On the Necessity of the Invariance Conditions for Reach Control on Polytopes ☆

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Abstract

We study the Reach Control Problem (RCP) to make the solutions of an affine system defined on a polytopic state space reach and exit a prescribed facet of the polytope in finite time without first leaving the polytope. So-called *invariance conditions* are used to prevent solutions from leaving the polytope through facets which are not designated as the exit facet. These conditions are known to be necessary for solvability of the RCP on polytopes by continuous state feedback. We study whether the invariance conditions are also necessary for solvability of the RCP on polytopes by open-loop controls. We show by way of a counterexample that surprisingly the answer is negative. We identify a suitable class of polytopes for which the invariance conditions remain necessary conditions.

1. Introduction

We study the Reach Control Problem (RCP) for affine systems on polytopes. The problem is to design a state feedback to force closed-loop solutions starting anywhere in a polytopic state space $\mathcal{P}$ to leave the polytope from a prescribed exit facet of the polytope in finite time [7, 9, 10]. The RCP is a fundamental reachability problem for piecewise affine hybrid systems [2, 8]. The problem has been developed in [7, 8, 13, 4, 14, 5, 1] for simplices and [7, 11, 9, 10] for polytopes. In these papers the *invariance conditions* are used to prevent solutions from leaving the polytope from facets which are not designated as the exit facet. They were shown to be necessary conditions for solvability of the RCP on polytopes by continuous state feedback in [7] and to be necessary conditions for solvability of the RCP on simplices by open-loop controls in [5]. In this note we show that the invariance conditions are not necessary conditions for

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solvability of the RCP on polytopes by open-loop controls. We prove that for a special class of polytopes the invariance conditions remain necessary conditions using open-loop controls. The result extends both [7] and [5], and opens the door for determining the largest feedback class needed to solve the RCP on polytopes.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The notation $\mathcal{K}_1 \setminus \mathcal{K}_2$ denotes elements of the set $\mathcal{K}_1$ not contained in the set $\mathcal{K}_2$. The notation $\mathscr{B}$ denotes the open ball of radius 1 centered at the origin. For two vectors $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of the two vectors. The notation $\mathrm{co}\,\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. Let $T_{\mathcal{P}}(x)$ denote the Bouligand tangent cone to set $\mathcal{P} \subset \mathbb{R}^n$ at point x [6]. A set-valued map $\mathcal{Y} : \mathbb{R}^n \to 2^{\mathbb{R}^q}$ is said to be *upper semicontinuous* at $x \in \mathbb{R}^n$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - x'\| < \delta \;\Rightarrow\; \mathcal{Y}(x') \subset \mathcal{Y}(x) + \epsilon\mathscr{B}$.

2. Reach Control Problem

Consider an n-dimensional polytope $\mathcal{P} := \mathrm{co}\,\{v_1, \ldots, v_p\}$ with vertex set $V := \{v_1, \ldots, v_p\}$. An *edge* of $\mathcal{P}$ is a 1-dimensional face of $\mathcal{P}$, and a *facet* of $\mathcal{P}$ is an $(n-1)$-dimensional face of $\mathcal{P}$. Let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$ denote the facets of $\mathcal{P}$. The facet $\mathcal{F}_0$ is called the *exit facet* and facets $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are called the *restricted facets*. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J := \{1, \ldots, r\}$, and $J(x) := \{j \in J \mid x \in \mathcal{F}_j\}$. That is, $J(x)$ is the set of indices of the restricted facets that contain x. For each $x \in \mathcal{P}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \; y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in J(x) \; \right\}. \tag{1}$$

