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Reach control of single-input systems on simplices using multi-affine feedback

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Abstract— **The paper studies the reach control problem (RCP) to make trajectories of an affine system defined on a simplex reach and exit a prescribed facet of the simplex in finite time without first leaving the simplex. Affine feedbacks are generally used to solve RCP, and there is an emerging belief that affine feedback and continuous state feedback are equivalent with respect to solvability of RCP on simplices. This equivalence has been proved under an assumption on the triangulation of the state space. There remains the question of whether this result can also be proved under arbitrary triangulations. In this paper, we show that the answer is negative by constructing an example for which no solution based on affine feedback exists, yet a continuous state feedback solves the problem. Then for single-input affine systems we provide a constructive method for synthesis of multi-affine feedbacks to solve RCP on simplices for the case where affine feedbacks fail to solve the problem.**

I. Introduction

We study the reach control problem (RCP) for affine systems on simplices. The problem is to design a state feedback to make the closed-loop trajectories starting in a simplex reach and exit a prescribed facet of the simplex in finite time [7]. In contrast with [7], it is not required here that trajectories leave the simplex immediately after they reach the exit facet for the first time [8], [15]. RCP is a fundamental reachability problem for an important class of hybrid systems [6], namely piecewise affine hybrid systems [8]. For interesting applications of RCP, the reader is referred to [1], [2], [3].

RCP was first formulated in [7], and it has been developed in [8], [15], [4], [16], [17], [13], [9], [10], [11], [12]. The literature on RCP can be classified into two main streams. First, solvability of RCP on simplices by affine feedback was deeply studied in [8], [15], [4], [16], [17]. To apply the methods developed in these papers, the polytopic state space is first triangulated into simplices, and then an affine feedback is synthesized on each simplex, with the intention that the set of controllers on the simplices will collectively solve the problem on the polytopic region. Second, solvability of RCP on polytopes was directly studied in [7], [9], [10], [11], [12], where continuous piecewise affine (PWA) feedback is synthesized to solve the problem.

The question arises of whether continuous PWA feedback on polytopes (affine feedback on simplices) is the largest continuous feedback class needed to solve RCP. As a first step in answering this question, we focus in this paper on simplices and study the relationship between affine feedbacks and continuous state feedbacks for RCP on simplices. In [4], it has been shown that affine feedback and continuous state feedback are equivalent for solving RCP on simplices. The result holds under the assumption that the polytopic state space has been triangulated properly with respect to $\mathcal{O}$, the set of possible equilibria of the system. Specifically, for an n-dimensional simplex $\mathcal{S}$ of the triangulation, $\mathcal{S} \cap \mathcal{O}$ is either the empty set or a k-dimensional face of $\mathcal{S}$, where $0 \leq k \leq n$. There remains the question of whether this result holds under arbitrary triangulations of the state space.

In this paper we show that the answer is negative by constructing a counterexample in which affine feedbacks fail to solve RCP, yet a continuous state feedback solves the problem. Then, we investigate an alternative feedback class for solving RCP on simplices for the case where affine feedbacks fail. Since this research study is completely novel, we focus in this paper on single-input affine systems, and we provide a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices.

The paper is organized as follows. In the next section we review RCP. Section III provides the counterexample for the equivalence of affine feedback and continuous state feedback. In Section IV we explore the geometric properties of the set of open-loop equilibria in the simplex. In Section V we provide a constructive method for the synthesis of multi-affine feedbacks for RCP on simplices. In Section VI two examples are given illustrating the synthesis method. Section VII concludes the paper.

Notation. Let $\mathcal{S} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{S}}$, the interior is $\mathcal{S}^{\circ}$, and the boundary is $\partial \mathcal{S}$. The relative interior is denoted ri $(\mathcal{S})$ and the relative boundary of $\mathcal{S}$, denoted rb $(\mathcal{S})$, is $\overline{\mathcal{S}} \setminus$ ri $(\mathcal{S})$. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\mathscr{B}$ denotes the open ball of radius 1 centered at the origin. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$, and the notation aff $\{v_1, v_2, \ldots\}$ denotes the affine hull of a set of points $v_i \in \mathbb{R}^n$. The notation $L_f V(x)$ denotes the Lie derivative of function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to function $f : \mathbb{R}^n \to \mathbb{R}^n$. Finally, $T_{\mathcal{P}}(x)$ denotes the Bouligand tangent cone to set $\mathcal{P} \subset \mathbb{R}^n$ at point x [5].

II. Reach Control Problem

We consider an $n-$dimensional simplex $\mathcal{S} :=$ co $\{v_0, v_1, \cdots, v_n\}$ with vertex set $V := \{v_0, v_1, \cdots, v_n\}$ and facets $\mathcal{F}_0, \cdots, \mathcal{F}_n$ (the facet is indexed by the vertex it does not contain). Let h_i, $i \in \{0, \cdots, n\}$, be the unit normal

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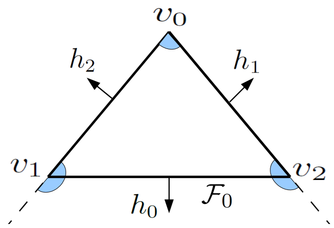

Fig. 1. The convex cones $\mathcal{C}(v_i)$ in a two-dimensional simplex.

vector to the facet $\mathcal{F}_i$ pointing outside the simplex, and let $\mathcal{F}_0$ be the *exit facet*. We call the other facets $\mathcal{F}_1, \cdots, \mathcal{F}_n$ the *restricted facets*. Define $I := \{1, \cdots, n\}$ to be the set of indices of the restricted facets of $\mathcal{S}$. Given $x \in \mathcal{S}$, let $I(x)$ be the minimal index set such that $x \in \text{co}\ \{v_i \mid i \in I(x)\}$.

Consider the affine control system defined on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\phi_u(t, x_0)$ denote the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{S}$ and evaluated at time t. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 2.1: (Reach Control Problem (RCP)) Consider system (1) defined on a simplex $\mathcal{S}$. Find a state feedback $u(x)$ such that for every $x_0 \in \mathcal{S}$, there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

RCP says that trajectories of (1) starting in $\mathcal{S}$ reach and exit $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{S}$. We write $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ if RCP is solved by some feedback. Notice that in order for the above problem definition to make sense, it is assumed that the dynamics (1) are extended to a neighborhood of $\mathcal{S}$.

For $x \in \mathcal{S}$, define the closed, convex cone $\mathcal{C}(x)$ by

$$\mathcal{C}(x) := \left\{\ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in I \setminus I(x)\ \right\}.$$

Observe that $I \setminus I(x)$ is the set of indices of the restricted facets in which x is a point. Figure 1 illustrates the cones $\mathcal{C}(v_i)$ as shaded cones attached at each v_i since they are used to characterize tangent velocity vectors. Notice that for v_0, $\mathcal{C}(v_0)$ is exactly the Bouligand tangent cone to $\mathcal{S}$ at v_0, $T_{\mathcal{S}}(v_0)$. Instead, at $v_i \in \mathcal{F}_0$, they are different since $\mathcal{C}(v_i)$ includes directions pointing out of $\mathcal{S}$. Indeed the definition of $\mathcal{C}(v_i)$ does not involve h_0 because $\mathcal{F}_0$ is the exit facet.

