ and* (3) *holds.*

Next suppose V is locally Lipschitz and $D_f^+ V(x) \leq -1$, $x \in \mathcal{P}$. We can again deduce that closed-loop trajectories leave $\mathcal{P}$. Since $V(x)$ is continuous and $\mathcal{P}$ is compact, $V(x)$ is bounded from below on $\mathcal{P}$. Since $D^+ V(\phi(t, x_0)) = D_f^+ V(x)$ with $x = \phi(t, x_0)$ for V locally Lipschitz [44], we can apply the Comparison Lemma [28] to obtain $V(\phi(t, x_0)) \leq V(x_0) - t$ for all $x_0 \in \mathcal{P}$ and $t \geq 0$ such that $\phi(\tau, x_0) \in \mathcal{P}$, $\tau \in [0, t]$. Then we can apply Theorem 4.1.

Further specializing these results, a form of V that appears to have special relevance to RCP and is investigated in Section 5 is as follows. Let $I_0 = \{1, \ldots, L\}$, and suppose for each $i \in I_0$, $V_i : \mathbb{R}^n \to \mathbb{R}$ is a $\mathscr{C}^1$ function. Define

$$V(x) := \max_{i \in I_0} \bigl\{ V_i(x) \bigr\}\,. \tag{4}$$

Also, for each $x \in \mathbb{R}^n$ define the index set $I_0(x) := \bigl\{ i \in I_0 \mid V_i(x) = V(x) \bigr\}$. Then $V(x)$ is locally Lipschitz [16]



(a) (b)

Fig. 1. Examples in which Assumption 5.1 holds.

and

$$D_f^+ V(x) = \max_{i \in I_0(x)} L_f V_i(x) \,. \tag{5}$$

Then we obtain the following result.

Corollary 4.2 *Consider the system* (1) *defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Let V be as in* (4)*. All closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time if $D_f^+ V(x) < 0$, $x \in \mathcal{P}$.*

Finally, we examine the case when a flow function has not been found, but we have identified a locally Lipschitz function V satisfying $D_f^+ V(x) \leq 0$ for all $x \in \mathcal{P}$. The question is whether this information is enough to deduce that closed-loop trajectories exit $\mathcal{P}$. For this we use an argument similar to LaSalle Principle, but we use it in the opposite way to how LaSalle Principle is normally applied. Thus, the novelty is in showing that a LaSalle Principle is meaningful in the context of RCP, despite RCP imposing a radically different requirement from equilibrium stability. As such, the proof is almost identical to the standard LaSalle Principle, so it is omitted.

Theorem 4.3 (LaSalle) *Consider the system* (1) *defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Suppose there exists $V : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on a neighborhood of $\mathcal{P}$ and satisfies $D_f^+ V(x) \leq 0$, $x \in \mathcal{P}$. Let $\mathcal{M} := \{x \in \mathcal{P} \mid D_f^+ V(x) = 0\}$. If $\mathcal{M}$ does not contain an invariant set, then all trajectories starting in $\mathcal{P}$ leave it in finite time.*

5 PWA Feedback

In this section we focus on (continuous) PWA feedback, a widely studied feedback class to solve RCP on polytopes [21,22,25], not to mention other control problems [8,14]. In the literature there are currently two techniques to solve RCP on polytopes by PWA feedback: the monotonic reach control problem (MRCP) [24,25] and so-called simplex-based methods [22]. A detailed comparison of the two methods was given in [24]. Here we highlight the main findings.

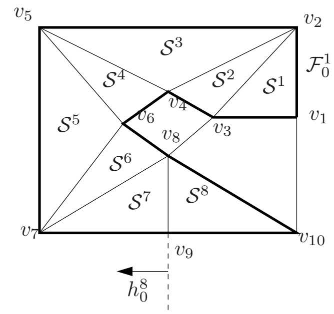


Fig. 2. Example in which Assumption 5.1 does not hold.

MRCP contains the same problem statement as RCP but it additionally imposes that a linear flow function strictly decreases along the closed-loop trajectories. *Simplex-based methods* or simply simplex methods involve, first, propitiously triangulating the polytope (using any a priori knowledge about the system dynamics), and, second, solving RCP for each simplex $\mathcal{S}^i$ of the triangulation using affine feedback $u^i(x) = K^i x + g^i$ [22]. Returning to our immediate inquiry to classify flow functions for verification of a candidate PWA feedback, for MRCP, there is nothing to be done - the flow function has been ordained to be linear. Instead we ask: *what class of flow functions emerges when RCP is solved by the simplex-based method?* We study this question for a specific topology: chains of simplices that form possibly non-convex polyhedra. Such topologies are of practical interest because of their applications in reach-avoid control problems [22,33] such as motion control of robots in complex environments [5] and anesthesia [17].

Definition 5.1 *Let $I_0 := \{1, \ldots, L\}$. Let $\{\mathcal{S}^k \mid k \in I_0\}$ be a collection of n-dimensional simplices. Define $\mathcal{P} := \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^L$. We say that $\mathcal{P}$ is a* chain *if the following holds:*

(i) If $\mathcal{S}^k \cap \mathcal{S}^j \neq \emptyset$ for $k, j \in I_0$, then $\mathcal{S}^k \cap \mathcal{S}^j$ is a face of $\mathcal{S}^k$ and of $\mathcal{S}^j$.
(ii) For each $k \in I_0$, denote the exit facet of $\mathcal{S}^k$ by $\mathcal{F}_0^k$. Then for $k = 2, \ldots, L$, $\mathcal{F}_0^k := \mathcal{S}^k \cap \mathcal{S}^{k-1}$.
(iii) The exit facet of $\mathcal{P}$ is $\mathcal{F}_0^1$.