Note that the index 0 never appears in $J(x)$ since $\mathcal{F}_0$ is the exit facet. For any $x \in \mathcal{P} \setminus \mathcal{F}_0$, $\mathcal{C}(x) = T_{\mathcal{P}}(x)$, the Bouligand tangent cone to $\mathcal{P}$ at x. Instead, at $x \in \mathcal{F}_0$, $\mathcal{C}(x)$ and $T_{\mathcal{P}}(x)$ are different since $\mathcal{C}(x)$ includes directions pointing out of $\mathcal{P}$. See Figure 1. We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{P}\,, \tag{2}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\mathrm{rank}(B) = m$. Let $\mathcal{B} = \mathrm{Im}\, B$, the image of B. Also define $\mathcal{O} := \{ \; x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B} \; \}$ and $\mathcal{O}_{\mathcal{P}} := \mathcal{P} \cap \mathcal{O}$, the set of all possible equilibrium points of (2) in $\mathcal{P}$. We say that a function $\mu : [0, \infty) \to \mathbb{R}^m$ is an *open-loop control* for (2) if it is bounded on any compact time interval and it is measurable. These standard conditions ensure the existence and uniqueness of solutions of (2). Let $\phi_u(t, x_0)$ denote the trajectory of (2) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 2.1 (Reach Control Problem (RCP)). *Consider system (2) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that for every $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.*

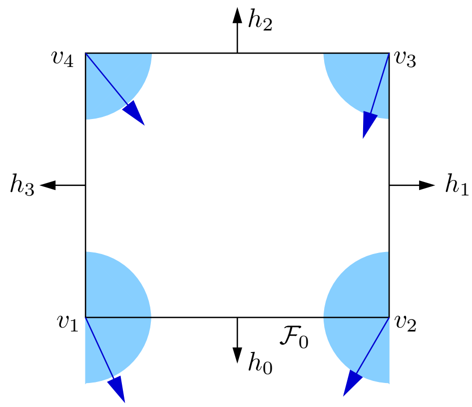

Figure 1: The convex cones $\mathcal{C}(v_i)$ in a two-dimensional polytope.

The RCP says that the solutions of (2) starting from initial conditions in $\mathcal{P}$ reach and exit $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. In the sequel we use the shorthand notation $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ to denote that the RCP is solvable by some control.

Definition 2.1. *We say the* invariance conditions are solvable *if for each $v \in V$, there exists $u \in \mathbb{R}^m$ such that*

$$Av + Bu + a \in \mathcal{C}(v)\,. \tag{3}$$

Equation (3) is referred to as the *invariance conditions* either for a specific vertex, or collecting all conditions for all vertices, for a polytope.

3. Counterexample

The invariance conditions (3) are known to be necessary for solvability of the RCP on polytopes by continuous state feedback [7] and on simplices by open-loop controls [5]. We show by way of a counterexample that for general polytopes and open-loop controls, the invariance conditions are, however, no longer necessary.

We consider the polytope $\mathcal{P} = \text{co }\{v_1, \cdots, v_5\} \subset \mathbb{R}^3$ shown in Figure 2 with vertices $v_1 = (\frac{1}{2}, 1, 1)$, $v_2 = (0, 1, 0)$, $v_3 = (1, 1, 0)$, $v_4 = (1, 0, 0)$, and $v_5 = (0, 0, 0)$. The exit facet is $\mathcal{F}_0 = \text{co }\{v_1, v_2, v_3\}$, depicted as a hatched region in the figure. The restricted facets are $\mathcal{F}_1 = \text{co }\{v_1, v_2, v_5\}$, $\mathcal{F}_2 = \text{co }\{v_1, v_3, v_4\}$, and $\mathcal{F}_3 = \text{co }\{v_1, v_4, v_5\}$. Also, $h_1 = (\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$, $h_2 = (\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$, and $h_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

