Monotonic Reach Control on Polytopes

Mohamed K. Helwa and Mireille E. Broucke

***Abstract*—The paper studies the problem of making the trajectories of an affine system defined on a polytopic state space reach a prescribed facet of the polytope in finite time without first leaving the polytope. The focus is on solvability by continuous piecewise affine feedback, and we formulate a variant of the problem in which trajectories exit in a monotonic sense. This allows to obtain necessary and sufficient conditions for solvability in certain geometric situations.**

I. INTRODUCTION

We study the reach control problem (RCP) for affine systems on polytopes. The problem is to design a feedback to force closed-loop trajectories starting anywhere in a polytopic state space to leave the polytope from a prescribed exit facet of the polytope in finite time [11]. Unlike previous work, here it is not required that trajectories leave the polytope at the first time they reach the exit facet. The main motivation behind RCP for affine systems on polytopes is the control of a subclass of hybrid systems called piecewise affine hybrid systems [12], [5]. A piecewise affine hybrid system consists of a discrete automaton such that each discrete mode is equipped with its own continuous-time affine dynamics defined on a polytope. When the continuous state crosses a facet of a polytope, the system is transferred to a new discrete mode. Reach control for piecewise affine hybrid systems requires at each discrete mode to prevent transitions to certain discrete modes, and to force a transition to a desired discrete mode. This requirement is translated at the continuous level to force all the state trajectories of a continuous-time affine system defined on a polytope to leave the polytope through a prescribed exit facet in finite time - that is, to solve RCP for an affine system on a polytope [12]. Interesting applications of RCP can include motion of robots in complex environments [2], aircraft and underwater vehicles [3], anesthesia, genetic networks [4], smart buildings, process control, among others [10].

The most definitive results on RCP are focused on reach control on simplices by affine feedback [12], [17], [6]. Results for polytopes come in one of two forms. Either one must perform a triangulation of the polytope and apply simplex-based reach control methods [12], [17]. Alternatively, one may impose conditions so that the design can be carried out in two independent steps: first one assigns control inputs at the vertices of the polytope guaranteeing propitious closed-loop behavior; second, one selects any triangulation of the polytope and one forms a (continuous) piecewise affine feedback based on the vertex control values of step one. The distinction between these two approaches is that simplex-based methods require stronger conditions on the vector field at the vertices of the triangulation; for instance, closed-loop trajectories can only exit from one exit facet of a given simplex. Instead, the second approach imposes weaker conditions at the vertices; for example, trajectories could exit from more than one facet of a given simplex. The penalty for the more relaxed requirements of the second method is that trajectories may not actually achieve the specification to exit the polytope. To guarantee this, an extra, exogenous condition must be added. It can be shown both theoretically and via examples that the two methods are complementary.

Past research on reach control on polytopes has either required strong sufficient conditions or restrictive assumptions on the system dynamics [11], [15]. This paper initiates a study of the reach control problem in which such restrictions are removed; instead geometric properties of the system are exploited to the best possible extent. In particular, the placement of $\mathcal{O}$, the set of possible equilibria, relative to the polytope $\mathcal{P}$ plays a key role, and in certain cases, clear necessary and sufficient conditions can be obtained which remove the conservativism or restrictiveness of previous work. We then formulate the *monotonic reach control problem* (MRCP) where it is required that trajectories exit the polytopic state space in a monotonic sense relative to a foliation of parallel hyperplanes.

Recent results on RCP include [7], [8], [1], [9], [18], [19] . Those results exploit system structure on simplices, particularly the reach control indices [9] and the concept of reach controllability [19]. Such structure is inexistent on polytopes due to their high combinatorial complexity. Instead in this paper we adopt a more computational approach. Finally, a preliminary version of this note appeared in [13]. While [13] focused on the differences between MRCP and the simplex-based approach [12], [6] and on the relationship between MRCP and solvability of RCP by arbitrary triangulations, this note focuses on providing solution methods for MRCP. Also, here we include all proofs, important technical remarks (e.g. Remark 4.1), and an efficient algorithm for MRCP (Section V).

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$.

II. REACH CONTROL PROBLEM

Consider an n-dimensional polytope

$$\mathcal{P} := \text{co}\ \{v_1, \ldots, v_p\}$$

with vertex set $V := \{v_1, \ldots, v_p\}$ and facets $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$. The *target set* is the facet $\mathcal{F}_0$ of $\mathcal{P}$. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index

The authors are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: mkhelwa@scg.utoronto.ca, broucke@control.utoronto.ca). Supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

sets $I := \{1,\ldots,p\}$ and $J := \{1,\ldots,r\}$. For each $x \in \mathcal{P}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \, y \in \mathbb{R}^n \mid h_j \cdot y \le 0, \; j \in J \; s.t. \; x \in \mathcal{F}_j \right\}.$$

We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a \,, \qquad\qquad x \in \mathcal{P} \,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\phi_u(t, x_0)$ be the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the target set $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 2.1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \ge 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{P}$ reach and exit the target $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. Notice that in order for condition (i) to make sense it is assumed that the dynamics (1) are extended to a small neighborhood of $\mathcal{P}$. In the sequel we use the shorthand notation $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ to denote that condition (i) of Problem 2.1 holds for some control.