We denote by h_0^k the unit normal vector of $\mathcal{F}_0^k$ pointing out of $\mathcal{S}^k$. Also let $\alpha_k \in \mathbb{R}$ be such that $h_0^k \cdot x = \alpha_k$ for all $x \in \mathcal{F}_0^k$. Figures 1 and 2 illustrate the notion of a chain. Now we place an additional restriction on the types of chains to be studied.

Assumption 5.1 *Let $\mathcal{P} := \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^L$ be a chain. We assume that $\{x \in \mathcal{P} \mid h_0^k \cdot x \leq \alpha_k, h_0^{k+1} \cdot x \geq \alpha_{k+1}\} = \mathcal{S}^k$ for $k = 1, \ldots, L-1$; and $\{x \in \mathcal{P} \mid h_0^L \cdot x \leq \alpha_L\} = \mathcal{S}^L$.*

Example 5.1 *Assumption 5.1 says that only one simplex $\mathcal{S}^k$ can lie in the intersection of the two half-spaces $\{x \in \mathcal{P} \mid h_0^k \cdot x \leq \alpha_k\}$ and $\{x \in \mathcal{P} \mid h_0^{k+1} \cdot x \geq \alpha_{k+1}\}$. Figure 1 shows examples in which Assumption 5.1 is satisfied. For instance, in Figure 1(a), it can be seen that $\{x \in \mathcal{P} \mid h_0^{k+1} \cdot x \geq \alpha_{k+1}\} \cap \{x \in \mathcal{P} \mid h_0^k \cdot x \leq \alpha_k\} = \mathcal{S}^k$ for $k = 1, \ldots, 4$ and $\{x \in \mathcal{P} \mid h_0^5 \cdot x \leq \alpha_5\} = \mathcal{S}^5$. On the other hand, Figure 2 shows an example in which Assumption 5.1 is not satisfied because $\{x \in \mathcal{P} \mid h_0^8 \cdot x \leq \alpha_8\} \neq \mathcal{S}^8$. Intuitively, Assumption 5.1 requires that the given chain does not make a circulation in the state space. Such a circulation may be required by the control specification. Notice in this case the chain can be divided into two or more chains which do satisfy Assumption 5.1. Hence, Assumption 5.1 is not a significant restriction.* ◁

Theorem 5.1 *Consider the system* (1) *defined on a chain $\mathcal{P} = \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^L$, and suppose that Assumption 5.1 holds. Let $u(x)$ be a continuous PWA feedback which is affine on each $\mathcal{S}^i$, $i \in I_0$, and let $f(x) := Ax + Bu(x) + a$. If $\mathcal{S}^i \xrightarrow{\mathcal{S}^i} \mathcal{F}_0^i$ for $i \in I_0$ using $u(x)$, then there exist affine functions $V_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I_0$ such that*

$$V(x) = \max_{i \in I_0} \{V_i(x)\} \tag{6}$$

satisfies $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$.

PROOF. Because $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0^1$, by Corollary 9 of [37] there exists $\xi_1 \in \mathbb{R}^n$ such that

$$\xi_1 \cdot (Ax + Bu(x) + a) < 0\,, \qquad x \in \mathcal{S}^1\,. \tag{7}$$

We choose $V_1(x) := \xi_1 \cdot x$. Let $k \in \{2, \ldots, L\}$ and consider $\mathcal{S}^k = \mathrm{co}\,\{v_0^k, v_1^k, \ldots, v_n^k\}$, where v_0^k is the vertex of $\mathcal{S}^k$ not in $\mathcal{F}_0^k$. See Figure 3. By the geometry of the simplex (see Lemma 2.1(2) of [21]), there exist $\lambda_i > 0$, $i = 1, \cdots, n$, such that

$$h_0^k = -\lambda_1 h_1^k - \cdots - \lambda_n h_n^k \tag{8}$$

where h_i^k is the outward unit normal vector of the facet of $\mathcal{S}^k$ not containing vertex v_i^k for $i = 1, \ldots, n$. Then because $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$, the invariance conditions (2) are satisfied at v_0^k using $u(x)$; that is

$$h_j^k \cdot (Av_0^k + Bu(v_0^k) + a) \leq 0\,, \quad j = 1, \ldots, n\,. \tag{9}$$

Now by Lemma 2.1(1) of [21], $\{h_1, \ldots, h_n\}$ span $\mathbb{R}^n$. Hence, there must be some inequality among those in (9) which holds strictly (for if not, $Av_0^k + Bu(v_0^k) + a = 0$, which contradicts that $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$). Suppose w.l.o.g. that

$$h_1^k \cdot (Av_0^k + Bu(v_0^k) + a) < 0\,. \tag{10}$$

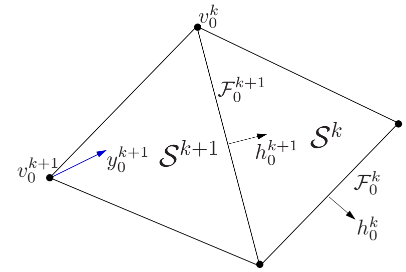


Fig. 3. Notation for the proof of Theorem 5.1.