Definition 2.1: We say the *invariance conditions are solvable* if there exist $u_0, \cdots, u_n \in \mathbb{R}^m$ such that

$$Av_i + Bu_i + a \in \mathcal{C}(v_i)\,, \quad i \in \{0, \cdots, n\}. \tag{2}$$

The invariance conditions (2) are used to construct affine feedbacks [7]; by convexity of affine feedbacks on $\mathcal{S}$, (2) ensures that all the restricted facets are blocked, so trajectories can exit $\mathcal{S}$ only through $\mathcal{F}_0$. More generally, we say a state feedback $u(x)$ satisfies the invariance conditions if

$$Ax + Bu(x) + a \in \mathcal{C}(x), \quad x \in \mathcal{S}\,. \tag{3}$$

Note that for $x \in \mathcal{S}^\circ \cup \text{ri}\ (\mathcal{F}_0)$, $\mathcal{C}(x) = \mathbb{R}^n$, so the invariance conditions hold trivially for those points.

The following necessary and sufficient conditions have been established for solvability of RCP by a given affine feedback.

Theorem 2.1 ([8],[15]): Given the system (1) on an n-dimensional simplex $\mathcal{S}$ and an affine feedback $u(x) = Kx + g$, where $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \cdots, u_n = u(v_n)$, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by $u(x)$ if and only if

(a) The invariance conditions (2) hold,
(b) There is no closed-loop equilibrium in $\mathcal{S}$.

Theorem 2.1 is not useful in control synthesis since it depends on having a candidate affine feedback. Instead, the geometry of the problem should be explored to find constructive conditions for RCP. To that end, let $\mathcal{B} = \text{Im}\ (B)$, the image of B. Define

$$\mathcal{O} := \{x \in \mathbb{R}^n | Ax + a \in \mathcal{B}\}.$$

One can show that if $\mathcal{O} \neq \emptyset$, then $\mathcal{O}$ is an affine space with dimension between m and n. Notice that at any $x \in \mathcal{O}$, the vector field $Ax+Bu+a$ can vanish by an appropriate choice of u. Indeed, $\mathcal{O}$ is the set of all possible equilibrium points of the system; that is, if x_0 is an equilibrium of (1) under feedback control, then $x_0 \in \mathcal{O}$. We also define

$$\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}.$$

Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_{\mathcal{S}} = \emptyset$ or $\mathcal{O}_{\mathcal{S}}$ is a convex polytope in $\mathcal{S}$ with a dimension $0 \leq \kappa \leq n$. Let $V_{\mathcal{O}_{\mathcal{S}}} = \{o_1, \cdots, o_{k+1}\}$ denote the set of vertices of $\mathcal{O}_{\mathcal{S}}$ and $I_{\mathcal{O}_{\mathcal{S}}} = \{1, \cdots, k+1\}$. We define

$$\text{cone}(\mathcal{O}_{\mathcal{S}}) := \bigcap_{i \in I_{\mathcal{O}_{\mathcal{S}}}} \mathcal{C}(o_i)\,.$$

Theorem 2.2 ([16]): Suppose $m = 1$. If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by continuous state feedback, then $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$.

Finally, we review an important version of LaSalle Theorem [10] which is used in our proofs in Sections III, V.

Theorem 2.3 (LaSalle): Consider the system (1) defined on a compact set $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Suppose there exists a continuously differentiable (C^1) function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies $L_f V(x) \leq 0$, $x \in \mathcal{P}$. Let $\mathcal{M} := \{x \in \mathcal{P} \mid L_f V(x) = 0\}$. If $\mathcal{M}$ does not contain an invariant set, then all trajectories starting in $\mathcal{P}$ leave it in finite time.

III. A Counterexample

There is a belief in the literature that affine feedback and continuous state feedback are equivalent from the point of view of solvability of RCP on simplices. This result has been proved in [4] for the case where the state space has been triangulated with respect to $\mathcal{O}$: namely, for any simplex $\mathcal{S}$ of the triangulation, $\mathcal{O}_{\mathcal{S}}$ is either the empty set or a κ-dimensional face of $\mathcal{S}$. There remains the question of whether this result can also be proved under arbitrary triangulations. In this section, we present an example first studied in [16] showing that under arbitrary triangulations, affine feedback and continuous state feedback are not equivalent for solving RCP on simplices. In particular, RCP is not solvable by affine feedback, yet a continuous state feedback can be devised to solve the problem.

Consider a simplex $\mathcal{S} = \text{co}\,\{v_0, \ldots, v_4\}$, where $v_0 = 0$, $v_i = e_i$ (e_i is the ith Euclidean coordinate vector), and $h_i = -e_i$, $i \in \{1, \cdots, 4\}$. Consider the affine system on $\mathcal{S}$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -3 & -6 & -3 & -2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} -3 \\ -5 \\ 8 \\ 4 \end{bmatrix} u + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}. \tag{4}$$

Let $b := (-3, -5, 8, 4)$. It can be verified that

$$\mathcal{O} \;=\; \left\{ x \in \mathbb{R}^4 \mid x_1 = x_4 + \frac{1}{4}, x_2 = x_4 + \frac{1}{4}, x_3 = -2x_4 + \frac{1}{4} \right\}.$$

Setting $x_4 = 0$ in the defining equations for $\mathcal{O}$, we get $o_1 := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$. Setting $x_3 = 0$, we get $o_2 := \left(\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{8}\right)$. Thus, $\mathcal{O}_{\mathcal{S}} = \text{co}\,\{o_1, o_2\}$ where $o_1 = \frac{1}{4}v_0 + \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 \in \mathcal{F}_4$, and $o_2 = \frac{1}{8}v_0 + \frac{3}{8}v_1 + \frac{3}{8}v_2 + \frac{1}{8}v_4 \in \mathcal{F}_3$. Clearly, $\mathcal{O}_{\mathcal{S}} \cap \mathcal{S}^\circ \neq \emptyset$ and $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$. Because $o_1 \in \mathcal{F}_4$ and $o_2 \in \mathcal{F}_3$, we have $\text{cone}(\mathcal{O}_{\mathcal{S}}) = \{y \in \mathbb{R}^4 \mid h_3 \cdot y \le 0,\; h_4 \cdot y \le 0\}$. Since $h_3 \cdot b < 0$ and $h_4 \cdot b < 0$, $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$, so solvability of RCP by continuous state feedback cannot be ruled out by Theorem 2.2. Also, it can be verified that $u = 0$ satisfies the invariance conditions (3), so solvability of RCP by continuous state feedback cannot be ruled out by Proposition 3.1 of [7]. Nevertheless, it was shown in [16] based on the concept of *reach controllability* that in this example RCP is not solvable by affine feedback. Here we present a direct argument that illuminates the reason behind this failure.

Lemma 3.1: Given simplex $\mathcal{S}$ and system (4), RCP is not solvable by affine feedback.

Proof: Suppose by way of contradiction that $u(x) = Kx + g$ achieves $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$. Define $u_i = Kv_i + g$ and $y_i = Av_i + Bu_i + a$, $i \in \{0, \ldots, 4\}$. Using (4) we have $y_0 = (-3u_0, -5u_0 + 3, 8u_0 + 1, 4u_0)$, $y_1 = (-3u_1 - 1, -5u_1, 1 + 8u_1, 4u_1)$, $y_2 = (-3u_2, -3 - 5u_2, 1 + 8u_2, 4u_2)$, and $y_3 = (1 - 3u_3, -5u_3, -3 + 8u_3, 4u_3)$. Since $u(x)$ satisfies the invariance conditions, we get $h_1 \cdot y_0 = 3u_0 \le 0$ and $h_4 \cdot y_0 = -4u_0 \le 0$, so $u_0 = 0$. Similarly, $h_2 \cdot y_1 = 5u_1 \le 0$ and $h_4 \cdot y_1 = -4u_1 \le 0$, so $u_1 = 0$; $h_1 \cdot y_2 = 3u_2 \le 0$ and $h_4 \cdot y_2 = -4u_2 \le 0$, so $u_2 = 0$; and $h_2 \cdot y_3 = 5u_3 \le 0$ and $h_4 \cdot y_3 = -4u_3 \le 0$, so $u_3 = 0$; By convexity, $u(x) = 0$ for all $x \in \mathcal{F}_4 = \text{co}\,\{v_0, \cdots, v_3\}$. Now we observe that

$$y(o_1) = \frac{1}{4}y_0 + \frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{4}y_3 = 0.$$

This contradicts $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$. ∎

The failure occurs because the affine feedback cannot achieve the invariance conditions at v_i, $i \in I(o_1)$, without having an equilibrium at o_1. To solve this problem, we must find a continuous state feedback $u(x)$ such that (i) $u(v_i) = 0$, $i \in I(o_1)$, to satisfy the invariance conditions at the vertices; and (ii) $u(o_1) > 0$ to remove the equilibrium point at o_1. To that end, consider the multi-affine state feedback

$$u(x) = x_1 x_2 x_3 .$$

We observe that $u(x) = 0$ for $x \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ and $u(x) > 0$ for $x \in \mathcal{S} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$. This means $u(x)$ meets the requirements (i)-(ii). Now we show that $u(x)$ indeed solves RCP on $\mathcal{S}$.

