Pattern Preserving Pole Placement and Stabilization for Linear Systems

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***Abstract*— A method to characterize the symmetries inherent within physical systems via automorphism groups has already been established. In this paper, we define a specific block diagonal form to which a matrix can be decomposed if and only if it conforms with a given set of symmetries. We employ this decomposition and resulting block diagonal form to define controllability and stabilizability and give a pole placement algorithm when the feedback matrix is constrained by the inherent patterns of the system. Finally, an example is given which demonstrates our controllability notion and pole placement algorithm.**

I. INTRODUCTION

The majority of decentralization techniques in the control literature depend on optimal control procedures with decentralizing constraints on the feedback matrices. For example, the work of Shah and Parillo applies this approach to the decentralized control of systems with a poset-causal structure [1]. These numerical methods may not provide necessary and sufficient conditions to guarantee decentralization. Moreover, they most often fail to directly exploit the inherent patterning and structure which render the system decentralizable in the first place.

There have been some attempts in the literature to take advantage of system structure to find decentralized controllers. An early work by Brockett and Willems analyzes a class of matrices which can be block diagonalized by a common base matrix [2]. This class of so-called *block circulant systems* is defined by a group of identical subsystems which are coupled to each other in a ring structure.

A general notion of patterned linear systems was introduced by Hamilton and Broucke [3], adopting the framework of linear geometric control [4]. The methodology in [3] was further pursued by Sniderman and Broucke [5]. In the course of this research, a new definition of patterning arose: the patterning of a matrix could be encoded through commuting properties with other matrices. Namely, a matrix A is an element of a set of commuting matrices $\mathcal{C}(U,V)$ if and only if $UA = AV$. Using this notion of patterning, a full set of geometric results was elucidated for *block circulant systems*, including a pole placement algorithm capable of generating a stabilizing feedback which preserves aspects of patterning in the original system (those encoded by the commuting property). In one notable example, this algorithm was also shown to generate decentralizing results [6].

Recently, in a paper by Consolini and Tosques [7], the commutative notion of patterning was extended beyond a single commutative relationship to a set of commutative operations defined in terms of a finite group. A key insight of this paper is the use of the automorphism group of an equivalent graph of subsystems. This finite group is a standard way of representing symmetries inherent in mathematical structures [8]. In the paper, deep results of group theory, representation theory, and character theory were employed to demonstrate that there exists a stabilizing feedback which preserves the system property of commutativity with the matrix representations of the finite group. However, no explicit pole placement algorithm was given and complete results in terms of standard geometric problems were lacking. Even more recently, Consolini and Tosques have presented results which prove that it is possible to find a decentralized output feedback stabilizer if and only if there exists a controller that respects the symmetry of the graph automorphism group of the system [9].

This paper consolidates the group theoretic results of Consolini [7] with the algorithms and pole placement results of Sniderman et al. [5] to formulate a more general theory of patterned linear systems with the goal of developing a formal algebraic notion of decentralization.

II. BACKGROUND

We review some notions from group theory and representation theory [10]. An *isomorphism* is a homomorphism in which the defined map is bijective (characterized by an invertible linear map in our case). Two mathematical objects are said to be *isomorphic* (denoted by congruency symbol $\cong$) if there exists a linear invertible map between them. An *automorphism* is an isomorphism mapping a mathematical object to itself. The *General Linear Group* ($\mathbf{GL}_n(\mathcal{V})$) is the group of isomorphisms of a given vector space $\mathcal{V}$ onto itself. The *Symmetric Group* ($Sym(n)$) is the group of all permutations of a set of cardinality n [11]. Let $A \otimes B$ denote the *Kronecker product* between two matrices A and B of any dimension. The *direct sum of matrices* is defined to be $\bigoplus_{i=1}^{n} A_i = diag(A_1, A_2, \ldots, A_n)$ [12].

Let G be a multiplicative group of finite order. A *linear representation* ρ of G in a *representation space* $\mathcal{V} \subseteq \mathbb{R}^n$ is a map $\rho : G \longrightarrow GL(\mathcal{V})$, where $GL(\mathcal{V})$ is the general linear group of isomorphisms on $\mathcal{V}$. Given elements $a, b, c \in G$ such that $a \cdot b = c$, we have that $\rho(a) \cdot \rho(b) = \rho(c)$. We call two different representations $\rho_1 : G \longrightarrow GL(\mathcal{V}_1)$ and $\rho_2 : G \longrightarrow GL(\mathcal{V}_2)$ *isomorphic* (denoted by $\rho_1 \cong \rho_2$) if there exists a linear transformation $\tau : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ such that $\tau\rho_1(g) = \rho_2(g)\tau$ for all $g \in G$. A representation is said to be *irreducible* if there does not exist a non-trivial subspace $\mathcal{V}' \subset$