Combining (8), (9), and (10), we obtain

$$(-h_0^k) \cdot (Av_0^k + Bu(v_0^k) + a) < 0\,, k \in \{2, \ldots, L\}\,. \tag{11}$$

Next we define

$$\xi_k := \xi_{k-1} - c_k h_0^k\,, k \in \{2, \ldots, L\}\,. \tag{12}$$

Using (11), we choose $c_k > 0$ sufficiently large such that

$$\xi_k \cdot (Av_0^k + Bu(v_0^k) + a) < 0\,, \qquad k \in \{2, \ldots, L\}\,. \tag{13}$$

Now we show that for each $k \in I_0$ and for all $x \in \mathcal{S}^k$, $\xi_k \cdot (Ax + Bu(x) + a) < 0$. We argue by induction. For the base step, we have (7). Next, suppose that

$$\xi_k \cdot (Ax + Bu(x) + a) < 0\,, \qquad x \in \mathcal{S}^k\,. \tag{14}$$

We must show $\xi_{k+1} \cdot (Ax + Bu(x) + a) < 0$ for all $x \in \mathcal{S}^{k+1}$. Referring to Figure 3, because $\mathcal{F}_0^{k+1}$ is a facet of $\mathcal{S}^k$ which is not its exit facet, $u(x)$ is continuous, and $\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k$, the invariance conditions (2) for $\mathcal{S}^k$ hold at vertices $v_i \in \mathcal{F}_0^{k+1}$. That is,

$$(-h_0^{k+1}) \cdot (Av_i + Bu(v_i) + a) \leq 0\,, \quad v_i \in \mathcal{F}_0^{k+1}\,. \tag{15}$$

Using (12), (14), and (15), we get

$$\xi_{k+1} \cdot (Av_i + Bu(v_i) + a) < 0\,, \quad v_i \in \mathcal{F}_0^{k+1}\,. \tag{16}$$

Since $u(x)$ is affine on $\mathcal{S}^{k+1}$ and $\mathcal{S}^{k+1} = \mathrm{co}\,\{v_0^{k+1}, v_i \mid v_i \in \mathcal{F}_0^{k+1}\}$, (13) and (16) together imply $\xi_{k+1} \cdot (Ax + Bu(x) + a) < 0$ for all $x \in \mathcal{S}^{k+1}$, as desired. Now we choose $V_k(x) := \xi_k \cdot x + \sum_{j=1}^{k-1} c_{j+1}\alpha_{j+1}$ for $k \in \{2, \ldots, L\}$, and we let $V(x)$ be as in (6).

It remains to show $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$. Our results above give

$$L_f V_k(x) = \xi_k \cdot (Ax + Bu(x) + a) < 0\,,\ x \in \mathcal{S}^k,\ k \in I_0\,. \tag{17}$$

Recall that $h_0^k \cdot x = \alpha_k$ for $x \in \mathcal{F}_0^k$ and that h_0^k points outside of $\mathcal{S}^k$. Also, by definition $V_k(x) - V_{k+1}(x) =$

$(\xi_k - \xi_{k+1}) \cdot x - c_{k+1}\alpha_{k+1}$. Therefore, for any $x \in \mathcal{P}$,

$$V_k(x) \geq V_{k+1}(x) \iff h_0^{k+1} \cdot x \geq \alpha_{k+1}\,, k \in \{1, \ldots, L-1\} \quad (18a)$$
$$V_k(x) \geq V_{k-1}(x) \iff h_0^{k} \cdot x \leq \alpha_{k}\,, k \in \{2, \ldots, L\}\,. \quad (18b)$$

Define the sets $\Gamma^k = \left\{x \in \mathcal{P} \mid V_k(x) \geq V_j(x), \quad j \in I_0\right\}$ for $k \in I_0$. If we compare (18) with Assumption 5.1, we find that $\Gamma^k \subseteq \mathcal{S}^k$, $k \in I_0$. Consider any $x \in \mathcal{P}$ and suppose $k \in I_0(x)$. By the definition of $I_0(x)$, $V_k(x) \geq V_j(x)$, $j \in I_0$, which implies $x \in \Gamma^k \subseteq \mathcal{S}^k$. It follows from (17) that for all $x \in \mathcal{P}$, if $k \in I_0(x)$, then $L_f V_k(x) < 0$. Applying (5), we conclude $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$, as desired. □

The previous result plays the same role as a converse Lyapunov theorem, and as with certain converse Lyapunov theorems, it may appear to be only of theoretical interest. The result is of practical interest when simplex methods fail, yet a flow function of the form (6) may still be relevant. A typical scenario is when simplex methods fail because the invariance conditions of one or more simplices are not solvable. Hence, there is practical interest to have a numerical procedure to construct a flow function of the form (6). For simplicity, in the next result we present a numerical procedure for two simplices only.

Corollary 5.2 *Consider the system* (1) *defined on a polytope* $\mathcal{P}$*, and a triangulation* $\mathbb{T} = \left\{\mathcal{S}^1, \mathcal{S}^2\right\}$ *of* $\mathcal{P}$*, where* $\mathcal{S}^1 = \mathrm{co}\,\{v_1, \ldots, v_{n+1}\}$*, and* $\mathcal{S}^2 = \mathrm{co}\,\{v_2, \ldots, v_{n+2}\}$*. The exit facet of* $\mathcal{S}^1$ *is* $\mathcal{F}_0 = \mathrm{co}\,\{v_1, \ldots, v_n\}$ *and the exit facet of* $\mathcal{S}^2$ *is* $\mathcal{F} = \mathcal{S}^1 \cap \mathcal{S}^2 = \mathrm{co}\,\{v_2, \ldots, v_{n+1}\}$*. Let h be the unit normal vector to $\mathcal{F}$ pointing out of $\mathcal{S}^2$. Let $u(x)$ be a continuous PWA feedback on $\mathbb{T}$ and let $f(x) := Ax + Bu(x) + a$. Suppose that $u(x)$ satisfies the invariance conditions* (2) *of* $\mathcal{P}$*, and for some* $2 < k \leq n+2$ *it satisfies* (2) *of* $\mathcal{S}^1$ *at vertices* $v_2, \ldots, v_{k-1}$*. Suppose the following linear programming (LP) problem is solvable*

$$\begin{bmatrix} f(v_1)^T & 0 \\ \vdots & \vdots \\ f(v_{n+1})^T & 0 \\ f(v_k)^T & -h \cdot f(v_k) \\ \vdots & \vdots \\ f(v_{n+2})^T & -h \cdot f(v_{n+2}) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ c \end{bmatrix} < 0\,. \quad (19)$$