Lemma 3.2: The closed-loop vector field $y(x) := Ax + Bu(x) + a$ satisfies the invariance conditions (3).

Proof: Suppose $x \in \mathcal{F}_1$. Since $h_1 = (-1, 0, 0, 0)$ and $x_1 = 0$, we have $h_1 \cdot y(x) = x_1 - x_3 + 3x_1x_2x_3 = -x_3 \le 0$. Suppose $x \in \mathcal{F}_2$. Since $h_2 = (0, -1, 0, 0)$, $x_2 = 0$, and $x_1 + x_3 + x_4 \le 1$, we have $h_2 \cdot y(x) = 3x_1 + 3x_3 + 2x_4 - 3 \le 3(x_1 + x_3 + x_4) - 3 \le 0$. Suppose $x \in \mathcal{F}_3$. Since $h_3 = (0, 0, -1, 0)$ and $x_3 = 0$, we have $h_3 \cdot y(x) = -1 \le 0$. Suppose $x \in \mathcal{F}_4$. Since $h_4 = (0, 0, 0, -1)$ and $x_4 = 0$, we have $h_4 \cdot y(x) = -4x_1x_2x_3 \le 0$. Combining these facts, we conclude (3). ∎

Proposition 3.3: Given simplex $\mathcal{S}$ and system (4), $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using $u(x)$.

Proof: Consider the *flow function* $V(x) := h_4 \cdot x$ [10], [12]. We compute $\dot{V}(x) = h_4 \cdot y(x) = -4x_4 - 4x_1x_2x_3 \le 0$, $x \in \mathcal{S}$. Also, $\dot{V}(x) = 0$ when $x_4 = 0$ and $x_1x_2x_3 = 0$. To apply the LaSalle Principle, Theorem 2.3, we must show $\mathcal{M} := \left\{ x \in \mathcal{S} \mid \dot{V}(x) = 0 \right\}$ does not contain an invariant set. In this example, we have $\mathcal{M} = \mathcal{F}_4 \cap (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$. On $\mathcal{M}$, $x_4 = 0$ and $x_1x_2x_3 = 0$, so the dynamics for states $\tilde{x} := (x_1, x_2, x_3)$ reduce to:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{a} = \begin{bmatrix} -1 & 0 & 1 \\ -3 & -6 & -3 \\ 0 & 0 & -4 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

Define $\overline{x} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and observe that $\tilde{A}\overline{x} + \tilde{a} = 0$. Consider the function $\tilde{V}(\tilde{x}) := (\tilde{x} - \overline{x})^T P (\tilde{x} - \overline{x})$, and consider the Lyapunov equation $\tilde{A}^T P + P\tilde{A} = -I$, where P is a symmetric matrix and I is the identity matrix. Because $\sigma(\tilde{A}) \subset \mathbb{C}^-$, a unique solution exists, given by

$$P = \begin{bmatrix} \frac{17}{28} & \frac{-1}{28} & \frac{4}{25} \\ \frac{-1}{28} & \frac{1}{12} & \frac{-1}{35} \\ \frac{4}{25} & \frac{-1}{35} & \frac{261}{1400} \end{bmatrix}.$$

Then one can verify $\dot{\tilde{V}}(\tilde{x}) = -\|\tilde{x} - \overline{x}\|^2 < 0$ for all $(\tilde{x}, 0) \in \mathcal{M}$. Since $\mathcal{M}$ is compact, by a standard argument, all trajectories must leave $\mathcal{M}$ [10], [12]. In particular, there is no invariant set in $\mathcal{M}$. By Theorem 2.3, all trajectories initiated in $\mathcal{S}$ leave it in finite time. Then, since the invariance conditions (3) are achieved (Lemma 3.2), $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using $u(x)$. ∎

The above example shows that under arbitrary triangulations, affine feedback and continuous state feedback are not equivalent for solving RCP on simplices. Also, it gives a hint that multi-affine feedbacks may be a suitable class to solve RCP on simplices when affine feedbacks fail. The above analysis is specific to this example. We aim to find a general methodology for constructing multi-affine feedbacks to solve RCP on simplices.

IV. The Equilibrium Set

The objective of the rest of this paper is to elaborate a synthesis method for multi-affine feedbacks that works in cases like the previous example when affine feedbacks fail but RCP is still solvable by continuous state feedback. Since multi-affine feedback synthesis for RCP on simplices has not been studied before, we focus in this paper on single-input affine systems.

Our synthesis method strongly relies on properties of $\mathcal{E}_{\mathcal{S}}$, the set of equilibria in $\mathcal{S}$ of the open-loop system $\dot{x} = Ax+a$. Let

$$\mathcal{E} := \{x \in \mathbb{R}^n \mid Ax + a = 0\}.$$

Also define $\mathcal{E}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{E}$. Clearly $\mathcal{E} \subseteq \mathcal{O}$ and $\mathcal{E}_{\mathcal{S}} \subseteq \mathcal{O}_{\mathcal{S}}$. In this section we present our main assumptions, review some technical results about $\mathcal{E}_{\mathcal{S}}$ in [16], [17], and finally investigate further properties of $\mathcal{E}_{\mathcal{S}}$.

First, we present the main assumptions used in the rest of the paper. As stated above, we assume $m = 1$. In [4] it was assumed that if $\mathcal{O}_{\mathcal{S}} \neq \emptyset$, then $\mathcal{O}_{\mathcal{S}}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \le \kappa \le n$. Here, we consider the general case where $\mathcal{O}$ intersects the interior of $\mathcal{S}$. In general, this intersection is a convex polytope. However, in this paper we assume $\mathcal{O}_{\mathcal{S}}$ is a simplex, and we restrict $\mathcal{O}_{\mathcal{S}}$ so that it does not intersect $\mathcal{F}_0$.

Assumption 4.1:

(A1) $\mathcal{O}_{\mathcal{S}} = \text{co}\,\{o_1, \cdots, o_{\kappa+1}\}$, a κ-dimensional simplex with $1 \le \kappa \le n$.
(A2) If $\mathcal{E}_{\mathcal{S}} \neq \emptyset$, then $\mathcal{E}_{\mathcal{S}} = \text{co}\,\{\epsilon_1, \cdots, \epsilon_{\kappa_0+1}\}$, a κ_0-dimensional simplex with $0 \le \kappa_0 \le \kappa$.
(A3) $\mathcal{O}_{\mathcal{S}} \cap \mathcal{S}^{\circ} \neq \emptyset$.
(A4) $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$.