Definition 2.1: We say the *invariance conditions are solvable* if for each $v \in V$ there exists $u \in \mathbb{R}^m$ such that

$$Av + Bu + a \in \mathcal{C}(v) \,. \tag{2}$$

Equation (2) is referred to as the *invariance conditions* either for a specific vertex, or collecting all conditions for all vertices, for a polytope. The relevance of the invariance conditions to RCP is that they ensure that trajectories only exit $\mathcal{P}$ via $\mathcal{F}_0$ under affine or PWA feedback [11].

Let $\mathcal{B} = \text{Im}\; B$, the image of B. Define the set

$$\mathcal{O} := \{ \, x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B} \, \} \,. \tag{3}$$

Notice that the vector field $Ax + Bu + a$ can vanish at any $x \in \mathcal{O}$ for an appropriate choice of $u \in \mathbb{R}^m$, so $\mathcal{O}$ is the set of all possible equilibrium points of (1). That is, if x_0 is an equilibrium of (1) under feedback control, then $x_0 \in \mathcal{O}$. It can be verified that either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ is an affine space with dimension between m and n. We also define the set of possible equilibrium points of (1) on $\mathcal{P}$ by

$$\mathcal{O}_{\mathcal{P}} := \mathcal{P} \cap \mathcal{O} \,. \tag{4}$$

Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_{\mathcal{P}} = \emptyset$ or $\mathcal{O}_{\mathcal{P}}$ is a κ-dimensional polytope in $\mathcal{P}$. If $\mathcal{O}_{\mathcal{P}} \neq \emptyset$, we define the vertex set of $\mathcal{O}_{\mathcal{P}}$ to be $V_{\mathcal{O}} = \{o_1, \ldots, o_q\}$, where o_i are the vertices of $\mathcal{O}_{\mathcal{P}}$ (not necessarily vertices of $\mathcal{P}$). Also define the index set $I_{\mathcal{O}} = \{1, \ldots, q\}$. Finally, we review the definition of triangulation [14]. A triangulation $\mathbb{T}$ of a polytope $\mathcal{P}$ is a subdivision of $\mathcal{P}$ into full dimensional simplices $\mathcal{S}_1, \cdots, \mathcal{S}_L$ such that (i) $\mathcal{P} = \cup_{i=1}^{L} \mathcal{S}_i$, (ii) For all $i, \; j \in \{1, \cdots, L\}$ with $i \neq j$, the intersection $\mathcal{S}_i \cap \mathcal{S}_j$ is either empty or a common face of $\mathcal{S}_i$ and $\mathcal{S}_j$.

III. From Simplices to Polytopes

In this section, we focus on continuous state feedbacks. It is known that for simplices, RCP is solvable by affine feedback if and only if two conditions hold: (a) the invariance conditions (2) are solvable, and (b) the unique affine feedback built from one such solution does not admit a closed-loop equilibrium in the simplex [12], [17]. The no-equilibrium requirement can also be expressed as a so-called *flow condition*, which gives an equivalent numerical test [17]. We are interested to obtain the most immediate extension of this result for polytopes. First, we restrict our attention to continuous piecewise affine (PWA) feedback. Assuming PWA feedback, the invariance conditions remain necessary conditions for solvability of RCP on polytopes [11]. Instead, the flow condition is no longer necessary for solvability on polytopes. Indeed the statement that there is no closed-loop equilibrium is no longer equivalent to existence of a flow condition when dealing with general polytopes, because the equivalence relies on the convexity of the closed-loop vector field. Convexity is preserved with affine feedback, but it may not be with PWA feedback. On the other hand, the flow condition affords useful properties; particularly that trajectories exit the polytope in an orderly way. In this section we begin an exploration of the extent to which results for simplices carry over to polytopes. Guided by these insights, we formulate in Section IV a restricted version of RCP: we incorporate the requirement of a flow condition into the problem statement, and we call this restricted problem *monotonic reach control*.

Let $\mathbb{T}$ be a triangulation of polytope $\mathcal{P}$. A point $x \in \mathcal{P}$ lies in the interior of precisely one simplex $\mathcal{S}_x$ in $\mathbb{T}$ whose vertices are, say, $v_1, \ldots, v_k$. Then $x = \sum_{i=1}^{k} \lambda_i v_i$, where $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. Coefficients $\lambda_1, \ldots, \lambda_k$ are called the *barycentric coordinates* of x. Given a state feedback $u(x)$ on $\mathcal{P}$, we say u is a *piecewise affine feedback* associated with $\mathbb{T}$ if for any $x \in \mathcal{P}$, $x = \sum_i \lambda_i v_i$ implies $u(x) = \sum_i \lambda_i u(v_i)$, where $\{v_i\}$ are the vertices of $\mathcal{S}_x$ and the λ_i are the barycentric coordinates of x. It is easy to show that $u(x)$ is a continuous state feedback on $\mathcal{P}$.

Remark 3.1: If $u(x)$ is a piecewise affine feedback on $\mathcal{P}$, then for each n-dimensional simplex $\mathcal{S}_k \in \mathbb{T}$, there exist $K_k \in \mathbb{R}^{m\times n}$ and $g_k \in \mathbb{R}^m$ such that $u(x)$ takes the form $u(x) = K_k x + g_k$, $x \in \mathcal{S}_k$.