Then there exist affine functions $V_1 : \mathbb{R}^n \to \mathbb{R}$*,* $V_2 : \mathbb{R}^n \to \mathbb{R}$*, and a function V of the form* (6) *such that* $D_f^+ V(x) < 0$ *for all* $x \in \mathcal{P}$*.*

Corollary 5.2 provides a simple tool for verifying that all closed-loop trajectories initiated in $\mathcal{P}$ leave it in finite time for the case where existing techniques fail. Analogous to the case of two simplices (Corollary 5.2), it is in general possible to verify the existence of a flow function of the form (6) by solving an LP problem in the decision variables $\xi_1, c_2, \cdots, c_L$.

6 Control Flow Functions

We have emphasized an analogy between Lyapunov functions for the equilibrium stability problem and flow functions for the reach control problem. The analogy will be deepened in this section, where we examine the Artstein-Sontag theorem based on control Lyapunov functions and we reinterpret it in the context of RCP. We introduce the notion of a *control flow function*. Like control Lyapunov functions, control flow functions convert a tool for analysis, flow functions, into a tool for synthesis. We begin with a non-constructive result on synthesis of PWA feedback following [2]. We then turn to constructive methods - the inspiration is the universal formulas of [27,42,41].

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{C}^1$ function. We say that V is a *control flow function* if for each $x \in \mathcal{P}$ there exists $u \in \mathbb{R}^m$ such that $Ax + Bu + a \in \mathcal{C}(x)$ and

$$\frac{\partial V}{\partial x}(x)(Ax + Bu + a) < 0\,. \quad (20)$$

Suppose it is known that a control flow function exists for the system (1) on $\mathcal{P}$. We are interested in the question of whether this implies that RCP is solvable on $\mathcal{P}$. Second, if it is solvable, is it possible to construct a feedback law? To that end, for each $x \in \mathcal{P}$ define

$$\mathcal{U}(x) := \left\{u \in \mathbb{R}^m \mid Ax + Bu + a \in \mathcal{C}(x)\right\}$$
$$\mathcal{U}^o(x) := \left\{u \in \mathbb{R}^m \mid Ax + Bu + a \in \mathcal{C}^\circ(x)\right\}$$
$$\mathcal{U}^{flow}(x) := \left\{u \in \mathcal{U}(x) \mid \frac{\partial V}{\partial x}(x)(Ax + Bu + a) < 0\right\}$$
$$(\mathcal{U}^{flow})^\circ(x) := \left\{u \in \mathcal{U}^o(x) \mid \frac{\partial V}{\partial x}(x)(Ax + Bu + a) < 0\right\}.$$

Assuming $\mathcal{U}(x) \neq \emptyset$ for all $x \in \mathcal{P}$, then $\mathcal{U} : \mathcal{P} \to 2^{\mathbb{R}^m}$ is a set-valued map with closed, convex values. If $(\mathcal{U}^{flow})^\circ(x)$ is non-empty, then it is a convex set consisting of all control inputs satisfying both (20) and a strict form of the invariance conditions. In particular, velocity vectors must lie in the interior of the n-dimensional cone $\mathcal{C}(x)$. In the sequel we assume that for each $x \in \mathcal{P}$, $\mathcal{U}^{flow}(x) \neq \emptyset$. Moreover in Theorems 6.2 and 6.5 we assume $(\mathcal{U}^{flow})^\circ(x) \neq \emptyset$, $x \in \mathcal{P}$. The additional requirement to satisfy strict invariance conditions is needed when the flow function is not sufficiently smooth; whether it can be removed is an open problem.

We begin with a fact about $\mathcal{C}(x)$. We have already discussed that $\mathcal{C}(x) = T_{\mathcal{P}}(x)$, assuming x is not in $\mathcal{F}_0$;

they differ at points in $\mathcal{F}_0$. On compact, convex sets $\mathcal{X}$, $x \mapsto T_{\mathcal{X}}(x)$ is a lower semi-continuous set-valued map [15]. Not surprisingly, this also holds for $x \mapsto \mathcal{C}(x)$.

Lemma 6.1 *The map $x \mapsto \mathcal{C}(x)$ is lower semi-continuous on $\mathcal{P}$. Moreover, for each $x \in \mathcal{P}$, there exists $\delta > 0$ such that for all $x' \in \mathscr{B}_\delta(x) \cap \mathcal{P}$, $\mathcal{C}(x) \subset \mathcal{C}(x')$.*

PROOF. The second statement implies the first one, so we only prove the second statement. Let $x \in \mathcal{P}$. If $x \in \mathcal{P}^\circ$ then there exists $\delta > 0$ such that for all $x' \in \mathscr{B}_\delta(x) \cap \mathcal{P}$, $\mathcal{C}(x) = \mathcal{C}(x') = \mathbb{R}^n$. If $x \in \partial\mathcal{P}$, suppose w.l.o.g. $x \in \cap_{i=1}^k \mathcal{F}_i$ for some $1 \le k \le r$. Then $h_j \cdot x < \alpha_j$ for $j = k+1, \ldots, r$, where $\{z \mid h_j \cdot z = \alpha_j\}$ is the hyperplane containing $\mathcal{F}_j$, and $\mathcal{C}(x) = \{y \in \mathbb{R}^n \mid h_j \cdot y \le 0, j = 1, \ldots, k\}$. There exists $\delta > 0$ such that for all $x' \in \mathscr{B}_\delta(x) \cap \mathcal{P}$, $h_j \cdot x' < \alpha_j$, for $j = k+1, \ldots, r$. That is, w.l.o.g. $x' \in \cap_{i=1}^{k'} \mathcal{F}_i$ for some $1 \le k' \le k$. Hence, $\mathcal{C}(x) \subset \mathcal{C}(x') = \{y \in \mathbb{R}^n \mid h_j \cdot y \le 0, j = 1, \ldots, k'\}$. $\square$