Suppose without loss of generality (w.l.o.g.) that $v_0 = 0$. From Proposition 3.1 of [7], we know that the invariance conditions are necessary for solvability of RCP by continuous state feedback. Here, we assume w.l.o.g. that the invariance conditions are solvable using $u = 0$. If the invariance conditions are solvable using $u(x) = Fx + g \neq 0$, then we consider the new affine system $\dot{x} = \tilde{A}x + \tilde{a} + Bu = (A+BF)x+(a+Bg)+Bu$, and apply the proposed design procedure to it to obtain the multi-affine feedback. The final control law is $u(x) = Fx + g + w(x)$, where $w(x)$ is the obtained multi-affine feedback.

In [16], [17] the same assumptions were used to find geometric necessary and sufficient conditions for solvability of RCP by affine feedback. The reader is referred to Remark 1 of [17] for more discussion on Assumption 4.1. Assumption 4.1 enables us to study an interesting geometric case that is more general than the one studied in [4] and to make use of foundational results from [16], [17]. The results required from [16], [17] are as follows.

Theorem 4.1 ([16], [17]): Suppose that Assumption 4.1 holds and $m = 1$. Also, suppose $Av_i + a \in \mathcal{C}(v_i)$ for $i \in \{0, \cdots, n\}$ and $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$. If $\mathcal{E}_{\mathcal{S}} \neq \emptyset$, then $\mathcal{E}_{\mathcal{S}} \subset \text{rb}\,(\mathcal{O}_{\mathcal{S}}) \subset \partial\mathcal{S}$.

Theorem 4.1 says that if the necessary conditions for solvability of RCP on $\mathcal{S}$ by continuous state feedback are achieved, then equilibrium points can appear only on the boundary of $\mathcal{S}$.

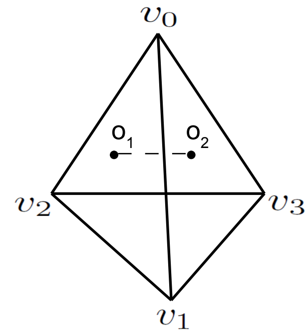

Fig. 2. The set $\mathcal{E}_{\mathcal{S}}$ in Example 4.1

Lemma 4.2 ([16], [17]): Suppose that Assumption 4.1 holds and $m = 1$. Also, suppose $Av_i + a \in \mathcal{C}(v_i)$ for $i \in \{0, \cdots, n\}$ and $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$. If $\mathcal{E}_{\mathcal{S}} \neq \emptyset$, then $\mathcal{E}_{\mathcal{S}}$ is a κ_0-dimensional face of $\mathcal{O}_{\mathcal{S}}$, where $0 \le \kappa_0 < \kappa$.

Lemma 4.3 ([16], [17]): Suppose $m = 1$ and $Av_i + a \in \mathcal{C}(v_i)$ for $i \in \{0, \cdots, n\}$. Also, suppose there exists $\overline{x} \in \mathcal{E}_{\mathcal{S}}$ such that $I(\overline{x}) = \{0, \cdots, q\}$, where $1 \le q < n$. Then there exists a coordinate transformation $z = T^{-1}x$ such that the transformed system has the form

$$\dot{z} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} z + \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u\,, \qquad (5)$$

where $A_1 \in \mathbb{R}^{q\times q}$, $A_{12} \in \mathbb{R}^{q\times(n-q)}$, $a_1 \in \mathbb{R}^q$, $b_1 \in \mathbb{R}^q$, $A_2 \in \mathbb{R}^{(n-q)\times(n-q)}$, and $b_2 \in \mathbb{R}^{n-q}$ for $q > 0$.

In [16], [17] it was also shown that $T = [v_1 \cdots v_n]$. Lemma 4.3 says that we can always decompose the dynamics into those contributing to equilibria and transversal dynamics.

Now we are ready to go beyond [16], [17] and discover an important property of the equilibrium set $\mathcal{E}_{\mathcal{S}}$. We show that if Assumption 4.1 and the necessary conditions for solvability of RCP by continuous state feedback hold, then not only is $\mathcal{E}_{\mathcal{S}}$ in the boundary of $\mathcal{S}$, but also it is a single point.

Theorem 4.4: Consider the system (1) defined on simplex $\mathcal{S}$. Suppose that $m = 1$, Assumption 4.1 holds, $Av_i + a \in \mathcal{C}(v_i)$, $i \in \{0, \dots, n\}$, and $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$. If $\mathcal{E}_{\mathcal{S}} \neq \emptyset$, then $\dim(\mathcal{E}_{\mathcal{S}}) = 0$.

The technical proof of this result is found in the Appendix. Here we present an example that illustrates the proof idea.

Example 4.1: Consider a canonical simplex $\mathcal{S} = \text{co}\,\{v_0, \cdots, v_4\} \subset \mathbb{R}^4$ with $v_0 = 0$ and $v_i = e_i$, $i = 1, \dots, 4$. Consider the system (1) defined on $\mathcal{S}$ and suppose Assumption 4.1 holds, $Ax + a \in \mathcal{C}(x)$ for all $x \in \mathcal{S}$, and $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$. Also suppose $\mathcal{E}_{\mathcal{S}} = \text{co}\,\{o_1, o_2\}$ where $I(o_1) = \{0, 1, 2\}$ and $I(o_2) = \{0, 1, 3\}$ as shown in Figure 2. Observe that $\mathcal{E}_{\mathcal{S}} \subset \text{co}\,\{v_0, \cdots, v_3\} \subset \partial\mathcal{S}$ as required by Theorem 4.1 and that $\mathcal{E}_{\mathcal{S}} \cap \text{ri}\,(\text{co}\,\{v_0, \dots, v_3\}) \neq \emptyset$. Now consider $o_1 \in \mathcal{F}_3 \cap \mathcal{F}_4$. Clearly

$$h_j \cdot (Ao_1 + a) = h_j \cdot 0 = 0\,, \qquad j = 3, 4\,.$$

Also since $Ax + a \in \mathcal{C}(x)$, $x \in \mathcal{S}$,

$$h_j \cdot (Av_i + a) \le 0\,, \qquad i = 0,1,2, \quad j = 3,4\,.$$

Now $o_1 = \sum_{i=0}^{2} \alpha_i^{o_1} v_i$ with $\alpha_i^{o_1} > 0$ and $\sum_i \alpha_i^{o_1} = 1$. Combined with the previous two inequalities, we get

$$h_j \cdot (Av_i + a) = 0\,, \qquad i = 0,1,2, \quad j = 3,4\,.$$

Since $v_0 = 0$, this becomes

$$h_j \cdot a = 0\,, \quad h_j \cdot Av_i = 0\,, \quad i = 1,2, \quad j = 3,4\,. \tag{6}$$

In like manner for $o_2 \in \mathcal{F}_2 \cap \mathcal{F}_4$, we obtain

$$h_j \cdot a = 0\,, \quad h_j \cdot Av_i = 0\,, \quad i = 1,3, \quad j = 2,4\,. \tag{7}$$

Using the fact that $h_j = -e_j$, $j \in \{1, \ldots, 4\}$, (6) and (7) imply that the system has the form

$$\dot{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} x + Bu + \begin{bmatrix} \gamma_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

Consider the intersection of $\mathcal{E}$ with the subspace $\mathcal{H}_4 :=$ aff $(\mathcal{F}_4) = \{x \in \mathbb{R}^4 \mid x_4 = 0\}$. It is characterized by the linear constraints $Ax + a = 0$ and $x_4 = 0$. That is,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \gamma_1 \;=\; 0 \tag{9a}$$
$$a_{22}x_2 \;=\; 0 \tag{9b}$$
$$a_{33}x_3 \;=\; 0\,. \tag{9c}$$