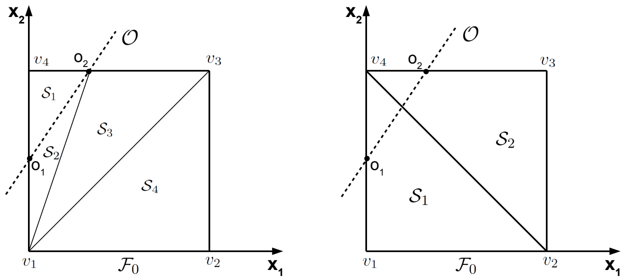
We say $\mathbb{T}$ is a *triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$* if $\mathbb{T}$ is a refinement of a subdivision of the point set $V \cup V_{\mathcal{O}}$ such that $\mathcal{O}_{\mathcal{P}}$ is a union of simplices in $\mathbb{T}$.

Example 3.1: Consider the polytope in Figure 1. In Figure 1(a) $\mathcal{O}_{\mathcal{P}} = co\{o_1, o_2\}$ is a 1-dimensional simplex in $\mathbb{T}$, so we say $\mathbb{T}$ is a triangulation with respect to $\mathcal{O}$. In Figure 1(b) $\mathcal{O}_{\mathcal{P}}$ cannot be expressed as a union of simplices in $\mathbb{T}$, so $\mathbb{T}$ is not a triangulation with respect to $\mathcal{O}$.

Suppose we are given a triangulation $\mathbb{T}$ of $\mathcal{P}$ with respect to $\mathcal{O}$ and we are given $u(x)$, a piecewise affine feedback defined on $\mathbb{T}$ which satisfies the invariance conditions of $\mathcal{P}$. Define

$$b_i := Ao_i + Bu(o_i) + a \in \mathcal{B} \cap \mathcal{C}(o_i) \,, \qquad\qquad i \in I_{\mathcal{O}} \,. \tag{5}$$

If we want to exclude closed-loop equilibria in $\mathcal{P}$, then we only need to concentrate on the behavior of the closed-loop



(a) (b)

Fig. 1. Two triangulations of $\mathcal{P}$. (a) $\mathbb{T}$ is a triangulation with respect to $\mathcal{O}$, and (b) $\mathbb{T}$ is not.

vector field in $\mathcal{O}_{\mathcal{P}}$. A basic result of convex analysis says that there are no closed-loop equilibria in $\mathcal{O}_{\mathcal{P}}$ if there is a flow condition on $\mathcal{O}_{\mathcal{P}}$.

Lemma 3.1: Let $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B}\}$ be such that $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$. Then there exists $\beta \in \mathcal{B}$ such that $\beta \cdot b_i < 0$, $i = 1, \ldots, q$.

The condition that $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$ can be related to the existence of closed-loop equilibria in $\mathcal{P}$.

Theorem 3.2: Consider the system (1) defined on a polytope $\mathcal{P}$. Let $\mathbb{T}$ be a triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$, $u(x)$ be a piecewise affine feedback defined on $\mathbb{T}$, and b_i be as in (5). If $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$, then the closed-loop system has no equilibrium in $\mathcal{P}$.

Proof: Let $x \in \mathcal{O}_{\mathcal{P}}$, and without loss of generality, suppose $x = \sum_{i=1}^{k} \lambda_i o_i$, where λ_i are the barycentric coordinates of x such that $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Let $\beta \in \mathcal{B}$ be as in Lemma 3.1. Since $\mathcal{O}_{\mathcal{P}}$ is a union of simplices in $\mathbb{T}$, and $y(x) := Ax + Bu(x) + a$ is affine on each simplex, we have $\beta \cdot y(x) = \beta \cdot \left(\sum_{i=1}^{k} \lambda_i y(o_i)\right) = \sum_{i=1}^{k} \lambda_i (\beta \cdot b_i) < 0$. Thus, $y(x) \neq 0$ for all $x \in \mathcal{O}_{\mathcal{P}}$. Since $y(x) \neq 0$ for all $x \in \mathcal{P} \setminus \mathcal{O}_{\mathcal{P}}$, the result is obtained. ∎

The previous theorem gives a general condition in order that the closed-loop system has no equilibrium in $\mathcal{P}$. In [6], two geometric sufficient conditions were presented to guarantee that there are no closed-loop equilibria in a given simplex. The first condition was that $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, where, $\text{cone}(\mathcal{S})$ is the tangent cone to simplex $\mathcal{S}$ at the vertex not contained in the exit facet $\mathcal{F}_0$. The second condition was that there is a set of linearly independent vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$, where it is assumed that $v_1, \ldots, v_q$ are the vertices of $\mathcal{S} \cap \mathcal{O}$. We would like to translate these two geometric conditions for simplices to the more general setting of polytopes. To that end, define

$$\text{cone}(\mathcal{O}_{\mathcal{P}}) := \bigcap_{o \in V_{\mathcal{O}}} \mathcal{C}(o)\,.$$

In particular, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ is the cone of directions in $\mathcal{B}$ that simultaneously satisfy the union of all invariance conditions at all vertices of $\mathcal{O}_{\mathcal{P}}$.

Lemma 3.3: Suppose $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$. Then there exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$ such that $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$.

Proof: Select any $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ and set $b_i = b$ for $i \in I_{\mathcal{O}}$. Since $\text{cone}(\mathcal{O}_{\mathcal{P}}) \subset \mathcal{C}(o_i)$, we have $b_i \in \mathcal{B} \cap \mathcal{C}(o_i)$ for all $o_i \in V_{\mathcal{O}}$. Clearly $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$. ∎

Next, consider the condition for a simplex $\mathcal{S}$ that there is a linearly independent set of vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$. Removing the restriction that vertices of $\mathcal{O}_{\mathcal{P}}$ are vertices of $\mathcal{P}$, we have the following analogous condition for polytopes.