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$\mathcal{V}$ that is invariant to the action of G, i.e. $\rho(g)\mathcal{V}' \subset \mathcal{V}'$ for all $g \in G$. We henceforth denote any irreducible representation with the tilde, i.e. $\tilde{\rho}$. We recall Schur's Lemma, which is one of the most fundamental concepts of representation theory,

Lemma 1 (Schur's Lemma)**.** Let $\tilde{\rho}^1 : G \longrightarrow GL(\mathcal{V}_1)$ and $\tilde{\rho}^2 : G \longrightarrow GL(\mathcal{V}_2)$ be two irreducible representations of G, and let $A : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be a linear map such that $\tilde{\rho}^2(g) \cdot A = A \cdot \tilde{\rho}^1(g)$ for all $g \in G$. Then:

(1) If $\tilde{\rho}^1(g)$ is not isomorphic to $\tilde{\rho}^2(g)$, then $A = 0$.

(2) If $\mathcal{V}_1 = \mathcal{V}_2$ and $\rho^1 = \rho^2$, then A is a scalar multiple of identity.

There are a finite number h of irreducible representations $\tilde{\rho}_i : G \longrightarrow GL(\mathcal{W}_i)$, $i = 1...h$, for a given group G that are unique up to isomorphism [10]. Thus any irreducible representation is isomorphic to exactly one of the $\tilde{\rho}_i$. Henceforth, the irreducible representations will be fixed to be $\{\tilde{\rho}_1, \dots, \tilde{\rho}_h\}$, with representation spaces $\{\mathcal{W}_1, \dots, \mathcal{W}_h\}$, respectively. The following Lemma from [10, Proposition 1.8] regards the decomposition of a representation into irreducible constituents.

Lemma 2. Suppose we have a representation of G in $\mathcal{V}$ given by $\rho : G \longrightarrow GL(\mathcal{V})$. Then there exists a decomposition,

$$\mathcal{V} = \mathcal{W}_1^{\oplus \eta_1} \oplus \dots \oplus \mathcal{W}_h^{\oplus \eta_h}$$

where the $\mathcal{W}_i$ are the distinct irreducible representation spaces of G such that $\rho(g)\mathcal{W}_i \subset \mathcal{W}_i$. Each η_i is the multiplicity of $\mathcal{W}_i$ in $\mathcal{V}$. Note that it is possible that $\eta_i = 0$.

In the following lemma we show how this decomposition is extended to the representation itself.

Lemma 3. Given a real vector space $\mathcal{V} \sim \mathbb{R}^n$ and a representation of G in $\mathcal{V}$ given by $\rho : G \longrightarrow GL(\mathcal{V})$, there exists an invertible matrix T such that for all $g \in G$,

$$\tilde{\rho}(g) = T^{-1}\rho(g)T = \bigoplus_{i=1}^{h} \bigoplus_{j=1}^{\eta_i} \tilde{\rho}_i(g)$$

$$= \left[\begin{array}{ccc|c|ccc} \tilde{\rho}_1(g) & & & & & & \\ & \ddots & & & & & \\ & & \tilde{\rho}_1(g) & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & \tilde{\rho}_h(g) & & \\ & & & & & \ddots & \\ & & & & & & \tilde{\rho}_h(g) \end{array}\right], \tag{1}$$

where the $\tilde{\rho}_i(g) \in \mathbb{R}^{n_i \times n_i}$ are the h irreducible representations each repeated with multiplicity η_i and n_i is the dimension of the corresponding representation space $\mathcal{W}_i$.

The proof of this lemma is omitted as it clearly follows from Lemma 2. We call the matrix T the *irreducible decomposition transformation*. A computationally tractable method by which a specific transformation matrix T can be found for a given representation is available, but was excluded due to space limitations.

III. The Automorphism Group

We consider the linear time-invariant system,

$$\dot{x} = Ax + Bu \tag{2}$$

where $x \in \mathcal{X}$, $u \in \mathcal{U}$, $\mathcal{X} \sim \mathbb{R}^n$, $\mathcal{U} \sim \mathbb{R}^m$, and $\text{rank}(B) = m$. We seek a method to characterize the patterns that are inherent within the structure of a large scale linear system. In a number of canonical examples, patterns may be characterized via a commutative property between system matrices and a set of base matrices representing a symmetry operation [5], [13], [7]. As presented in [7], we define a finite group of symmetry operations which commute with the system matrices of (2). This group can be found via the following procedure:

1. Partition the state space of the large scale system into its known subsystems: $\mathcal{X} = \mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_m$, where each $\mathcal{X}_i$ corresponds to the state space of i^{th} subsystem.
2. Partition the system matrices (A, B) according to the partition of the state space above. The blocks along the diagonal of A represent the system matrix A_i for each subsystem i. The off-diagonal blocks of the A matrix represent the coupling between the subsystems. Finally, the non-zero blocks of the partitioned B matrix represent inputs into the subsystems.
3. We define a graph $\mathcal{G}$ with a set of vertices $V = \{1, ..., m\}$ corresponding to the subsystems and a set of edges $E = \{(i, j) \in V \times V : \ i \text{ and } j \text{ adjacent}\}$ corresponding to the non-zero A_{ij} blocks.
4. Identify the set of permutation operations on the vertex set V for which the edge set E remains unchanged. This is akin to finding the subsystem states which can be interchanged without affecting the overall system. This set of operations is known as the *automorphism group* [11], which we denote as G. This group is a subgroup of the finite m-dimensional symmetric group $Sym(m)$ and is therefore finite [11].

The automorphism group G is represented in both the state $\mathcal{X}$ and input $\mathcal{U}$ spaces by a set of permutation matrices as shown in [7].

IV. Patterned Decomposition of a Matrix

Armed with the finite group characterization of symmetries, we endeavor to further characterize the properties of a matrix that possesses the patterning prescribed by a given finite group G. We begin with a fundamental definition.

Definition 1 (G-Patterned Matrix)**.** Let G be a finite group, $\mathcal{V}_1 = \mathbb{R}^n$ and $\mathcal{V}_2 = \mathbb{R}^m$ be vector spaces, and $\rho^1 : G \longrightarrow GL(\mathcal{V}_1)$ and $\rho^2 : G \longrightarrow GL(\mathcal{V}_2)$ be two representations of G. We say the matrix $A \in \mathbb{R}^{n \times m}$ is *G-patterned* if $\rho^1(g) \cdot A = A \cdot \rho^2(g)$ for all $g \in G$.

This definition is similar to the *G-equivariant* definition given in [7]. We now begin the main contribution of this paper by stating a powerful theorem which equates G-patterning to a special block diagonal form. We recall that a given representation space $\mathcal{V}$ can decomposed into a direct sum of the h irreducible representation spaces $\mathcal{W}_i$, each repeated with multiplicity η_i and having dimension n_i.

Theorem 1 (Patterned Matrix Structure)**.** *We are given $\rho^1 : G \longrightarrow GL(\mathcal{V}_1)$ and $\rho^2 : G \longrightarrow GL(\mathcal{V}_2)$ such that $\mathcal{V}_1 \sim \mathbb{R}^n$ and $\mathcal{V}_2 \sim \mathbb{R}^m$, as well as T_1 and T_2, the associated irreducible decomposition transformations. A matrix $A \in \mathbb{R}^{n\times m}$ is G-patterned if and only if the decomposed matrix $\tilde{A} = T_1^{-1}AT_2$ has the following block diagonal form,*

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 & \dots & 0 \\ 0 & \tilde{A}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{A}_h \end{bmatrix} \tag{3}$$

where

$$\tilde{A}_i = \hat{A}_i \otimes I_{n_i}\,, \quad \hat{A}_i \in \mathbb{R}^{\eta_i^1\times\eta_i^2}\,.$$

The integers η_i^1 and η_i^2 are the multiplicities of each $\mathcal{W}_i$ in $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively.

Proof. We first partition $\tilde{A}$ as

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \dots & \tilde{A}_{1h} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{h1} & \dots & \tilde{A}_{hh} \end{bmatrix}, \qquad \tilde{A}_{ij} \in \mathbb{R}^{\eta_i^1\cdot n_i\times\eta_j^2\cdot n_j}.$$

We will show that A is G-patterned if and only if,

$$\tilde{A}_{ij} = \begin{cases} \hat{A}_i \otimes I_{n_i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

From (1), we know that for each representation ρ^1 and ρ^2, there exist transformation matrices T_1 and T_2 such that for $j = 1, 2$,

$$\tilde{\rho}^1(g) = T_j^{-1}\rho^j(g)T_j = \begin{bmatrix} R_1^j(g) & & & \\ & R_2^j(g) & & \\ & & \ddots & \\ & & & R_h^j(g) \end{bmatrix},$$

where the blocks $R_i^j(g)$ are given, for all $g \in G$, by

$$R_i^j(g) = \bigoplus_{k=1}^{\eta_i^j} \tilde{\rho}_i(g)\,, \quad i = 1,\dots,h\,, \quad j = 1, 2\,. \tag{4}$$