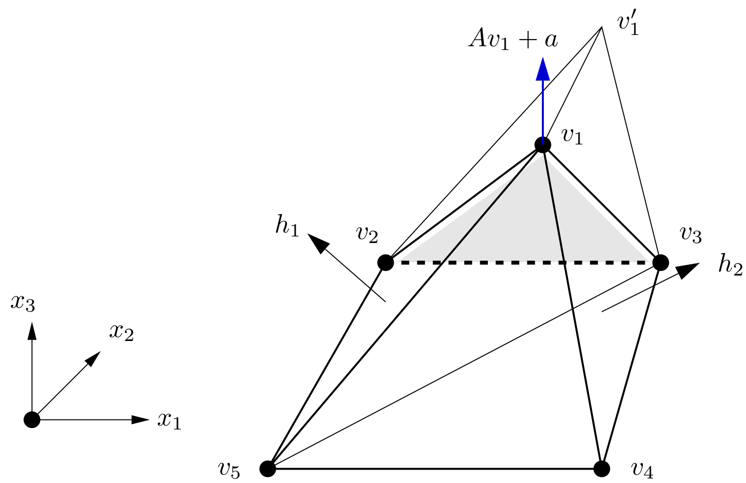

Figure 2: The invariance conditions are not solvable but the RCP is solvable by open-loop controls.

The affine control system is

$$\dot{x} = Ax + Bu + a = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ \frac{1}{10} & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} .$$

The set of possible equilibria of this system is $\mathcal{O} = \{x \in \mathbb{R}^3 \mid \frac{1}{10}x_1 + x_3 = -1\}$. It can be verified that $\mathcal{P} \cap \mathcal{O} = \emptyset$. This means that for all $x \in \mathcal{P}$ and all $u \in \mathbb{R}^2$, $Ax + Bu + a \neq 0$.

We show that the invariance conditions of $\mathcal{P}$ at v_1 are not solvable. We observe that $v_1 \in \mathcal{F}_0 \cap \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$, so $J(v_1) = \{1, 2, 3\}$. The condition (3) at v_1 states that there must exist $u_1 \in \mathbb{R}^2$ such that

$$y_1 := Av_1 + Bu_1 + a \in \mathcal{C}(v_1) = \{y \in \mathbb{R}^3 \mid h_j \cdot y \leq 0,\ j \in \{1, 2, 3\}\ \} .$$

Letting $u_1 = (u_{11}, u_{12})$ and substituting numerical values, we have $y_1 = (u_{11}, u_{12}, 2 + \frac{1}{20})$. The condition $h_1 \cdot y_1 \leq 0$ becomes $u_{11} \geq 1.025$, and the condition $h_2 \cdot y_1 \leq 0$ becomes $u_{11} \leq -1.025$. Clearly these two conditions cannot be solved simultaneously for u_{11}, so there does not exist $u_1 \in \mathbb{R}^2$ so that (3) holds at v_1. We conclude the invariance conditions of $\mathcal{P}$ are not solvable at v_1.

Second, we show there exist open-loop controls solving the RCP on $\mathcal{P}$. In fact, we construct a piecewise affine feedback that solves the problem. First we triangulate $\mathcal{P}$ using the triangulation $\mathbb{T} = \{\mathcal{S}_1, \mathcal{S}_2\}$, where $\mathcal{S}_1 = \text{co}\ \{v_1, v_2, v_3, v_5\}$ and $\mathcal{S}_2 = \text{co}\ \{v_1, v_3, v_4, v_5\}$ are two simplices, as shown in Figure 2. Second, we split the control objective as $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback, where $\mathcal{F} := \mathcal{S}_1 \cap \mathcal{S}_2 = \text{co}\ \{v_1, v_3, v_5\}$, and $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback.

It is well-known that the RCP is solvable by affine feedback on simplices if and only if the invariance conditions of the simplex are solvable and the unique affine feedback constructed from one choice of solution of the invariance conditions does not admit a closed-loop equilibrium in the simplex [8, 13]. Since $\mathcal{P} \cap \mathcal{O} = \emptyset$, neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ can have equilibria, so we must only construct affine feedbacks satisfying the invariance conditions of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. For the vertices of $\mathcal{S}_2$ we select control values $u_1 = (-5, 10)$, $u_3 = (-12, 10)$, $u_4 = (-12, 12)$, and $u_5 = (5, 12)$. Then we construct the unique affine feedback $u(x)$ on $\mathcal{S}_2$ satisfying $u(v_i) = u_i$, $v_i \in \mathcal{S}_2$ [7]. Similarly, we construct the affine feedback on $\mathcal{S}_1$ that achieves $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$. We conclude by Theorem 9 of [11] that the following discontinuous piecewise affine feedback solves the RCP on $\mathcal{P}$.