Lemma 3.4: Suppose there exists a linearly independent set of vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$. Then $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$.

The previous two conditions provide the analogy for polytopes of the related geometric conditions for simplices. Then based on Theorem 3.2, both of the previous conditions imply there is no closed-loop equilibrium in $\mathcal{P}$, assuming $\mathcal{P}$ is triangulated with respect to $\mathcal{O}$. Unfortunately, in contrast with the situation for simplices, a no-equilibrium condition is not enough to deduce that RCP is solved, as discussed before. For this reason we bring in the flow condition explicitly in the problem statement; this approach is developed next.

IV. Monotonic Reach Control Problem

Problem 4.1 (Monotonic Reach Control Problem (MRCP)): Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

(ii) There exists $\xi \in \mathbb{R}^n$ such that for all $x \in \mathcal{P}$, $\xi \cdot (Ax + Bu(x) + a) < 0$.

The new condition (ii) is called a *flow condition*, and the problem is called "monotonic" because trajectories flow through the polytope in a common sense with respect to a foliation of parallel hyperplanes with normal vector ξ. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically if properties (i)-(ii) of Problem 4.1 hold.

Although MRCP is a restricted version of RCP, it is more general than the existing technique that depends on imposing the sufficient condition $h_0 \cdot (Ax + Bu(x) + a) > 0,\ x \in \mathcal{P}$ [11]. Clearly, this strong sufficient condition is a special case of the flow condition in which $\xi = -h_0$.

Remark 4.1: It can be easily recognized that algebraic necessary and sufficient conditions for solvability of MRCP by continuous PWA feedback can be obtained directly based on the control values at the vertices of $\mathcal{P}$. In particular, MRCP is solvable by continuous PWA feedback if and only if there exist $\xi \in \mathbb{R}^n$ and $u(v_i) \in \mathbb{R}^m,\ i \in I$, such that

$$\xi \cdot (Av_i + Bu(v_i) + a) < 0, \quad i \in I \tag{6}$$

$$h_j \cdot (Av_i + Bu(v_i) + a) \leq 0, \quad i \in I, \quad j \in J\ s.t.\ v_i \in \mathcal{F}_j. \tag{7}$$

However, the inequalities (6) are bilinear inequalities whose solving is NP hard. In [12] and [17], numerical algorithms were proposed for RCP on simplices to convert the bilinear inequalities associated with the flow condition to a series of linear programming (LP) problems whose number increases exponentially with the system dimension. Instead of these computationally expensive techniques, we explore the geometric conditions for solvability of MRCP in this section, which will lead to efficient synthesis methods.

Now we investigate geometric necessary and sufficient conditions for solvability of MRCP under assumptions on the placement of $\mathcal{O}$ with respect to $\mathcal{P}$. The first result when $\mathcal{O}_{\mathcal{P}} = \emptyset$ is based on the following technical lemma.

Lemma 4.1: Consider the system (1) defined on a compact, convex set $\mathcal{A}$. If $\mathcal{A} \cap \mathcal{O} = \emptyset$, then there exists $\beta \in$ Ker (B^T) such that $\beta \cdot (Ax + Bu + a) < 0$, for all $x \in \mathcal{A}$ and $u \in \mathbb{R}^m$.

Proof: Since $\mathcal{A}$ is compact and convex, the image of $\mathcal{A}$ under the affine map $x \mapsto Ax + a$, denoted $\mathcal{W}_1 = A(\mathcal{A}) + a$ is also compact and convex. Also, $\mathcal{W}_1 \cap \mathcal{B} = \emptyset$. For suppose not. Then there is a point $x \in \mathcal{A}$ such that $Ax + a \in \mathcal{B}$. Then $x \in \mathcal{O}$, by definition, which contradicts $\mathcal{A} \cap \mathcal{O} = \emptyset$. Note that both $\mathcal{W}_1$ and $\mathcal{B}$ are convex sets, and that $\mathcal{W}_1$ is bounded. By Corollary 11.4.2 in [16], there exists a hyperplane $\mathcal{H}$ separating $\mathcal{B}$ and $\mathcal{W}_1$ strongly. This implies $\mathcal{B}$ is parallel to $\mathcal{H}$ since $\mathcal{B}$ is a subspace. Let β be the normal vector to $\mathcal{H}$ pointing to the side containing $\mathcal{B}$. Then, $\beta \in$ Ker (B^T) and $\beta \cdot (Ax + a) < 0$ for all $x \in \mathcal{A}$. Since $\beta \cdot B = 0$, the result follows. ■

Theorem 4.2: Consider the system (1) defined on a polytope $\mathcal{P}$, and suppose $\mathcal{O}_{\mathcal{P}} = \emptyset$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by continuous piecewise affine feedback if and only if the invariance conditions (2) are solvable.