If $a_{22} \neq 0$, then any $x \in \mathcal{E} \cap \mathcal{H}_4$ must have $x_2 = 0$ to satisfy the second equation, a contradiction to $\mathcal{E}_{\mathcal{S}} \cap$ ri (co $\{v_0, \ldots, v_3\}) \neq \emptyset$. Therefore, $a_{22} = 0$. Similarly, $a_{33} = 0$. This implies (9) provides at most 1 independent equation to characterize $\mathcal{E} \cap \mathcal{H}_4$, so $\dim(\mathcal{E} \cap \mathcal{H}_4) \ge 2$. Then since $\mathcal{E} \cap \mathcal{H}_4$ is an affine space in $\mathbb{R}^3$ intersecting the interior of co $\{v_0, \ldots, v_3\}$, a simplex in $\mathbb{R}^3$, we have $\dim(\mathcal{E}_{\mathcal{S}}) = \dim(\mathcal{E} \cap \mathcal{H}_4) \ge 2$, a contradiction to $\mathcal{E}_{\mathcal{S}} =$ co $\{o_1, o_2\}$. $\lhd$

V. Multi-affine Feedback Synthesis

In this section we present a general method for synthesis of multi-affine feedbacks to solve RCP on simplices. We assume throughout the section that (possibly after an affine coordinate transformation) the simplex is in *canonical form*:

$$v_0 = 0\,, v_i = e_i\,, \qquad i = 1, \ldots, n\,,$$

where e_i is the ith Euclidean coordinate vector. Note that this is no loss of generality as any simplex can be transformed to canonical form.

Definition 5.1: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is multi-affine if it is a polynomial in $x_1, \cdots, x_n$ with the property that the degree of f in any x_i, $i \in I$, is either 0 or 1. Equivalently, f has the form:

$$f(x_1, \cdots, x_n) = \sum_{i_1, \cdots, i_n \in \{0,1\}} c_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

with $c_{i_1, \cdots, i_n} \in \mathbb{R}^m$ for all $i_1, \cdots, i_n \in \{0, 1\}$ and using the convention that if $i_k = 0$, then $x_k^{i_k} = 1$.

If $\mathcal{E}_{\mathcal{S}} = \emptyset$, then the conditions $Av_i + a \in \mathcal{C}(v_i)$, $i \in \{0, \ldots, n\}$, imply from Theorem 2.1 that $u = 0$ solves RCP on $\mathcal{S}$. Therefore, we focus our attention on the case when $\mathcal{E}_{\mathcal{S}} \neq \emptyset$. Following Theorem 4.4, let $\mathcal{E}_{\mathcal{S}} = \{o_1\}$ where $I(o_1) = \{0, \ldots, q\}$. Since $\mathcal{E}_{\mathcal{S}} \subset \partial\mathcal{S}$ (by Theorem 4.1), we have $q < n$. We consider a multi-affine feedback of the form

$$u(x) = x_1 x_2 \cdots x_q.$$

We show that this multi-affine feedback solves RCP on $\mathcal{S}$. First, we show that $u(x)$ satisfies the invariance conditions.

Lemma 5.1: Consider the system (1) defined on a simplex $\mathcal{S}$. Suppose Assumption 4.1 holds, $\mathcal{E}_{\mathcal{S}} = \{o_1\}$ with $I(o_1) = \{0, \cdots, q\}$, where $q < n$, and $m = 1$. Also, suppose

(N1) $Av_i + a \in \mathcal{C}(v_i)$, $i = 0, \ldots, n$.
(N2) $B \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$.

Then the multi-affine feedback $u(x) = x_1 \cdots x_q$ satisfies the invariance conditions (3).

Proof: Due to the assignment of vertices, $x_j = 0$ when $x \in \mathcal{F}_j$ for $j \in \{1, \ldots, n\}$. First, consider $j = 1, \ldots, q$. We have $u(x) = 0$ for $x \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_q$. By (N1) and convexity,

$$h_j \cdot (Ax + Bu(x) + a) = h_j \cdot (Ax + a) \le 0,\ x \in \mathcal{F}_j,\ j = 1, \ldots, q\,.$$

Next consider $j = q+1, \ldots, n$. By (N2), $B \in \mathcal{C}(o_1)$. That is,

$$h_j \cdot B \le 0\,,\ j \in I \setminus I(o_1) = \{q+1, \ldots, n\}\,. \tag{10}$$

Notice that with our assignment of vertices of $\mathcal{S}$, $u(x) \ge 0$ for all $x \in \mathcal{S}$. Then by combining (N1), convexity, and (10), we get

$$h_j \cdot (Ax + Bu(x) + a) \le 0\,, \quad x \in \mathcal{F}_j\,, \quad j = q+1, \ldots, n\,.$$

We conclude $u(x)$ satisfies (3). ■

Now we show that using our proposed multi-affine feedback $u(x) = x_1 \cdots x_q$, all closed-loop trajectories initiated in $\mathcal{S}$ leave it in finite time, and so $u(x)$ solves RCP on $\mathcal{S}$. For this we require a technical lemma.

Lemma 5.2: Let $\mathcal{H} := \{x \in \mathbb{R}^n \mid \rho_1 \cdot x = c_1, \ldots, \rho_r \cdot x = c_r\}$ be an affine space, where $\rho_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$. Consider the system $\dot{x} = Ax + a$ and suppose there exist $x_0 \in \mathcal{H}$ and $T > 0$ such that $\phi(t, x_0) \in \mathcal{H}$ for all $t \in [0, T)$. Then $\phi(t, x_0) \in \mathcal{H}$ for all t.

Proof: Let $\mathcal{H}_1$ be the smallest affine set containing $\{\phi(t, x_0) \mid t \in (0, T)\}$. Then $\mathcal{H}_1 \subseteq \mathcal{H}$, so it can be expressed as $\mathcal{H}_1 = \{x \in \mathbb{R}^n \mid \rho_1 \cdot x = c_1, \cdots, \rho_s \cdot x = c_s\}$, with $r \le s$. Now we have $\rho_j \cdot \phi(t, x_0) = c_j$ for $t \in (0, T)$ and $j \in \{1, \cdots, s\}$. Taking the derivative, $\rho_j \cdot (A\phi(t, x_0) + a) = 0$ for $t \in (0, T)$ and $j \in \{1, \cdots, s\}$. By definition, $\mathcal{H}_1$ is the affine hull of $\{\phi(t, x_0) \mid t \in (0, T)\}$, and so there exist $t_i \in (0, T)$, $i = 1, \ldots, z$, such that $\mathcal{H}_1 =$ aff $\{\phi(t_1, x_0), \ldots, \phi(t_z, x_0)\}$. That is, for each $x \in \mathcal{H}_1$, $x = \sum_{i=1}^{z} \alpha_i^x \phi(t_i, x_0)$ with $\sum_{i=1}^{z} \alpha_i^x = 1$. Then we have

$$\rho_j \cdot (Ax + a) = \sum_{i=1}^{z} \alpha_i^x \rho_j \cdot (A\phi(t_i, x_0) + a) = 0,$$
$$x \in \mathcal{H}_1,\ j = 1, \ldots, s.$$

By Nagumo's Theorem [5], $\mathcal{H}_1$ is an invariant set. Hence, $\phi(t, x_0) \in \mathcal{H}_1 \subseteq \mathcal{H}$ for all t. ■

The following is the main result of this section.

Theorem 5.3: Consider the system (1) defined on a simplex $\mathcal{S}$. Suppose Assumption 4.1 holds, $\mathcal{E}_{\mathcal{S}} = \{o_1\}$ with $I(o_1) = \{0, \cdots, q\}$, where $q < n$, and $m = 1$. Also suppose

(N1) $Av_i + a \in \mathcal{C}(v_i)$, $i = 0, \ldots, n$.