$$u(x) = \begin{cases} \begin{bmatrix} -22 & 5 & 6 \\ 0 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 5 \\ 12 \end{bmatrix}, & x \in \mathcal{S}_1 \\ \begin{bmatrix} -17 & 0 & -\frac{3}{2} \\ 0 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 5 \\ 12 \end{bmatrix}, & x \in \mathcal{S}_2 \setminus \mathcal{S}_1 . \end{cases}$$

This feedback has a discontinuity along $\mathcal{F}$, but it does not have sliding modes. This is because $u(x)$ satisfies the invariance conditions of $\mathcal{S}_1$, and so once solutions initiated in $\mathcal{S}_2 \setminus \mathcal{S}_1$ enter $\mathcal{S}_1$, they cannot return to $\mathcal{S}_2 \setminus \mathcal{S}_1$ before leaving $\mathcal{P}$ through $\mathcal{F}_0$.

Using this feedback solution of the RCP we can now understand how the problem is solvable using open-loop controls. In particular, solutions starting in $\mathcal{S}_2 \setminus \mathcal{S}_1$ near v_1 cross through $\mathcal{F} = \mathcal{S}_1 \cap \mathcal{S}_2$ into $\mathcal{S}_1$ because $\mathcal{S}_2$ contains no equilibrium and because $\mathcal{F}_2$ and $\mathcal{F}_3$ are restricted for $\mathcal{S}_2$. For example, to see that $\mathcal{F}_2$ and $\mathcal{F}_3$ are restricted for $\mathcal{S}_2$ we verify the invariance conditions of $\mathcal{S}_2$ at v_1. We have $y_1 = Av_1 + Bu(v_1) + a) = (-5, 0, 2 + \frac{1}{20})$ where $u(v_1) = (-5, 10)$ using the controller for $\mathcal{S}_2 \setminus \mathcal{S}_1$. Then we obtain

$$\begin{aligned} h_2 \cdot y(v_1) &= \frac{1}{\sqrt{5}}(2, 0, 1) \cdot (-5, 0, 2 + \frac{1}{20}) = \frac{1}{\sqrt{5}}(-8 + \frac{1}{20}) \leq 0 \\ h_3 \cdot y(v_1) &= \frac{1}{\sqrt{2}}(0, -1, 1) \cdot (-5, 0, 2 + \frac{1}{20}) = \frac{1}{\sqrt{2}}(-8 + \frac{1}{20}) \leq 0 . \end{aligned}$$

The invariance conditions of $\mathcal{S}_2$ at v_3, v_4 and v_5 can similarly be verfied to show that $\mathcal{F}_2$ and $\mathcal{F}_3$ are restricted using the affine feedback for $\mathcal{S}_2 \setminus \mathcal{S}_1$. Once the solution starting in $\mathcal{S}_2 \setminus \mathcal{S}_1$ reaches $\mathcal{F}$, the affine controller for $\mathcal{S}_1$ takes over. This controller drives the solutions through $\mathcal{F}_0$, the only facet not restricted in $\mathcal{S}_1$.

The proposed piecewise affine controller and corresponding open-loop controls work despite the fact that the invariance conditions of $\mathcal{P}$ at v_1 are not solvable. The invariance conditions of $\mathcal{P}$ at v_1 include two incompatible constraints, $h_1 \cdot (Av_1 + Bu_1 + a) \leq 0$ and $h_2 \cdot (Av_1 + Bu_1 + a) \leq 0$, which cannot be solved simultaneously for $u_1 \in \mathbb{R}^2$. Instead, using piecewise affine control these constraints are split between $\mathcal{S}_1$ and $\mathcal{S}_2$. The constraint regarding h_1 is part of the invariance conditions of $\mathcal{S}_1$ at v_1, while the constraint regarding h_2 is part of the invariance conditions of $\mathcal{S}_2$ at v_1.