Proof: ($\Longrightarrow$) Follows from the necessity of the invariance conditions [11]. ($\Longleftarrow$) Select the control $u_i \in \mathbb{R}^m$ for each vertex $v_i \in V$ to satisfy the invariance conditions (2). Form a triangulation $\mathbb{T}$ of $\mathcal{P}$. Using the method of [11], one can find unique K_j and g_j corresponding to the affine feedback $u(x) = K_j x + g_j$ on each simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = u_i$, $i = 1, \ldots, p$. We obtain the piecewise affine closed-loop system $\dot{x} = (A + BK_j)x + (a + Bg_j)$. (Note that since $\mathcal{P} \cap \mathcal{O} = \emptyset$, the closed-loop system has no equililbria in $\mathcal{P}$.) By Lemma 4.1 there exists $\beta \in$ Ker (B^T) such that $\beta \cdot (Ax + Bu(x) + a) = \beta \cdot (Ax + a) < 0$, $x \in \mathcal{P}$. By a standard argument, all trajectories exit $\mathcal{P}$ in finite time. Moreover, since the invariance conditions (2) hold, trajectories in $\mathcal{P}$ exit through $\mathcal{F}_0$ [11]. Thus, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback. ■

Suppose we have found $\xi \in \mathbb{R}^n$ and $\epsilon > 0$ such that $\xi \cdot (Ax + Bu(x) + a) < -\epsilon$, $x \in \mathcal{P}$. Then, it can be easily shown that an upper bound on the time to leave $\mathcal{P}$ is $T_u = \frac{\max_{x \in \mathcal{P}} \xi \cdot x - \min_{x \in \mathcal{P}} \xi \cdot x}{\epsilon}$. For more details, refer to Remark 4.9 of [11].

In [6] necessary and sufficient conditions for solvability of RCP on simplices were obtained based on the assumption that $\mathcal{S} \cap \mathcal{O}$ is a face of the simplex. The same assumption for polytopes makes possible a straightforward generalization to polytopes for solvability of MRCP.

Assumption 4.1: Polytope $\mathcal{P}$ and system (1) satisfy the following condition: $\mathcal{O}_{\mathcal{P}}$ is a κ-dimensional face of $\mathcal{P}$, where $0 \le \kappa \le n$. In particular, $\mathcal{O}_{\mathcal{P}} =$ co $\{v_1, \ldots, v_q\}$, where v_i is a vertex of $\mathcal{P}$, and let $V_{\mathcal{O}} := \{v_1, \ldots, v_q\}$.

Theorem 4.3: Consider the system (1) defined on $\mathcal{P}$ and suppose Assumption 4.1 holds. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by continuous piecewise affine feedback if and only if

(i) The invariance conditions (2) are solvable.
(ii) There exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ such that $0 \notin$ co $\{b_1, \ldots, b_q\}$.

Proof: ($\Longrightarrow$) Let $y(x) := Ax + Bu(x) + a$, where $u(x)$ is the PWA feedback achieving $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. Since $u(x)$ is a continuous state feedback, the invariance conditions are solvable [11]. Now suppose that condition (ii) does not hold. This implies $0 \in$ co $\{y(v_1), \ldots, y(v_p)\}$. On the other hand, by the assumption that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y(v_i) < 0$ for $i \in I$. This implies $\mathbf{0}$ and co $\{y(v_1), \ldots, y(v_p)\}$ are strongly separated, a contradiction.

($\Longleftarrow$) For each vertex $v_i \in V \setminus \mathcal{O}_{\mathcal{P}}$, select a control $u_i \in \mathbb{R}^m$ to satisfy the invariance conditions (2). For $v_i \in V_{\mathcal{O}}$, select $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. Form a triangulation $\mathbb{T}$ of $\mathcal{P}$. Using the method of [11], one can find unique K_j and g_j corresponding to the affine feedback $u(x) = K_j x + g_j$ on each n-dimensional simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = u_i$, $i = 1, \ldots, p$ and $y(v_i) = b_i$, $i = 1, \ldots, q$. We obtain the piecewise affine closed-loop system $\dot{x} = (A + BK_j)x + (a + Bg_j) =: y(x)$, $x \in \mathcal{P}$. We show a flow condition holds on $\mathcal{P}$. First, by Lemma 3.1, a flow condition holds for the closed loop vector field $y(x) := (A + BK_i)x + Bg_i + a$ at vertices of $\mathcal{O}_{\mathcal{P}}$. That is, there exists $\beta_1 \in \mathcal{B}$ such that $\beta_1 \cdot y(v_i) = \beta_1 \cdot b_i < 0$, $i = 1, \ldots, q$. Next let $\mathcal{P}' :=$ co $\{v_i \mid v_i \in V \setminus V_{\mathcal{O}}\}$. Note that because $\mathcal{O}_{\mathcal{P}}$ is a face of $\mathcal{P}$, $\mathcal{P}' \cap \mathcal{O} = \emptyset$. According to Lemma 4.1, there exists $\beta_2 \in$ Ker (B^T) such that for all $x \in \mathcal{P}'$, $\beta_2 \cdot (Ax + Bu(x) + a) < 0$. Define $\beta = \alpha\beta_1 + (1-\alpha)\beta_2$ for some $\alpha \in (0,1)$. Consider $v_i \in V_{\mathcal{O}}$. Using the fact that $\beta_2 \cdot b_i = 0$, we have $\beta \cdot y(v_i) = \alpha\beta_1 \cdot y(v_i) < 0$. Next consider $v_i \in V \setminus V_{\mathcal{O}}$. We have

$$\beta \cdot (Av_i + Bu_i + a) = \alpha\beta_1 \cdot (Av_i + Bu_i + a) + (1-\alpha)\beta_2 \cdot (Av_i + a)\,.$$

The term $\beta_1 \cdot (Av_i + Bu_i + a)$ is a constant of unknown sign, whereas we know $\beta_2 \cdot (Av_i + a) < 0$. Therefore it is possible to select α sufficiently small so that $\beta \cdot (Av_i + Bu_i + a) < 0$ for all $v_i \in V \setminus V_{\mathcal{O}}$. We conclude that for all $v_i \in V$, $\beta \cdot y(v_i) < 0$.