Theorem 6.2 *Consider the system* (1) *defined on a polytope $\mathcal{P}$. Suppose there exists a $\mathscr{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in \mathcal{P}$, $(\mathcal{U}^{flow})^\circ(x) \neq \emptyset$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by continuous PWA feedback.*

PROOF. The proof follows the construction in [2]. Let $u_0(x)$ be any selection from the multi-valued map $(\mathcal{U}^{flow})^\circ(x)$, $x \in \mathcal{P}$. Because V is $\mathscr{C}^1$, $Ax + Bu + a$ is smooth in x, and by Lemma 6.1, it follows that for each $x \in \mathcal{P}$ and $y \in \mathcal{P}$ sufficiently close to x, $\frac{\partial V}{\partial x}(y)(Ay + Bu_0(x) + a) < 0$ and $Ay + Bu_0(x) + a \in \mathcal{C}^\circ(x) \subset \mathcal{C}^\circ(y)$. Consequently, for each $x \in \mathcal{P}$ and $y \in \mathcal{P}$ sufficiently close to x, $u_0(x) \in (\mathcal{U}^{flow})^\circ(y)$. In particular, for each $x \in \mathcal{P}$, we can find a ball $\mathscr{B}_0(x) \cap \mathcal{P}$ centered at x and open in $\mathcal{P}$, such that for all $y \in \mathscr{B}_0(x) \cap \mathcal{P}$, $u_0(x) \in (\mathcal{U}^{flow})^\circ(y)$. Then $\{\mathscr{B}_0(x)\}$ is an open cover of $\mathcal{P}$ and since $\mathcal{P}$ is compact, there exists a finite subcover $\{\mathscr{B}_0^i \mid i \in I_0\}$, where $I_0 = \{1, \ldots, L\}$ and $\mathscr{B}_0^i = \mathscr{B}_0(x^i)$ for some $x^i \in \mathcal{P}$. By Theorem 16.4 of [31], there exists a triangulation $\mathbb{T}$ of $\mathcal{P}$ with vertex set $\mathbb{T}_0$ such that $\mathbb{T}$ *refines* the open cover $\{\mathscr{B}_0^i \mid i \in I_0\}$. That is, for each $w \in \mathbb{T}_0$, $\overline{\mathrm{st}}(w) \subset \mathscr{B}_0^j$ for some $j \in I_0$.

Now we assign control values at the vertices of $\mathbb{T}$. For each $w \in \mathbb{T}_0$, define $u(w) := u_0(x^k)$, where $k \in I_0$ is any index such that $\overline{\mathrm{st}}(w) \subset \mathscr{B}_0^k$. Let $\kappa(w)$ denote this choice of $k \in I_0$. Next consider any $x \in \mathcal{P}$. Then $x = \sum_{i=1}^l \beta_i^x w_{j_i}$ where $(\beta_1^x, \ldots, \beta_l^x)$ are the barycentric coordinates of x and $\{w_{j_i}\}$ are the vertices of $\mathcal{S}_x$ as reviewed in Section 2. Define the control $u(x) := \sum_{i=1}^l \beta_i^x u(w_{j_i})$. Clearly, $u : \mathcal{P} \to \mathbb{R}^m$ is a continuous PWA feedback on $\mathcal{P}$. Moreover, because $x \in \overline{\mathrm{st}}(w_{j_i}) \subset \mathscr{B}_0^{\kappa(w_{j_i})}$, we have $u(w_{j_i}) = u_0(x^{\kappa(w_{j_i})}) \in (\mathcal{U}^{flow})^\circ(x)$, $i = 1, \ldots, l$. By convexity of $(\mathcal{U}^{flow})^\circ(x)$ we conclude that $u(x) \in (\mathcal{U}^{flow})^\circ(x)$.

In sum, we have shown there exists a continuous PWA feedback $u(x)$ on $\mathcal{P}$ such that for all $x \in \mathcal{P}$,

$$
\begin{aligned}
Ax + Bu(x) + a &\in \mathcal{C}(x)\,, \\
\frac{\partial V}{\partial x}(x)(Ax + Bu(x) + a) &< 0\,.
\end{aligned}
$$

The PWA feedback $u(x)$ can be affinely extended to a neighborhood of $\mathcal{P}$ such that it is locally Lipschitz on this neighborhood [9]. Therefore the closed-loop vector field $Ax + Bu(x) + a$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. By Theorem 4.1 of [21], closed-loop trajectories cannot exit $\mathcal{P}$ from non-exit facets. By Proposition 3.5, Chapter 7, of [36], closed-loop trajectories must exit $\mathcal{P}$ in finite time. We conclude $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ using $u(x)$. $\square$

The next problem we investigate is whether a control flow function can be used to explicitly construct the feedback solving RCP. We seek a "universal formula" associated with RCP. Because the research problem of finding a universal formula for RCP has not been posed before, the main result (Theorem 6.4) is stated only for single-input systems. Already the formulas in [27,42,41] provide feedbacks that ensure that a function V is strictly decreasing along closed-loop trajectories. We are not able to directly adopt those formulas because in RCP we have the added requirement that the invariance conditions must hold for the proposed feedback. The latter involves a more careful analysis of the range of control values permissible for each $x \in \mathcal{P}$. To that end, we begin with the following technical result.