($\Longrightarrow$) Let A be G-patterned. Applying the irreducible decomposition transformations to the commutative property, we have that $\rho^1(g)\cdot A = A\cdot\rho^2(g)$ if and only if $\tilde{\rho}^1(g)\tilde{A} = \tilde{A}\tilde{\rho}^2(g)$, where $\tilde{A} = T_1^{-1}AT_2$. Rewriting the commutative formula explicitly in terms of the R_i^j blocks, we have for all $g \in G$,

$$R_i^1(g)\tilde{A}_{ij} = \tilde{A}_{ij}R_j^2(g)\,, \quad \forall\, i, j\,. \tag{5}$$

Equivalently,

$$\underbrace{\begin{bmatrix} \tilde{\rho}_i(g) & & \\ & \ddots & \\ & & \tilde{\rho}_i(g) \end{bmatrix}}_{\eta_i^1 \text{ times}} \tilde{A}_{ij} = \tilde{A}_{ij} \underbrace{\begin{bmatrix} \tilde{\rho}_j(g) & & \\ & \ddots & \\ & & \tilde{\rho}_j(g) \end{bmatrix}}_{\eta_j^2 \text{ times}}.$$

Now we make a further partition of $\tilde{A}_{ij}$ as follows:

$$\tilde{A}_{ij} = \begin{bmatrix} (\tilde{A}_{ij})_{11} & \dots & (\tilde{A}_{ij})_{1\eta_j^2} \\ \vdots & \ddots & \vdots \\ (\tilde{A}_{ij})_{\eta_i^1 1} & \dots & (\tilde{A}_{ij})_{\eta_i^1\eta_j^2} \end{bmatrix}, \quad (\tilde{A}_{ij})_{kl} \in \mathbb{R}^{n_i\times n_j}.$$

Applying this decomposition to (5), we obtain a completely reduced form of the commutative property; namely

$$\tilde{\rho}_i(g)\cdot(\tilde{A}_{ij})_{kl} = (\tilde{A}_{ij})_{kl}\cdot\tilde{\rho}_j(g), \qquad \forall i, j, k, l. \tag{6}$$

First, suppose $i \neq j$. Then $\tilde{\rho}_i$ is not isomorphic to $\tilde{\rho}_j$, so by Schur's lemma, $(\tilde{A}_{ij})_{kl} = 0$, for all k, l. This implies that $\tilde{A}_{ij} = 0$. Second, suppose $i = j$. Then $\tilde{\rho}_i$ is equal to $\tilde{\rho}_j$, so by Schur's Lemma,

$$(\tilde{A}_{ij})_{kl} = a_{kl}^i \cdot I_{n_i},$$

where $a_{kl}^i \in \mathbb{R}$. We can factor the identity matrix and employ an abuse of notation to obtain

$$\tilde{A}_i = \tilde{A}_{ii} = \hat{A}_i \otimes I_{n_i} = \begin{bmatrix} a_{11}^i & \dots & a_{1\eta_j^2}^i \\ \vdots & \ddots & \vdots \\ a_{\eta_i^1 1}^i & \dots & a_{\eta_i^1\eta_j^2}^i \end{bmatrix} \otimes I_{n_i}.$$

As a result $\hat{A}_i \in \mathbb{R}^{\eta_i^1\times\eta_i^2}$.
($\Longleftarrow$) Note that we only need to prove equation (6) for the commutative property to be true overall. First, suppose $i \neq j$. Then, by assumption, $(\tilde{A}_{ij})_{kl} = 0$. Therefore, $\tilde{\rho}_i(g)(\tilde{A}_{ij})_{kl} = 0 = (\tilde{A}_{ij})_{kl}\tilde{\rho}_j(g)$ for all i, j, k, l such that $i \neq j$ and for all $g \in G$. Therefore, the commutative property is satisfied.
Next, suppose $i = j$. For all k, l and for all $g \in G$ we have,

$$\begin{aligned} \tilde{\rho}_i(g)\cdot(\tilde{A}_{ij})_{kl} &= \tilde{\rho}_i(g)\cdot a_{kl}^i\cdot I_{n_i} \\ &= a_{kl}^i\cdot I_{n_i}\cdot\tilde{\rho}_i(g) = (\tilde{A}_{ij})_{kl}\cdot\tilde{\rho}_j(g). \end{aligned}$$

Thus the commuting property holds in both cases and A is G-patterned.