Now let $x \in \mathcal{P}$, and without loss of generality, suppose $x = \sum_{i=1}^{k} \lambda_i v_i$, where λ_i are the barycentric coordinates of x such that $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since $y(x)$ is affine on simplices of $\mathbb{T}$, we have $y(x) = \sum_{i=1}^{k} \lambda_i y(v_i)$. Therefore, for $x \in \mathcal{P}$, $\beta \cdot y(x) = \sum_{i=1}^{k} \lambda_i \beta \cdot y(v_i) < 0$. By a standard argument, all trajectories exit $\mathcal{P}$ in finite time, and by Proposition 3.1 of [11], they do so through $\mathcal{F}_0$. Thus, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback. ■

Lemmas 3.3 and 3.4 provide sufficient geometric conditions for condition (ii) of Theorem 4.3. These provide the analog to the results for simplices appearing in [6]. Finally, we consider the general case when $\mathcal{O}_{\mathcal{P}} \cap \mathcal{P}^{\circ} \neq \emptyset$. This case is considerably more difficult; indeed a complete solution is not known even for simplices. Therefore, we study only single-input systems. The starting point for this study is a necessary condition whose proof for the case of simplices is provided in [19].

Theorem 4.4: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_{\mathcal{P}} \neq \emptyset$. If RCP is solvable by continuous state feedback, then $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$.

Starting from Theorem 4.4, we create a monotonic flow by "pushing" a vector $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ onto each of the vertices of $\mathcal{P}$ while preserving the invariance conditions. We show that if MRCP is solvable, then it is solvable by this b-extremal solution. This then leads to a design procedure for constructing the appropriate controls, to be developed in Section V.

Let $y \in \mathbb{R}^n$ and define the index set $I_y := \{i \in I \mid y \in \mathcal{C}(v_i)\}$. That is, I_y is the index set of vertices for which the velocity vector y satisfies the invariance conditions of that vertex. By Theorem 4.4, $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) \neq \mathbf{0}$ is a necessary condition for solvability of RCP when $m = 1$, so we assume we have such a $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$. For the indices $i \notin I_b$, let $\overline{u}_i$ be such that $\overline{y}_i := Av_i + B\overline{u}_i + a \in \mathcal{C}(v_i)$ contains the maximal b component. In particular, $\overline{u}_i$ is the solution of the following LP

$$\max_{u \in \mathbb{R}} \quad b \cdot (Av_i + Bu + a) \tag{8a}$$
$$\text{subject to:} \quad Av_i + Bu + a \in \mathcal{C}(v_i). \tag{8b}$$

Since $b \notin \mathcal{C}(v_i)$ and $m = 1$, the maximum exists and is unique, and it corresponds to one or more invariance conditions evaluating to zero at v_i. Given a triangulation $\mathbb{T}$ of $\mathcal{P}$, let $\overline{u}(x)$ denote any PWA feedback such that $\overline{u}(v_i) = \overline{u}_i$, $i \notin I_b$.

The following result tells us that a b-extremal controller, in the sense just described, can always be selected to solve MRCP, if it is solvable by PWA feedback. The second condition (ii) below, presently less meaningful, will be seen to provide a useful tool in the algorithmic solution of MRCP, to be developed in Section V.

Theorem 4.5: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_\mathcal{P} \neq \emptyset$. Suppose $\mathbb{T}$ is a triangulation and $u(x)$ an associated continuous piecewise affine control such that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $u(x)$. Then there exist $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$, $\overline{y}_i = Av_i + B\overline{u}_i + a$, $i \notin I_b$, where $\overline{u}_i$ is the solution of the LP (8), and $\overline{u}(x)$ as above such that:

(i) $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $\overline{u}(x)$,
(ii) $0 \notin \text{co } \{b, \ \overline{y}_i \mid i \notin I_b\}$.

Proof: Since $u(x)$ is a continuous state feedback solving $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$, $u(x)$ satisfies the invariance conditions [11]. Let $\mathcal{O}_\mathcal{P} = \text{co } \{o_1, \ldots, o_q\}$. Define $b = Ao_1 + Bu(o_1) + a \in \mathcal{B} \cap \mathcal{C}(o_1)$. Then we must have $Ao_i + Bu(o_i) + a = \alpha_i b$ with $\alpha_i > 0$ for $i = 2, \ldots, q$. Otherwise, by the same argument as in the proof of Theorem 4.4, there is an equilibrium in $\mathcal{P}$ using $u(x)$. We conclude $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$.
Secondly, we show that the solution of the LP (8), $\overline{u}_i$, $i \notin I_b$, exists. Because the invariance conditions are solvable, a feasible solution of the constraint (8b) exists. Suppose, by contradiction, that for some $i \notin I_b$ the maximum does not exist under the constraint (8b). Because $i \notin I_b$, $b \notin \mathcal{C}(v_i)$ so there exists $j \in J$ such that $v_i \in \mathcal{F}_j$ and $h_j \cdot b > 0$. Then we have that the b-component in Bu can be made arbitrary large, while also $h_j \cdot (Av_i + Bu + a) \leq 0$. This is clearly impossible since $h_j \cdot b > 0$. Because $m = 1$, it can be easily shown that the solution $\overline{u}_i$ is also unique.
Now we show that (i)-(ii) are achieved. Since $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $u(x)$, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot (Ax + Bu(x) + a) < 0$, $x \in \mathcal{P}$. In particular, $\xi \cdot (Ao_1 + Bu(o_1) + a) = \xi \cdot b < 0$. Now define $\overline{u}(x) := u(x) + w(x)$, where $w(x)$ is determined by $Bw(x) = c(x)b$, such that the positive PWA function $c(x) \geq 0$ associated with $\mathbb{T}$ arises from assigning $\overline{u}_i$, $i \notin I_b$. Also, set $c(v_i) = 0$, $i \in I_b$. Then the invariance conditions still hold, and for $x \in \mathcal{P}$,