Lemma 6.3 *Consider the system* (1) *defined on a polytope $\mathcal{P}$. Suppose there exists a $\mathscr{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in \mathcal{P}$, $\mathcal{U}^{flow}(x) \neq \emptyset$. Then there exists $\alpha > 0$ such that for each $x \in \mathcal{P}$, there exists $u \in \mathbb{R}^m$ satisfying*

$$
\begin{aligned}
\frac{\partial V}{\partial x}(x)(Ax + Bu + a) &\le -\alpha && \text{(21a)} \\
Ax + Bu + a &\in \mathcal{C}(x)\,. && \text{(21b)}
\end{aligned}
$$

PROOF. Define the extended real-valued function $\chi : \mathcal{P} \to [-\infty, +\infty]$ by $\chi(x) := \inf_{u \in \mathcal{U}(x)} \frac{\partial V}{\partial x}(x)(Ax + Bu + a)$, $x \in \mathcal{P}$. Since $\mathcal{U}^{flow}(x) \neq \emptyset$ for all $x \in \mathcal{P}$, we know that $\chi(x) \in [-\infty, 0)$, $x \in \mathcal{P}$. We claim that $\chi(x)$ is an upper semi-continuous function on $\mathcal{P}$.

First consider $x_0 \in \mathcal{P}$ such that $-\infty < \chi(x_0) < 0$. We must show that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathscr{B}_\delta(x_0) \cap \mathcal{P}$, then $\chi(x) < \chi(x_0) + \epsilon$. It can be shown that there exists $u_0 \in \mathcal{U}(x_0)$ such that $\frac{\partial V}{\partial x}(x_0)(Ax_0 +$

$Bu_0 + a) = \chi(x_0)$ (the argument involves taking a sequence $\{u_0^k \in \mathcal{U}(x_0)\}$ such that $\lim_{k\to\infty} \frac{\partial V}{\partial x}(x_0)(Ax_0 + Bu_0^k + a) = \chi(x_0)$ and then using the fact that $\mathcal{C}(x_0)$ and $\mathcal{B}$ are closed). By assumption there exist $u_i \in \mathbb{R}^m$, $i = 1,\dots,p$, an assignment of control inputs at the vertices of $\mathcal{P}$ satisfying (2). (Note that if $x_0 = v_i$, a vertex of $\mathcal{P}$, then assign $u_i := u_0$). If x_0 is not a vertex of $\mathcal{P}$, consider the point set $\{x_0, v_1, \dots, v_p\}$. If x_0 is a vertex of $\mathcal{P}$, then consider the point set $\{v_1,\dots,v_p\}$. We can construct a triangulation $\mathbb{T}$ of $\mathcal{P}$ based on the selected point set which satisfies that for each simplex $\mathcal{S}_j \in \mathbb{T}$, if $x_0 \in \mathcal{S}_j$, then x_0 is a vertex of $\mathcal{S}_j$ [30]. Based on the control inputs $\{u_0, u_1, \dots, u_p\}$ (or $\{u_1,\dots,u_p\}$ if x_0 is a vertex of $\mathcal{P}$), we form on each n-dimensional simplex in the triangulation the unique affine feedback [21]. This yields a continuous PWA feedback on $\mathcal{P}$; denote it $u'(x)$. By a standard convexity argument, $u'(x)$ satisfies the invariance conditions (2) of $\mathcal{P}$. Also, $u'(x_0) = u_0$, so $\frac{\partial V}{\partial x}(x_0)(Ax_0 + Bu'(x_0) + a) = \chi(x_0) < 0$. Because $V(x)$ is $\mathscr{C}^1$ and u' is continuous on $\mathcal{P}$, $\frac{\partial V}{\partial x}(x)(Ax + Bu'(x) + a)$ is continuous on $\mathcal{P}$. Hence, for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathscr{B}_\delta(x_0) \cap \mathcal{P}$, $\frac{\partial V}{\partial x}(x)(Ax + Bu'(x) + a) < \chi(x_0) + \epsilon$. Since $u'(x)$ satisfies (2), $u'(x) \in \mathcal{U}(x)$. Then by definition of $\chi(x)$, $\chi(x) \le \frac{\partial V}{\partial x}(x)(Ax + Bu'(x) + a) < \chi(x_0) + \epsilon$, $x \in \mathscr{B}_\delta(x_0) \cap \mathcal{P}$. Second, consider $x_0 \in \mathcal{P}$ such that $\chi(x_0) = -\infty$. We must show that for each $c < 0$, there exists $\delta > 0$ such that for all $x \in \mathscr{B}_\delta(x_0) \cap \mathcal{P}$, $\chi(x) < c$. Because $\chi(x_0) = -\infty$, there exists a sequence $\{u^k\}$ such that $Ax_0 + Bu^k + a \neq 0$ and

$$\frac{\partial V}{\partial x}(x_0)(Ax_0 + Bu^k + a) \longrightarrow -\infty \tag{22a}$$