(N2) $B = Ao_{\kappa+1} + a \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$.

Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by $u(x) = x_1 \cdots x_q$.

Proof: The first step of the proof is to show there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot (Ax + Bu(x) + a) \leq 0$ for all $x \in \mathcal{S}$. The image of $\mathcal{S}^\circ$ under the affine map $x \mapsto Ax + a$, denoted $\mathcal{C}_1$, is convex and relatively open by Theorems 3.4 and 6.6 of [14]. By assumption, $\mathcal{E}_{\mathcal{S}} = \{o_1\} \subset \partial\mathcal{S}$. Therefore, $Ax + a \neq 0$ for all $x \in \mathcal{S}^\circ$. Thus, the nonempty relatively open convex set $\mathcal{C}_1$ and the nonempty affine set $\{0\}$ do not intersect. By Theorem 11.2 of [14], there exists a hyperplane $\mathcal{H}$ containing $\{0\}$ such that one of the open half-spaces associated with $\mathcal{H}$ contains $\mathcal{C}_1$. That is, there exists $\xi \in \mathbb{R}^n$ such that

$$\xi \cdot (Ax + a) < 0\,, \quad x \in \mathcal{S}^\circ\,. \tag{11}$$

By continuity of $x \mapsto (Ax + a)$, we get

$$\xi \cdot (Ax + a) \leq 0\,, \quad x \in \mathcal{S}\,. \tag{12}$$

Next we claim that $\xi \cdot B < 0$. For this we first show that $Ao_i + a = \gamma_i B$ with $\gamma_i \geq 0$, for all $i \in I_{\mathcal{O}_{\mathcal{S}}}$. First, $Ao_1 + a = 0$ so $\gamma_1 = 0$. Second, $\gamma_2, \ldots, \gamma_{\kappa+1}$ must all have the same sign. Otherwise, by convexity there exists $x \in$ co $\{o_2, \ldots, o_{\kappa+1}\}$ such that $Ax + a = 0$, which contradicts that $\mathcal{E}_{\mathcal{S}} = \{o_1\}$. Since by (N2), $\gamma_{\kappa+1} = 1$, we have $\gamma_i > 0$, $i = 2, \ldots, \kappa + 1$. Now let $x_o \in \mathcal{S}^\circ \cap \mathcal{O}_{\mathcal{S}}$. By Theorem 4.1, $x_o \in$ ri $(\mathcal{O}_{\mathcal{S}})$. That is, $x_o = \sum_{i=1}^{\kappa+1} \beta_i o_i$ with $\beta_i > 0$ and $\sum_{i=1}^{\kappa+1} \beta_i = 1$. Then $Ax_o + a = \sum_{i=1}^{\kappa+1} \beta_i(\gamma_i B) = \gamma B$ with $\gamma > 0$. Finally, by (11) we have

$$\xi \cdot B = \frac{1}{\gamma} \xi \cdot (Ax_o + a) < 0\,. \tag{13}$$

By the assignment of vertices for a canonical simplex, $u(x) \geq 0$ for all $x \in \mathcal{S}$. Combining (12) with (13) we have

$$\xi \cdot (Ax + Bu(x) + a) \leq 0\,, \qquad x \in \mathcal{S}\,. \tag{14}$$

Moreover,

$$\xi \cdot (Ax + Bu(x) + a) = 0 \iff \xi \cdot (Ax + a) = 0 \text{ and } u(x) = 0\,.$$

The second step of the proof is to use the flow-like condition (14) in the LaSalle Principle for RCP (Theorem 2.3) to show that all closed-loop trajectories exit $\mathcal{S}$. For this we identify the set

$$\mathcal{M} = \{x \in \mathcal{S} \mid \xi \cdot (Ax + a) = 0, u(x) = 0\}\,.$$

According to the LaSalle Principle for RCP, we must show that $\mathcal{M}$ does not contain any invariant set. Since $\mathcal{M} \subset \{x \in \mathcal{S} \mid u(x) = 0\} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_q$, the dynamics on $\mathcal{M}$ reduce to $\dot{x} = Ax + a$. Suppose by the way of contradiction that there exists $x_0 \in \mathcal{M}$ such that $\phi(t, x_0) \in \mathcal{M}$ for all $t \geq 0$. Since $\mathcal{M} \subset \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_q$, w.l.o.g. suppose $x_0 \in \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_k$ and $x_0 \notin \mathcal{F}_{k+1} \cap \cdots \cap \mathcal{F}_q$ for some $1 \leq k \leq q$. There are two cases. First, if $k = q$ then $\phi(t, x_0)$ cannot leave $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_q$ instantaneously (otherwise $\phi(t, x_0)$ leaves $\mathcal{M}$, a contradiction). Second, suppose $k < q$. Since $\mathcal{F}_{k+1} \cup \cdots \cup \mathcal{F}_q$ is a compact set and $x \mapsto Ax + a$ is locally Lipschitz, there exists $t' > 0$ such that $\phi(t, x_0) \notin (\mathcal{F}_{k+1} \cup \cdots \cup \mathcal{F}_q)$ for $t \in [0, t']$. Then $\phi(t, x_0)$ cannot leave $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ instantaneously (for if so, $\phi(t, x_0)$ leaves $\cup_{i=1}^q \mathcal{F}_i$ instantaneously, so it leaves $\mathcal{M}$, a contradiction). We conclude there exist $j \in \{1, \ldots, k\}$ and a time $T' > 0$ such that $\phi(t, x_0) \in \mathcal{F}_j$ for all $t \in [0, T')$. Applying Lemma 5.2, $\phi(t, x_0) \in$ aff $(\mathcal{F}_j)$ for all t. But by assumption $\phi(t, x_0) \in \mathcal{M}$ for $t \geq 0$, so we have $\phi(t, x_0) \in \mathcal{F}_j$ for $t \geq 0$. Now we show this is impossible.

Let $\mathcal{C}_2$ be the image of $\mathcal{F}_j$ under the mapping $x \mapsto Ax + a$. Since $\mathcal{F}_j$ is convex and compact, $\mathcal{C}_2$ is also convex and compact [14]. Since $\mathcal{E}_{\mathcal{S}} = \{o_1\}$, $I(o_1) = \{0, \ldots, q\}$, and $j \in \{1, \ldots, q\}$, we have $\mathcal{E}_{\mathcal{S}} \cap \mathcal{F}_j = \emptyset$. Therefore, $Ax + a \neq 0$, $x \in \mathcal{F}_j$. Thus, the compact convex sets $\mathcal{C}_2$ and $\{0\}$ do not intersect. By Corollary 11.4.2 of [14], there exists a hyperplane $\mathcal{H}$ that strongly separates $\mathcal{C}_2$ and $\{0\}$. In particular, there exists $\rho \in \mathbb{R}^n$ such that $\rho \cdot (Ax + a) < 0$ for $x \in \mathcal{F}_j$. This implies $\phi(t, x_0)$ leaves $\mathcal{F}_j$ in finite time, a contradiction. We conclude $\mathcal{M}$ does not contain invariant sets. Then by LaSalle Theorem (Theorem 2.3), all closed-loop trajectories initiated in $\mathcal{S}$ leave it in finite time. Since the invariance conditions hold (Lemma 5.1), the trajectories can do so only via $\mathcal{F}_0$. We conclude $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by $u(x)$. ■

Remark 5.1:
1) The above result is still true for the case when in condition $(N2)$, $B = c(Ao_{\kappa+1} + a)$, where $c > 0$, $c \neq 1$. If $-B = Ao_{\kappa+1} + a \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by $u(x) = -x_1 \cdots x_q$.
2) If the given simplex is not in canonical form, then the obtained multi-affine feedback $u(x)$ should be converted back to original coordinates. The control law in the original coordinates is continuous but not necessarily multi-affine (see Example 6.2).