□

Definition 2 (Pattern Reduced Form)**.** We can decompose any G-patterned matrix A into the form given in Theorem 1. We refer to the direct matrix sum of the $\hat{A}_i$ from Theorem 1 as the *Pattern Reduced Form* and denote this form by $\hat{A} \in \mathbb{R}^{\gamma_1\times\gamma_2}$ such that $\hat{A} = \bigoplus_{i=1}^{h} \hat{A}_i$ (with $\gamma_1 = \sum_{i=1}^h \eta_i^1$ and $\gamma_2 = \sum_{i=1}^h \eta_i^2$),

We now state some of the properties of the eigenvalue spectrum of a square G-patterned matrix. Let σ^n denote the disjoint union of n repetitions of a given spectrum σ. Note that the spectral sets are allowed to contain repeated elements.

Lemma 4. Let $A \in \mathbb{R}^{n\times n}$ be a G-patterned matrix. The spectrum $\sigma(A)$ is equivalent to the disjoint union of the spectra of the $\hat{A}_i$ each repeated n_i times, where n_i is the dimension of the irreducible representation associated with $\hat{A}_i$,

$$\sigma(A) = \biguplus_{i=1}^{h} \sigma^{n_i}(\hat{A}_i). \tag{7}$$

Proof. Let $M = \overset{h}{\underset{i=1}{\oplus}} M_i$ be an arbitrary square matrix. We note first that $\sigma(M) = \sigma(\overset{h}{\underset{i=1}{\oplus}} M_i) = \overset{h}{\underset{i=1}{\uplus}} \sigma(M_i)$, since M is block diagonal. We also note that by a property of the Kronecker product, $\sigma(M \otimes I_n) = \sigma^n(M)$. Therefore $\sigma(A) = \sigma(\tilde{A}) = \sigma(\overset{h}{\underset{i=1}{\oplus}} \tilde{A}_i) = \sigma(\overset{h}{\underset{i=1}{\oplus}} \hat{A}_i \otimes I_{n_i}) = \overset{h}{\underset{i=1}{\uplus}} \sigma(\hat{A}_i \otimes I_{n_i}) = \overset{h}{\underset{i=1}{\uplus}} \sigma^{n_i}(\hat{A}_i)$. □

We say that A has a *patterned spectrum* if its spectrum is as in Lemma 4 above.

V. Patterned Control Theory

A. Patterned Dynamical System

We now extend the concept of a G-patterned matrix to a LTI dynamical system.

Definition 3 (G-Patterned System)**.** Let G be a finite group. Let A and B in (2) above both be G-patterned such that there exist $\rho^x(G) : G \to \mathbf{GL}(\mathcal{X})$ and $\rho^u(G) : G \to \mathbf{GL}(\mathcal{U})$ such that $\rho^x(g)A = A\rho^x(g)$ and $\rho^x(g)B = B\rho^u(g)$ for all $g \in G$. Then we say the system (A, B) is a *G-Patterned System.*

Definition 4 (Pattern Reduced Form)**.** We say $(\hat{A}, \hat{B})$ is the *Pattern Reduced Form* of the system (A, B). Additionally, we call each $(\hat{A}_i, \hat{B}_i)$ a *pattern reduced subsystem* of the system (A, B).

B. Patterned Controllability

We are now positioned to define an appropriate notion of controllability for a given patterned system. That is, we characterize when a system is controllable via a feedback that preserves the patterning of the original system.

Definition 5 (Pattern Controllable)**.** We say that a G-patterned system (A, B) is *pattern controllable* if each pattern reduced subsystem $(\hat{A}_i, \hat{B}_i)$ is controllable in the standard linear control theory sense.

C. Patterned Pole Placement

In pursuit of our pole placement goal, we first consider the following two lemmas, which will bridge the gap between pattern controllability and patterned pole placement.

Lemma 5. Let A and B be as in (2) with $K \in \mathbb{R}^{n \times n}$ and let A,B, and K be G-patterned. Then the matrix $A + BK$ is also G-patterned. Moreover, $\widetilde{A + BK} = \tilde{A} + \tilde{B}\tilde{K}$ and $\widehat{A + BK} = \hat{A} + \hat{B}\hat{K}$. Furthermore, $\sigma(\widehat{A + BK}) = \overset{h}{\underset{i=1}{\uplus}} \sigma(\hat{A}_i + \hat{B}_i\hat{K}_i)$.

Proof. Due to space limitations, we provide only a proof sketch. Applying the appropriate transformations, we have $\widetilde{A + BK} = \tilde{A} + \tilde{B}\tilde{K} = \overset{h}{\underset{i=1}{\oplus}} \sigma(\hat{A}_i + \hat{B}_i\hat{K}_i) \otimes I_{n_i}$. The spectrum $\sigma(\widehat{A + BK}) = \overset{h}{\underset{i=1}{\oplus}} \sigma(\hat{A}_i + \hat{B}_i\hat{K}_i) = \overset{h}{\underset{i=1}{\uplus}} \sigma(\hat{A}_i + \hat{B}_i\hat{K}_i)$ by the proof of Lemma 5. □

Now we state our main pole placement result with a proof that constructively demonstrates how to determine patterned feedback K.