$$Ax_0 + Bu^k + a \in \mathcal{C}(x_0)\,. \tag{22b}$$

Define $b^k := \frac{Ax_0 + Bu^k + a}{\|Ax_0 + Bu^k + a\|} \in \mathcal{C}(x_0)$. Since $\{b^k\}$ is a bounded sequence, it has a convergent subsequence $\{b^{k_j}\}$ such that $b^{k_j} \longrightarrow \overline{b}$. Since $\mathcal{C}(x_0)$ is closed, $\overline{b} \in \mathcal{C}(x_0)$. However, (22a) implies $\|Ax_0 + Bu^{k_j} + a\| \to \infty$ so $\frac{Ax_0 + a}{\|Ax_0 + Bu^{k_j} + a\|} \to 0$. Then $\overline{b} = \lim_{k_j\to\infty} \frac{Bu^{k_j}}{\|Ax_0 + Bu^{k_j} + a\|} \in \mathcal{B} \cap \mathcal{C}(x_0)$. Also by (22a), $\frac{\partial V}{\partial x}(x_0)\overline{b} < 0$. Fix $c < 0$. By Lemma 6.1, there exists $\delta_1 > 0$ such that for all $x \in \mathscr{B}_{\delta_1}(x_0) \cap \mathcal{P}, \mathcal{C}(x_0) \subset \mathcal{C}(x)$. Hence $\overline{b} \in \mathcal{B} \cap \mathcal{C}(x)$. Also, there exists $\delta_2 > 0$ such that for all $x \in \mathscr{B}_{\delta_2}(x_0) \cap \mathcal{P}$, $\frac{\partial V}{\partial x}(x)\overline{b} < 0$. Let $\delta := \min\{\delta_1, \delta_2\}$. For $x \in \mathscr{B}_\delta(x_0) \cap \mathcal{P}$, let u' be such that $Ax + Bu' + a \in \mathcal{C}(x)$. Then define $u = u' + c'\overline{u}$, where $c' > 0$ and $\overline{b} = B\overline{u}$. Then for all $c' > 0$, $Ax + Bu + a = (Ax + Bu' + a) + c'\overline{b} \in \mathcal{C}(x)$, and for $c' > 0$ sufficiently large, $\frac{\partial V}{\partial x}(x)(Ax + Bu + a) = \frac{\partial V}{\partial x}(x)(Ax + Bu' + a) + c'\left(\frac{\partial V}{\partial x}(x)\overline{b}\right) < c$. We conclude for each $c < 0$, there exists $\delta > 0$ such that for all $x \in \mathscr{B}_\delta(x_0) \cap \mathcal{P}$, $\chi(x) < c$, as desired.

We have shown $\chi(x)$ is upper semi-continuous on $\mathcal{P}$. Since $\mathcal{P}$ is compact, by Theorem 2.1 of [40], $\chi(x)$ attains a maximum on $\mathcal{P}$. That is, there exists $\alpha > 0$ such that $\chi(x) \le -\alpha$ for all $x \in \mathcal{P}$. □

The next result provides a universal formula for RCP for the case of single-input systems.

Theorem 6.4 *Consider the system* (1) *defined on a polytope $\mathcal{P}$. Suppose that $m = 1$. Also, suppose there exists a $\mathscr{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in \mathcal{P}$, $\mathcal{U}^{flow}(x) \neq \emptyset$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by continuous state feedback.*

PROOF. By Lemma 6.3, there exists $\alpha > 0$ such that for each $x \in \mathcal{P}$, there exists $u \in \mathbb{R}$ such that (21) holds. Hence, there exist inputs $u_i \in \mathcal{U}(v_i)$, $i = 1,\dots,p$. Triangulate $\mathcal{P}$ into simplices using only vertices of $\mathcal{P}$, and form on each simplex the unique affine feedback based on the control values u_i, $i = 1,\dots,p$ [21]. This yields a continuous PWA feedback $u^{inv}(x)$ such that, by convexity, $Ax + Bu^{inv}(x) + a \in \mathcal{C}(x)$ for all $x \in \mathcal{P}$. Apply the feedback transformation $u = u^{inv}(x) + w$ to (1) to obtain the new system

$$\dot{x} = Ax + Bu^{inv}(x) + Bw + a\,. \tag{23}$$

We claim there exists $\alpha > 0$ such that for each $x \in \mathcal{P}$, there exists $w \in \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(x)(Ax + Bu^{inv}(x) + Bw + a) \le -\alpha \tag{24a}$$

$$Ax + Bu^{inv}(x) + Bw + a \in \mathcal{C}(x)\,. \tag{24b}$$

Proof of claim: Fix $x \in \mathcal{P}$ and let $u \in \mathbb{R}$ satisfy (21). Define $w := u - u^{inv}(x)$. Then (24) immediately follows.

Let $f(x) := Ax + Bu^{inv}(x) + a$ and $g(x) := B$. Define the feedback [1]

$$u^{flow}(x) = \begin{cases} 0\,, & L_f V(x) \le \frac{-\alpha}{2}\,, \\ \frac{-(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)}\,, & L_f V(x) > \frac{-\alpha}{2}\,. \end{cases} \tag{25}$$

Observe that by (24a), if $L_g V(x) = 0$, then $L_f V(x) \le -\alpha$. That is,

$$L_f V(x) > -\alpha \quad \Longrightarrow \quad L_g V(x) \neq 0\,. \tag{26}$$

Using this fact, it can be shown that $u^{flow}(x)$ is continuous and locally Lipschitz. Define the feedback

$$u(x) = u^{inv}(x) + u^{flow}(x)\,.$$

[1] Here we adopt the same notation used in the universal formulas in [42,41].