VI. Examples

Example 6.1: Consider again the example presented in Section III. The objective is to show how our results in the previous section can be used to systematically synthesize multi-affine feedback that solves RCP on $\mathcal{S}$.

First, we check whether the conditions of Theorem 5.3 are achieved. As shown before, $\mathcal{O}_{\mathcal{S}} =$ co $\{o_1, o_2\}$, where $o_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0) \in \mathcal{F}_4$ and $o_2 = (\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{8}) \in \mathcal{F}_3$. Also, we have $\mathcal{E}_{\mathcal{S}} = \{o_1\} \subset \mathcal{F}_4$. Hence, in this example $\dim(\mathcal{O}_{\mathcal{S}}) = 1$, $\dim(\mathcal{E}_{\mathcal{S}}) = 0$, $\mathcal{O}_{\mathcal{S}} \cap \mathcal{S}^\circ \neq \emptyset$, and $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$. Thus, Assumption 4.1 holds. Also it can be verified that $u = 0$ satisfies the invariance conditions at the vertices of $\mathcal{S}$. Then as shown before in Section III, $B = (-3, -5, 8, 4) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}})$. We conclude that the conditions of Theorem 5.3 are achieved, and so $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by multi-affine feedback. In this example $q = 3$, and so $u(x) = x_1 x_2 x_3$, which is the same feedback used in the example in Section III.

The advantage here is that Theorem 5.3 enables us to find the multi-affine feedback systematically. Also, we don't need to have a separate proof for each example as we did in Lemma 3.2 and Proposition 3.3. $\triangleleft$

Example 6.2: Consider the following affine system

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 1 & 3 \\ -3 & -6 & 0 & -1 & 13 \\ 0 & 0 & 5 & 0 & -1 \\ 0 & 0 & 9 & -4 & -1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 8 \\ 9 \\ 17 \\ 5 \end{bmatrix} u + \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{15}$$

defined on a simplex $\mathcal{S} = \text{co}\ \{v_0, \cdots, v_5\}$, where $v_0 = 0$, $v_1 = e_1$, $v_2 = e_2$, $v_3 = (0, 1, 0, 1, 0)$, $v_4 = (0, 0, 1, 1, 0)$, and $v_5 = (1, 1, 1, 1, 1)$. The objective is to find a continuous state feedback that solves RCP on $\mathcal{S}$.

First, it can be verified that $u = 0$ solves the invariance conditions at the vertices of $\mathcal{S}$ ($Av_i + a \in \mathcal{C}(v_i)$, $i \in \{0, \cdots, 5\}$). Secondly, we compute the sets $\mathcal{O}_\mathcal{S}$ and $\mathcal{E}_\mathcal{S}$. It can be verified that $\mathcal{O}_\mathcal{S} = \text{co}\ \{o_1, o_2\}$, where $o_1 = (0.25, 0.5, 0, 0.25, 0) \in (\mathcal{F}_4 \cap \mathcal{F}_5)$ and $o_2 = (0.5568, 0.572, 0.2614, 0.2614, 0.1818) \in \mathcal{F}_3$. Also, $Ao_1 + a = 0$, while $Ao_2 + a \neq 0$. Therefore, in this example $\dim(\mathcal{O}_\mathcal{S}) = 1$, $\dim(\mathcal{E}_\mathcal{S}) = 0$, $\mathcal{O}_\mathcal{S} \cap \mathcal{S}^\circ \neq \emptyset$, and $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$. So, Assumption 4.1 holds. Then we check whether there exists $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$. Since $o_1 \in (\mathcal{F}_4 \cap \mathcal{F}_5)$ and $o_2 \in \mathcal{F}_3$, we have

$$\text{cone}(\mathcal{O}_\mathcal{S}) = \left\{ y \in \mathbb{R}^5 \mid h_3 \cdot y \leq 0,\ h_4 \cdot y \leq 0,\ h_5 \cdot y \leq 0 \right\},$$

where $h_3 = (0, 0, 1, -1, 0)$, $h_4 = (0, 0, -1, 0, 1)$, and $h_5 = (0, 0, 0, 0, -1)$. It can be verified that $B = (2, 8, 9, 17, 5) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$.

Although the necessary conditions for solvability of RCP by continuous state feedback are achieved, it can be shown using an argument similar to the one used in Lemma 3.1 that RCP is not solvable by affine feedback. Instead, the conditions of Theorem 5.3 are achieved, and so RCP is solvable by continuous state feedback.

To find this feedback, we firstly construct the coordinate transformation matrix $T = [v_1 \ \cdots \ v_5]$, and define $z = T^{-1}x$. In the new coordinates, the transformed vertices are $e_i = T^{-1}v_i$, $i \in \{0, \cdots, 5\}$, and the transformed unit normal vectors are $-e_j$, $j \in \{1, \cdots, 5\}$. In this example $o_1 \in (\mathcal{F}_4 \cap \mathcal{F}_5)$, and so $I(o_1) = \{0, \cdots, 3\}$. Theorem 5.3 tells us that the multi-affine feedback $u(z) = z_1 z_2 z_3$ solves RCP on $\mathcal{S}$. In the original coordinates the continuous control law is $u(x) = (x_1 - x_5)(x_2 + x_3 - x_4 - x_5)(-x_3 + x_4)$. Note that another method of solving this example is to firstly transform the system to the new coordinates, then check the conditions of Theorem 5.3 in the new coordinates, and finally return the feedback to the original coordinates. $\triangleleft$

VII. CONCLUSION

In this paper we have shown that under arbitrary triangulations of the state space, affine feedback and continuous state feedback are not equivalent for RCP on simplices. We present an example for which no solution based on affine feedback exists, yet a continuous state feedback solves the problem. Then we have identified for single-input affine systems an alternative feedback class for RCP on simplices when affine feedbacks fail to solve the problem. We first explored significant geometric properties of the equilibrium set. Then we used these properties to synthesize multi-affine feedbacks for RCP on simplices.

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APPENDIX

Proof: [Proof of Theorem 4.4] Suppose, by the way of contradiction, that $\dim(\mathcal{E}_\mathcal{S}) = \kappa_0 > 0$. By Assumption 4.1 (A2), $\mathcal{E}_\mathcal{S}$ is a simplex. In particular, $\mathcal{E}_\mathcal{S} = \text{co}\ \{o_1, \cdots, o_{\kappa_0+1}\}$. From Theorem 4.1, we have $\mathcal{E}_\mathcal{S} \subset \text{rb}\ (\mathcal{O}_\mathcal{S}) \subset \partial\mathcal{S}$. Also, by Assumption 4.1 (A4), $\mathcal{E}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$. So, suppose w.l.o.g. that $\mathcal{E}_\mathcal{S} \subset \text{co}\ \{v_0, v_1, \cdots, v_q\}$, where $q < n$ is the smallest index satisfying that. Let $\mathcal{S}' := \text{co}\ \{v_0, \cdots, v_q\}$ and $\mathcal{E}_{\mathcal{S}'} := \mathcal{E} \cap \mathcal{S}'$. Clearly, $\mathcal{E}_{\mathcal{S}'} = \mathcal{E}_\mathcal{S}$ and $\mathcal{E}_{\mathcal{S}'} \cap \text{ri}\ (\mathcal{S}') \neq \emptyset$. Note that $\mathcal{E}_{\mathcal{S}'}$ can be expressed as the intersection of the affine space $\mathcal{E} \cap \text{aff}\ (\mathcal{F}_{q+1}) \cap \cdots \cap \text{aff}\ (\mathcal{F}_n)$