Theorem 2 (Patterned Pole Placement)**.** *Let (A, B) be G-patterned system. Let $\mathcal{L}$ be an arbitrary symmetric spectrum of size $|\sigma(\hat{A})|$. There exists a G-patterned matrix $K \in \mathbb{R}^{m \times n}$ such that $\sigma(\widehat{A + BK}) = \mathcal{L}$ if and only if (A, B) is pattern controllable.*

Proof. ($\Longrightarrow$) First we assume that we have a G-patterned matrix K such that $\sigma(\widehat{A + BK}) = \mathcal{L}$. Since A, B, and K are patterned we know that $A + BK$ is patterned by Lemma 5. Since, by Lemma 5 we have $\sigma(\widehat{A + BK}) = \overset{h}{\underset{i=1}{\uplus}} \sigma(\hat{A}_i + \hat{B}_i\hat{K}_i)$, clearly we can split the spectrum accordingly $\mathcal{L} = \overset{h}{\underset{i=1}{\uplus}} \mathcal{L}_i$ such that $\sigma(\hat{A}_i + \hat{B}_i\hat{K}_i) = \mathcal{L}_i$ for all $i = 1...h$. Since each $\mathcal{L}_i$ is freely assignable, $(\hat{A}_i, \hat{B}_i)$ must be controllable via feedback $\hat{K}_i$.
($\Longleftarrow$) Now assume (A, B) is pattern controllable. This implies that the $(\hat{A}_i, \hat{B}_i)$ are controllable for all $i = 1, ..., h$. Thus, for each $(\hat{A}_i, \hat{B}_i)$, there exists a $\hat{K}_i$ which can be found using standard pole placement such that $\sigma(\hat{A}_i + \hat{B}_i\hat{K}_i) = \mathcal{L}_i$, where $\mathcal{L}_i$ is again an arbitrary spectrum. We note that for each $\hat{K}_i$ we can define a matrix $\tilde{K}_i = \hat{K}_i \otimes I_{n_i}$ (recall that n_i corresponds to the dimension of the associated irreducible representation space $\mathcal{W}_i$). We can then form the direct sum $\tilde{K} = \overset{h}{\underset{i=1}{\oplus}} \tilde{K}_i$ and apply the appropriate irreducible decomposition transformations such that $K = T_u^{-1}\tilde{K}T_x$. From Theorem 1, $K : \mathcal{X} \longrightarrow \mathcal{U}$ is G-patterned by construction and by Lemma 5, $\mathcal{L} = \overset{h}{\underset{i=1}{\uplus}} \mathcal{L}_i = \overset{h}{\underset{i=1}{\uplus}} \sigma(\hat{A}_i + \hat{B}_i\hat{K}_i) = \sigma(\widehat{A + BK})$. □

Thus, if the system is pattern controllable, we will be able to use a patterned feedback to place all of the poles of the spectrum $\sigma(\widehat{A + BK})$. By Lemma 4, this implies that all of the poles of the system can be placed arbitrarily, albeit with some necessary multiplicity resulting from the pattern preservation.

D. Pattern Stabilizability

We can define a notion of stabilizability as follows,

Definition 6 (Pattern Stabilizability)**.** The G-patterned system (A, B) is *Pattern Stabilizable* if there exists a G-patterned matrix $K \in \mathbb{R}^{m \times n}$ such that $\sigma(A + BK) \subset \mathbb{C}^-$.

Theorem 3 (Pattern Stabilizable Subsystems)**.** *The G-patterned system (A, B) is Pattern Stabilizable if and only if the pattern reduced subsystems $(\hat{A}_i, \hat{B}_i)$ are stabilizable in the standard linear control sense for all $i = 1, ..., h$.*

Proof. By Lemma 4 and Lemma 5 we have that $\sigma(A + BK) = \overset{h}{\underset{i=1}{\uplus}} \sigma^{n_i}(\hat{A}_i + \hat{B}_i\hat{K}_i)$ for all $i = 1 \ldots h$. Therefore, $\sigma(A + BK) \subset \mathbb{C}^-$ if and only if $\overset{h}{\underset{i=1}{\uplus}} \sigma^{n_i}(\hat{A}_i + \hat{B}_i\hat{K}_i) \subset \mathbb{C}^-$, which is clearly true if and only if, for all $i = 1...h$, $\sigma(\hat{A}_i + \hat{B}_i\hat{K}_i) \subset \mathbb{C}^-$ with given $\hat{K}_i \in \mathbb{R}^{\eta_i^u \times \eta_i^x}$. This is, of course, the definition of stabilizability for $(\hat{A}_i, \hat{B}_i)$ in the standard linear control sense. $\square$