$$\xi \cdot (Ax + B\overline{u}(x) + a) = \xi \cdot (Ax + Bu(x) + a) + \xi \cdot c(x) b < 0\,. \tag{9}$$

We conclude $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $\overline{u}(x)$. By equation (9) and the fact that $\xi \cdot b < 0$, we obtain (ii). ■

Theorem 4.5 suggests a design procedure to synthesize a PWA control $\overline{u}(x)$ to achieve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. The procedure is simply to inject the largest possible $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$ component in any vertex with $i \notin I_b$, and to use a sufficiently large b component for vertices with $i \in I_b$. We present an algorithm in Section V.

V. Algorithm for MRCP

In this section we present an algorithm for solving MRCP by PWA feedback for single-input systems. It is assumed that $\mathcal{O}_\mathcal{P} \neq \emptyset$, for if $\mathcal{O}_\mathcal{P} = \emptyset$, then Theorem 4.2 provides a solution. Also, if $\mathcal{O}_\mathcal{P}$ is a face of $\mathcal{P}$, then Theorem 4.3 provides a solution. The algorithm, inspired by Theorem 4.5, is easily explained in words: for a single input system, there are only two control directions $b, -b \in \mathcal{B}$. Choose $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$ (step 1). At all those vertices v_i where $b \notin \mathcal{C}(v_i)$, we inject a maximal b component into the vector field by choice of control u_i (step 2). If MRCP is solvable, then Theorem 4.5 tells us that such an extremal solution exists. A flow condition must hold with extremal control values; that is, we can find a candidate $\xi \in \mathbb{R}^n$ for Problem 4.1 (step 3). Then we use ξ to select control values at the remaining vertices v_i where $b \in \mathcal{C}(v_i)$ (step 4). If ξ cannot be found, then the procedure is repeated with $-b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$ (step 5). Theorem 5.1 shows that this procedure is sound and complete.

Algorithm 1:

1. Select $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$.
2. For each $i \notin I_b$, solve the LP for $\overline{u}_i \in \mathbb{R}$:
$$\max_{u \in \mathbb{R}} \quad b \cdot (Av_i + a + Bu)$$
$$\text{subject to:} \quad Av_i + a + Bu \in \mathcal{C}(v_i) \tag{10}$$
3. Solve the LP for $\xi \in \mathbb{R}^n$:
$$\xi \cdot (Av_i + a + B\overline{u}_i) \ < \ 0\,, \qquad i \notin I_b \tag{11a}$$
$$\xi \cdot b \ < \ 0\,. \tag{11b}$$
4. If (11) is solvable, then for each $i \in I_b$, solve the LP for $\overline{u}_i \in \mathbb{R}$:
$$\xi \cdot (Av_i + a + B\overline{u}_i) \ < \ 0 \tag{12a}$$
$$Av_i + a + B\overline{u}_i \ \in \ \mathcal{C}(v_i)\,. \tag{12b}$$
5. If (11) is not solvable, select $-b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$ and repeat steps $2-4$ after replacing b by $-b$.
6. Form a triangulation $\mathbb{T}$ of $\mathcal{P}$ using only vertices of $\mathcal{P}$. Construct an affine feedback $u(x) = K_j x + g_j$ for each $n-$dimensional simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = \overline{u}_i, \ i = 1, \cdots, p$.

Theorem 5.1: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_\mathcal{P} \neq \emptyset$. MRCP is solvable by continuous PWA feedback if and only if Algorithm 1 terminates successfully.

Proof: ($\Longleftarrow$) Suppose the algorithm terminates successfully. It is required to show that the PWA feedback $u(x)$ calculated in step 6 solves MRCP on $\mathcal{P}$. From (10) and (12b), $u(x)$ satisfies the invariance conditions (2). From (11a) and (12a), a flow condition holds at the vertices of $\mathcal{P}$. By the same argument as at the end of the proof of Theorem 4.3 (with β replaced by ξ), $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by the PWA feedback $u(x)$.
($\Longrightarrow$) Suppose that MRCP is solvable by continuous PWA feedback. By way of contradiction, we show that if Algorithm 1 does not terminate successfully, then MRCP is not solvable by continuous PWA feedback. Let's consider all the cases where the algorithm does not terminate successfully. Let $\overline{y}_i := Av_i + B\overline{u}_i + a$, $i = 1, \ldots, p$.