We show that for each $x \in \mathcal{P}$,

$$\frac{\partial V}{\partial x}(x)(Ax + Bu(x) + a) \le -\frac{\alpha}{2} \tag{27a}$$
$$Ax + Bu(x) + a \in \mathcal{C}(x)\,. \tag{27b}$$

There are two cases: (i) $L_f V(x) \le \frac{-\alpha}{2}$. In this case $u(x) = u^{inv}(x)$. Then $\frac{\partial V}{\partial x}(x)(Ax + Bu(x) + a) = L_f V(x) \le -\frac{\alpha}{2}$. Also, by construction of $u^{inv}(x)$, $Ax + Bu(x) + a \in \mathcal{C}(x)$. (ii) $L_f V(x) > \frac{-\alpha}{2}$. In this case $u(x) = u^{inv}(x) - \frac{(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)}$. By direct substitution, $\frac{\partial V}{\partial x}(x)(Ax + Bu(x) + a) = -\frac{\alpha}{2}$. Next, there exists $w \in \mathbb{R}$ satisfying (24). If $L_g V(x) > 0$, then $w \le \frac{-(L_f V(x) + \alpha)}{L_g V(x)} < \frac{-(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)} = u^{flow}(x) < 0$. Otherwise, if $L_g V(x) < 0$, then $0 < u^{flow}(x) = \frac{-(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)} < \frac{-(L_f V(x) + \alpha)}{L_g V(x)} \le w$. Consequently, in either case there exists $\lambda \in (0, 1)$ such that $u^{flow}(x) = \lambda w$, so $u(x) = (1 - \lambda)u^{inv}(x) + \lambda(u^{inv}(x) + w)$. By the construction of $u^{inv}(x)$, (24b), and convexity of $\mathcal{C}(x)$, $Ax + Bu(x) + a \in \mathcal{C}(x)$.

By (27a), the continuous function $V(x)$ is strictly decreasing along the closed-loop trajectories of (1) in compact $\mathcal{P}$. Hence, all closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time. Because $u(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$, so is the closed-loop vector field. Then by Theorem 4.1 of [21], trajectories that leave $\mathcal{P}$ do so only through $\mathcal{F}_0$. We conclude $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by the continuous state feedback $u = u^{inv} + u^{flow}$. $\square$

The feedback $u = u^{inv} + u^{flow}$ consists of a continuous PWA feedback u^{inv} and a locally Lipschitz feedback u^{flow} so the closed-loop vector field $Ax + Bu(x) + a$ is locally Lipschitz. This fact relies on V being a $\mathscr{C}^2$ function. When only a $\mathscr{C}^1$ control flow function is available, then u^{flow} and the closed-loop vector field are continuous. In this case solutions exist for each initial condition in $\mathcal{P}$ [28], but uniqueness of solutions is not guaranteed. Results such as Proposition 3.5, Chapter 7, of [36] do not require uniqueness of solutions. On the other hand, Theorem 4.1 of [21], which tells us that closed-loop trajectories cannot exit $\mathcal{P}$ from non-exit facets when $Ax + Bu(x) + a \in \mathcal{C}(x)$, $x \in \mathcal{P}$, cannot be used because it requires that the closed-loop vector field be locally Lipschitz on a neighborhood of $\mathcal{P}$. In order to overcome this obstacle, we introduce in the next result the stronger requirement, already used in Theorem 6.2, that $Ax + Bu(x) + a \in \mathcal{C}^{\circ}(x)$, $x \in \mathcal{P}$. The requirement can be relaxed if it is known that u^{flow} is locally Lipschitz.

Theorem 6.5 *Consider the system* (1) *defined on a polytope* $\mathcal{P}$*. Suppose that* $m = 1$*. Also, suppose there exists a* $\mathscr{C}^1$ *function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that for each* $x \in \mathcal{P}$, $(\mathcal{U}^{flow})^{\circ}(x) \neq \emptyset$*. Then* $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ *by the continuous state feedback* $u(x) = u^{inv}(x) + u^{flow}(x)$*, where* $u^{inv}(x)$ *satisfies* $Ax + Bu^{inv}(x) + a \in \mathcal{C}^{\circ}(x)$*,* $x \in \mathcal{P}$*, and* $u^{flow}(x)$ *is given by* (25)*.*

Remark 6.1 *The analogy between the explicit continuous feedback* $u(x)$ *provided for RCP in the proof of Theorem 6.4 and the universal formulas for stabilization in [27,42,41] is evident. While the universal formulas in [27,42,41] ensure that a* $\mathscr{C}^1$ *function* V *is strictly decreasing along closed-loop trajectories, they do not necessarily satisfy the invariance conditions of* $\mathcal{P}$*, and so they do not solve RCP. The cost we pay for the added requirement of achieving the invariance conditions is that unlike [41,27], our explicit feedback* $u(x)$ *is not necessarily real analytic.*

Remark 6.2 *Once a control flow function is found, one can simply substitute in the universal formula in the proof of Theorem 6.4 to construct a continuous feedback law solving RCP. Therefore, the only remaining challenge is how to find control flow functions in a computationally efficient way. By looking at the literature of Lyapunov stability, we find that the same problem exists there. However, in the last decade, computationally efficient methods for constructing Lyapunov (or control Lyapunov) functions have been proposed [32,35,38]. Unfortunately, these methods cannot be directly adopted for finding control flow functions since in RCP the additional requirement of achieving the invariance conditions should be satisfied by the selected control inputs at the points in* $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r$*. Instead, one can use these efficient results to find an initial guess of the control flow function, a* $\mathcal{C}^1$ *function* V *such that for each* $x \in \mathcal{P}^{\circ}$*, there exists* $u \in \mathbb{R}$ *satisfying* (20)*. After identifying* V *that satisfies* (20)*, one can check whether* V *is a control flow function by studying points in* $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r$ *and verifying the existence of control inputs that satisfy both* (20) *and the invariance conditions. One important topic for future research in RCP is to create computationally efficient techniques for the construction of control flow functions instead of depending on the existing techniques for control Lyapunov functions.*

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