in $\mathbb{R}^q$ with $\mathcal{S}'$, a simplex in $\mathbb{R}^q$. So, we have rb $(\mathcal{E}_{\mathcal{S}'}) \subset$ rb $(\mathcal{S}')$ (Lemma 1 of [17]). Since $\mathcal{E}_{\mathcal{S}'} \subset \mathcal{S}'$, rb $(\mathcal{E}_{\mathcal{S}'}) \subset$ rb $(\mathcal{S}')$, $\mathcal{E}_{\mathcal{S}'} \cap$ ri $(\mathcal{S}') \neq \emptyset$, and by Assumption 4.1 (A4) $\mathcal{E}_{\mathcal{S}'} \cap \mathcal{F}_0 = \emptyset$, then each index set $I(o_i)$, $i \in \{1, \cdots, \kappa_0 + 1\}$, has an exclusive member in $\{1, \cdots, q\}$ (Lemma 2 of [17]). That is, there exists $k \in I(o_i),\ k \neq 0$, and $k \notin I(o_j),\ j \in \{1, \cdots, \kappa_0 + 1\} \setminus \{i\}$. Suppose w.l.o.g. that we reorder the vertex labeling $\{1, \cdots, q\}$ such that the indices belonging to more than one set $I(o_i)$, $i \in \{1, \cdots, \kappa_0 + 1\}$, come first. Then we bring the indices corresponding to the exclusive members of $I(o_1), \cdots, I(o_{\kappa_0+1})$ respectively.
Now we study a vertex $o_k \in \mathcal{E}_{\mathcal{S}'}$, $k \in \{1, \cdots, \kappa_0 + 1\}$. We have by definition $Ao_k + a = 0$, and so

$$h_j \cdot (Ao_k + a) = 0, \quad j \in I.$$

We know $o_k = \sum_{i \in I(o_k)} \alpha_i v_i$, where $\alpha_i > 0$ and $\sum_{i \in I(o_k)} \alpha_i = 1$. So,

$$h_j \cdot (A \sum_{i \in I(o_k)} \alpha_i v_i + a) = 0, \quad j \in I.$$

Since $\sum_{i \in I(o_k)} \alpha_i = 1$, this implies

$$\sum_{i \in I(o_k)} \alpha_i h_j \cdot (Av_i + a) = 0, \quad j \in I. \tag{16}$$

Since $Av_i + a \in \mathcal{C}(v_i),\ i \in \{0, \cdots, n\}$, we must have

$$h_j \cdot (Av_i + a) \leq 0, \quad i \in I(o_k), \quad j \in I \setminus I(o_k). \tag{17}$$

Since $\alpha_i > 0$, (16) and (17) imply that

$$h_j \cdot (Av_i + a) = 0,\ i \in I(o_k),\ j \in I \setminus I(o_k),\ k \in \{1, \cdots, \kappa_0 + 1\}$$

Since $0 \in I(o_k)$ and $v_0 = 0$, we have

$$h_j \cdot a = 0, \quad j \in I \setminus I(o_k), \tag{18a}$$

$$h_j \cdot Av_i = 0, \quad i \in I(o_k), \quad j \in I \setminus I(o_k), \tag{18b}$$

for all $k \in \{1, \cdots, \kappa_0 + 1\}$.
Then we define the coordinate transformation $z = T^{-1}x$, where $T = [v_1 \cdots v_n]$. Notice that since by definition $\{v_0, v_1, \cdots, v_n\}$ are affinely independent and $v_0 = 0$, T is non-singular [7]. It is easy to verify that the transformed vertices are $e_i = T^{-1}v_i, i \in \{0, \cdots, n\}$, and the transformed unit normal vectors are $-e_j = c_j T^T h_j,\ j \in \{1, \cdots, n\}$, where c_j are positive scalars. Also, using (18) it can be verified that the dynamics in the new coordinates are

$$\dot{z} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} & \cdots & \Gamma_{0(\kappa_0+1)} & \Gamma'_{02} \\ & \Gamma_{11} & & & \Gamma'_{12} \\ & & \ddots & & \vdots \\ & & & \Gamma_{(\kappa_0+1)(\kappa_0+1)} & \Gamma'_{(\kappa_0+1)2} \\ & & & & A_2 \end{bmatrix} \begin{bmatrix} Z_{10} \\ Z_{11} \\ \vdots \\ Z_{1(\kappa_0+1)} \\ Z_2 \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_{10} \\ b_{11} \\ \vdots \\ b_{1(\kappa_0+1)} \\ b_2 \end{bmatrix} u, \tag{19}$$

where empty entries are zeros, Z_{10} is a partition of z corresponding to the indices appearing in more than one set $I(o_i)$, $i \in \{1, \cdots, \kappa_0 + 1\}$, and $Z_{11}, \cdots, Z_{1(\kappa_0+1)}$ are partitions corresponding to the exclusive members of $I(o_1), \cdots, I(o_{\kappa_0+1})$ respectively. Based on the above discussion, $\dim(Z_{1i}) \geq 1,\ i \in \{1, \cdots, \kappa_0 + 1\}$. Finally, Z_2 corresponds to the indices $\{q + 1, \cdots, n\}$.
Then the affine set $\mathcal{E} \cap$ aff $(\mathcal{F}_{q+1}) \cap \cdots \cap$ aff $(\mathcal{F}_n)$ can be characterized by the equations characterizing $\mathcal{E}$ and $Z_2 = 0$, which reduce to the following set of equations

$$\begin{aligned} \Gamma_{00}Z_{10} + \Gamma_{01}Z_{11} + \cdots + \Gamma_{0(\kappa_0+1)}Z_{1(\kappa_0+1)} + \gamma_1 = 0, \\ \Gamma_{11}Z_{11} = 0, \\ \vdots \\ \Gamma_{(\kappa_0+1)(\kappa_0+1)}Z_{1(\kappa_0+1)} = 0. \end{aligned} \tag{20}$$

Suppose that $\dim(\Gamma_{ii}) = p_i \times p_i,\ i \in \{1, \cdots, \kappa_0 + 1\}$. If $\mathrm{rank}(\Gamma_{11}) = p_1$, then any $z \in \mathcal{E} \cap$ aff $(\mathcal{F}_{q+1}) \cap \cdots \cap$ aff $(\mathcal{F}_n)$ must have $Z_{11} = 0$, a contradiction to $\mathcal{E}_{\mathcal{S}'} \cap$ ri $(\mathcal{S}') \neq \emptyset$. So, $\mathrm{rank}(\Gamma_{11}) < p_1$. Similarly, $\mathrm{rank}(\Gamma_{ii}) < p_i,\ i \in \{2, \cdots, \kappa_0 + 1\}$. Therefore, (20) provides at most $q - (\kappa_0 + 1)$ independent equations to characterize $\mathcal{E} \cap$ aff $(\mathcal{F}_{q+1}) \cap \cdots \cap$ aff $(\mathcal{F}_n)$. So, $\dim(\mathcal{E} \cap$ aff $(\mathcal{F}_{q+1}) \cap \cdots \cap$ aff $(\mathcal{F}_n)) \geq \kappa_0 + 1$. Finally, $\mathcal{E} \cap$ aff $(\mathcal{F}_{q+1}) \cap \cdots \cap$ aff $(\mathcal{F}_n)$ is an affine space in $\mathbb{R}^q$ that intersects the interior of $\mathcal{S}'$, a simplex in $\mathbb{R}^q$. So, $\dim(\mathcal{E}_{\mathcal{S}'}) = \dim(\mathcal{E} \cap$ aff $(\mathcal{F}_{q+1}) \cap \cdots \cap$ aff $(\mathcal{F}_n)) \geq \kappa_0 + 1$, a contradiction. ■