1. The algorithm terminates in step 1 if $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) = \mathbf{0}$. By Theorem 4.4, MRCP is not solvable by continuous state feedback.
2. The algorithm terminates in step 2 if either (10) is not solvable, but then the invariance conditions (2) are not solvable. By Proposition 3.1 in [11], MRCP is not solvable by continuous state feedback. Alternatively, for some $i \notin I_b$, the maximum does not exist under (10). But, this is impossible as shown in the second part of the proof of Theorem 4.5.
3. The algorithm terminates in step 4 if the LP is not feasible. As above, if (12b) is not solvable, then MRCP is not solvable by continuous state feedback. Instead, suppose (12a) is not achievable simultaneously with (12b). This cannot happen because $\xi \cdot b < 0$ and $b \in \mathcal{C}(v_i)$, $i \in I_b$, so any sufficiently large b-component added to a velocity vector already satisfying (12b) solves the LP (see the remark below).
4. The algorithm terminates in step 5 if either $-b \notin \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$, or one of the LP problems in steps $2-4$ is not solvable (for $-b$). First, consider the cases where $-b \notin \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$, or the LP in step 3 is not solvable (for $-b$). For these cases, for every $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$ the LP problem in step 3 is not solvable. Equivalently, by a result analogous to Lemma 3.1, for every $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$, $0 \in \text{co}\ \{b,\ \overline{y}_i \mid i \notin I_b\}$. Then by Theorem 4.5(ii), MRCP is not solvable by continuous PWA feedback. Secondly, consider the cases where the LP problem in step 2, or 4 is not solvable (using $-b$). By a similar argument to the previous two points, MRCP is not solvable by continuous state feedback.

■

Remark 5.1: Algorithm 1 is most closely related to the procedure contained in Theorem 4.17 of [12]. Their algorithm is for simplices and multi-input systems, while our algorithm is for polytopes and single-input systems. Comparing only the single-input cases, their algorithm requires solving at most $2^{(n+1)} + 2(n+1)$ linear programming (LP) problems, where n is the dimension of the simplex. Instead, our algorithm requires solving at most $2p + 2$ LP problems, where p is the number of vertices of $\mathcal{P}$.

Second, we remark that if (12b) is satisfied using u_i', then it can be easily verified that a control that satisfies (12) is calculated as follows. For $i \in I_b$, let $c_i > \max\{0, -\frac{\xi \cdot (Av_i + a + Bu_i')}{\xi \cdot b}\}$. Select $\overline{u}_i = u_i' + w(v_i)$, where $w(v_i)$ is determined by $Bw(v_i) = c_i b$.

VI. Conclusion

We formulated the monotonic reach control problem (MRCP) and obtained intrinsic necessary and sufficient conditions for solvability of MRCP by PWA feedback. We presented a numerical algorithm to solve MRCP in the single-input case which is the first to overcome the exponential complexity in the system dimension of algorithms currently available in the literature [12].

References

[1] G. Ashford and M.E. Broucke. Time-varying affine feedback for reach control on simplices. *Automatica*. Accepted, April 2012.

[2] C. Belta, V. Isler, G. J. Pappas. Discrete abstractions for robot motion planning and control in polygonal environments. *IEEE Transactions on Robotics*, Vol. 21, no. 5, pp. 864–874, October 2005.

[3] C. Belta, L. C. J. M. Habets. Controlling a class of nonlinear systems on rectangles. *IEEE Transactions on Automatic Control*. Vol. 51, issue 11, pp. 1749-1759, Nov. 2006.

[4] C. Belta, L. C. J. M. Habets, V. Kumar. Control of multi-affine systems on rectangles with applications to biomolecular networks. *Proc. 41st IEEE Conference on Decision and Control*. Las Vegas, Nevada, USA, Dec. 2002, pp. 534-539.

[5] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Trans. Automatic Control*, vol. 45, no. 10, pp. 1864–1876, 2000.

[6] M.E. Broucke. Reach control on simplices by continuous state feedback. *SIAM Journal on Control and Optimization*. vol. 48, issue 5, pp. 3482-3500, February 2010.

[7] M.E. Broucke. On the reach control indices of affine systems on simplices. *8th IFAC Symposium on Nonlinear Control Systems*. pp. 96–101, August 2010.

[8] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *American Control Conference*. pp. 2633–2638, June 2011.

[9] M.E. Broucke and M. Ganness. Reach control on simplices by piecewise affine feedback. *SIAM Journal on Control and Optimization*. Submitted December 2012.

[10] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid Dynamical Systems. *IEEE Control Systems Magazine*. Vol. 29, no. 2, pp. 28–93, April 2009.

[11] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.

[12] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control*. Vol. 51, no. 6, pp. 938–948, 2006.

[13] M. K. Helwa and M. E. Broucke. Monotonic Reach Control on Polytopes. *IEEE Conference on Decision and Control*. pp. 4741–4746, December, 2011.

[14] C. W. Lee. Subdivisions and triangulations of polytopes. *Handbook of Discrete and Computational Geometry*. CRC Press Series Discrete Math. Appl., pp. 271–290, 1997.

[15] Z. Lin and M.E. Broucke. On a reachability problem for affine hypersurface systems on polytopes. *Automatica*. Vol. 47, issue 4, pp. 769-775, April 2011.

[16] R.T Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1970.

[17] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. Vol. 42, no. 11, pp. 1913–1918, November 2006.

[18] E. Semsar Kazerooni and M.E. Broucke. Reach controllability of single input affine systems. *IEEE Conference on Decision and Control*. pp. 4747-4752, December 2011.

[19] E. Semsar Kazerooni and M. E. Broucke. Reach controllability of single input affine systems. *IEEE Trans. Automatic Control*. In revision, December 2012.