VI. Example

We present a system that has the general structure of a two level binary tree. We define the system matrices as follows:

$$A = \begin{bmatrix} A_1 & A_4 & A_4 & 0 & 0 & 0 & 0 \\ A_4 & A_2 & 0 & A_4 & A_4 & 0 & 0 \\ A_4 & 0 & A_2 & 0 & 0 & A_4 & A_4 \\ 0 & A_4 & 0 & A_3 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & A_4 & 0 & 0 & A_3 & 0 \\ 0 & 0 & A_4 & 0 & 0 & 0 & A_3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix}; A_2 = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix};$$

$$A_3 = \begin{bmatrix} -7 & 8 \\ 3 & 2 \end{bmatrix}; A_4 = \begin{bmatrix} -5 & 0 \\ 0 & 3 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

As explained in Section III, this system can be represented by the following graph:

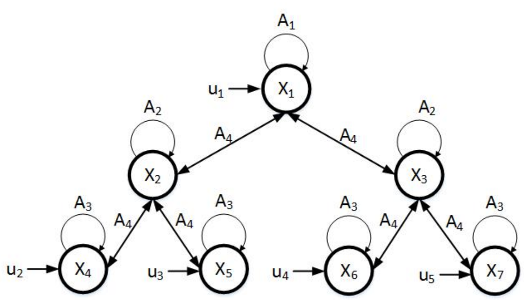

Fig. 1: Two level binary tree graph of example system

In the graph, the X_i represent the subsystems with directed arrows indicating the interactions between them as given by the A matrix. That is, a non-zero entry A_{ij} of the A matrix represents that subsystem X_j affects subsystem X_i and a line is therefore directed from X_j to X_i on the graph. The inputs u_j represent the inputs from the matrix B. The symmetry or pattern of the system is immediately apparent upon viewing the graph representation. However, for this and more complicated systems there exists software to aid in determining the full group of automorphic permutation operations (G) along with their matrix representations $\rho^x(g)$ and $\rho^u(g)$. To find the permissible permutations we use NAUTY [14] and for the remainder of the group theoretic information that we may need for a given system we rely on the well known Groups, Algorithms, Programming (GAP) System for Computational Discrete Algebra [15]. The symmetry inherent within this system was found to correspond to the dihedral D_8 group. This group is commonly denoted as follows: $G = \{e, a, a^2, a^3 = a^{-1}, x, ax, a^2x, a^3x\}$ where x and a are such that $a^4 = x^2 = e$ and $xax^{-1} = a^{-1}$ [16]. The system is, by definition, G-patterned with the D_8 group. For the purpose of exposition we demonstrate the state space and input space representations of a couple of the group elements (note that these are the permutation matrices corresponding to the permutations found via the procedure in Section III).

$$\rho^x(ax) = \begin{bmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & 0 & 0 & I_2 & 0 \end{bmatrix}; \rho^x(a^2x) = \begin{bmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2 & 0 & 0 \end{bmatrix};$$

$$\rho^u(ax) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \rho^u(a^2x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

where I_2 represents a two-dimensional identity matrix. Note that the system matrices will commute with these representations as prescribed by the definition of G-patterned system. From the GAP software we know that there are 5 irreducible representations that pertain to the D_8 group, denoted here as $\tilde{\rho}_1(g), \tilde{\rho}_2(g), \tilde{\rho}_3(g), \tilde{\rho}_4(g), \tilde{\rho}_5(g)$. The first four representations are one dimensional ($n_1 = n_2 = n_3 = n_4 = 1$), while the fifth has 2 dimensions ($n_5 = 2$) Additionally, we can employ group theoretic techniques, such as the canonical decomposition [17] to determine which irreducible representations compose the two representations above, along with the multiplicities of each irreducible representation. The multiplicities of the irreducible representations composing $\rho^x(g)$ are as follows: $\eta_1^x = 6, \eta_2^x = 0, \eta_3^x = 0, \eta_4^x = 4, \eta_5^x = 2$. The multiplicities of the irreducible representations composing $\rho^u(g)$ are as follows: $\eta_1^u = 2, \eta_2^u = 0, \eta_3^u = 0, \eta_4^u = 1, \eta_5^u = 1$. Knowing this, we can determine the irreducible decomposition transformations T_x and T_u such that,

$$\tilde{\rho}^x(g) = T_x^{-1}\rho^x(g)T_x$$

$$= diag\left(\underbrace{\tilde{\rho}_1(g), \ldots, \tilde{\rho}_1(g)}_{6\ \ times}, \underbrace{\tilde{\rho}_4(g), \ldots, \tilde{\rho}_4(g)}_{4\ \ times}, \tilde{\rho}_5(g), \tilde{\rho}_5(g)\right)$$

$$\tilde{\rho}^u(g) = T_u^{-1}\rho^u(g)T_u = diag\left(\tilde{\rho}_1(g), \tilde{\rho}_1(g), \tilde{\rho}_4(g), \tilde{\rho}_5(g)\right).$$

Now we can apply the decomposition matrices to the system matrices to give the following pattern decomposed forms of A and B,

$$\tilde{A} = T_x^{-1}AT_x = diag\left(\tilde{A}_1, \tilde{A}_4, \tilde{A}_5\right)$$

$$\tilde{A}_1 = \begin{bmatrix} 2 & 2 & 7.07 & 0 & 0 & 0 \\ 4 & 6 & 0 & -4.24 & 0 & 0 \\ 7.07 & 0 & -1 & 2 & -7.07 & 0 \\ 0 & -4.24 & 3 & -4 & 0 & 4.24 \\ 0 & 0 & -7.07 & 0 & -7 & 8 \\ 0 & 0 & 0 & 4.24 & 3 & 2 \end{bmatrix}$$

$$\tilde{A}_4 = \begin{bmatrix} -1 & 2 & -7.07 & 0 \\ 3 & -4 & 0 & 4.24 \\ -7.07 & 0 & -7 & 8 \\ 0 & 4.24 & 3 & 2 \end{bmatrix}; \tilde{A}_5 = \begin{bmatrix} [r]2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \\ 8 & 0 & -7 & 0 \\ 0 & 8 & 0 & -7 \end{bmatrix}$$

$$\tilde{B} = T_x^{-1}AT_u = diag\left(\tilde{B}_1, \tilde{B}_4, \tilde{B}_5\right)$$

$$\tilde{B}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \tilde{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \tilde{B}_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Employing Theorem 1, we note that the pattern reduced forms of $\tilde{A}$ and $\tilde{B}$ are as follows,

$$\hat{A}_1 = \tilde{A}_1; \quad \hat{A}_4 = \tilde{A}_4; \quad \hat{A}_5 = \begin{bmatrix} 2 & 3 \\ 8 & -7 \end{bmatrix}$$
$$\hat{B}_1 = \tilde{B}_1; \quad \hat{B}_4 = \tilde{B}_4; \quad \hat{B}_5 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Note that only the matrices of corresponding to the fifth irreducible representation ($\tilde{\rho}_5(g)$) have a reduced form. This is because only $\tilde{\rho}_5(g)$ has dimension ($n_5 = 2$) greater than one.

Each patterned-reduced subsystem $(\hat{A}, \hat{B})$ is controllable which implies that the entire system is pattern controllable. As a result we can place all of the eigenvalues of the patterned spectrum. We assign the spectra as follows: $\mathcal{L}_1 = \{-1, -2, -3, -4, -5, -6\}, \mathcal{L}_4 = \{-7, -8, -9, -10\}, \mathcal{L}_5 = \{-11, -12\}$. We use the MATLAB *place* function with the pattern reduced matrices to find pattern reduced feedback matrices,

$$\hat{K}_1 = \begin{bmatrix} -55.7805 & -15.8845 & -76.5477 & -1.6165 & 8.5844 & 26.7933 \\ 20.8369 & 1.7377 & 10.5936 & 0.0816 & -19.3971 & -3.1155 \end{bmatrix}$$
$$\hat{K}_4 = \begin{bmatrix} 46.475 & 7.7522 & -32.3612 & -24 \end{bmatrix}$$
$$\hat{K}_5 = \begin{bmatrix} 18 & 5.5 \end{bmatrix}.$$

Finally we put the feedback matrices in their unreduced form, collect them in a single matrix and reverse the pattern decomposition,

$$\tilde{K} = diag\left(\hat{K}_1, \hat{K}_4, \hat{K}_5 \otimes I_2\right)$$
$$K = T_u \tilde{K} T_x^{-1}.$$

The resulting K places the patterned spectrum and is, of course, G-patterned. The resulting spectrum of the system with feedback is as follows: $\sigma(A + BK) = \{-1, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11, -11, -12, -12\}$. As expected, we see repetition of the last two eigenvalues.

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