The defect in this example appears to be that the vertex v_1 is over-constrained with regard to the invariance conditions because it belongs to too many restricted facets of $\mathcal{P}$. If we can devise a mathematical condition to limit the number of restricted facets at each vertex of the polytope, then it may be possible to return to a situation when the invariance conditions are again necessary for solving the RCP. The question that remains is: how can we ensure that the structure of the polytope is such that no vertex is over-constrained with respect to invariance conditions? We propose a solution in the next section.

4. Main Result

In this section we propose a suitable class of polytopes for which the invariance conditions remain necessary conditions. To that end, an n-dimensional polytope $\mathcal{P}$ is said to be *simple* if each k-dimensional face of $\mathcal{P}$ is contained in exactly $n-k$ facets [3]. Figure 1 shows an example of a two-dimensional simple polytope.

Remark 4.1. *If $\mathcal{P}$ is a simple polytope, then $\mathcal{P}$ has the following properties [3]:*

(i) Each vertex of $\mathcal{P}$ is contained in exactly n edges.

(ii) Let $\mathcal{F}$ be a facet of $\mathcal{P}$ and v a vertex of $\mathcal{P}$ in $\mathcal{F}$. Then there are exactly $n-1$ edges in $\mathcal{F}$ containing v.

Theorem 4.1. *Let $\mathcal{P}$ be an n-dimensional simple polytope. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by open-loop controls, then the invariance conditions* (3) *are solvable.*

Proof. Define $\mathcal{Y}(x) := \{Ax + Bw + a \mid w \in \mathbb{R}^m\}$. Let $x_0 \in \mathcal{P} \setminus \mathcal{F}_0$. We show $\mathcal{C}(x_0) \cap \mathcal{Y}(x_0) \neq \emptyset$. By assumption there exists an open-loop control $\mu(t)$ and a time $T > 0$ such that $\phi_\mu(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$. Since $\mu(t)$ is an open-loop control, by definition there exists $c > 0$ such that $\|\mu(t)\| \le c$, for all $t \in [0, T]$. Consider the set $\mathcal{Y}_c(x) := \{Ax + Bw + a \mid w \in \mathbb{R}^m, \|w\| \le c\}$. One can easily show that both $x \mapsto \mathcal{Y}_c(x)$ and $x \mapsto \mathcal{Y}(x)$ are upper semicontinuous. Now take a sequence $\{t_i \mid t_i \in (0, T]\}$ with $t_i \to 0$. Since $\mathcal{Y}_c(x)$ is bounded on $\mathcal{P}$, there exists $M > 0$ such that $\|\phi_\mu(t_i, x_0) - x_0\| \le M t_i$. Therefore $\{\frac{\phi_\mu(t_i,x_0)-x_0}{t_i}\}$ is a bounded sequence and there exists a convergent subsequence (with indices relabeled) such that $\lim_{i\to\infty} \frac{\phi_\mu(t_i,x_0)-x_0}{t_i} =: v$. Since $\phi_\mu(t_i, x_0) \in \mathcal{P}$, $v \in T_{\mathcal{P}}(x_0)$ (see [6], p. 90). Now we show $v \in \mathcal{Y}(x_0)$. We have

$$\frac{\phi_\mu(t_i, x_0) - x_0}{t_i} = \frac{1}{t_i} \int_0^{t_i} \big[A\phi_\mu(\tau, x_0) + B\mu(\tau) + a\big] d\tau \,. \tag{4}$$

Taking the limit, we get

$$v = Ax_0 + a + B \lim_{i\to\infty} \frac{1}{t_i} \int_0^{t_i} \mu(\tau) d\tau \in \mathcal{Y}(x_0) \,.$$

Note that the limit exists by passing to a subsequence, if necessary, because μ is bounded on compact intervals. We conclude that $\mathcal{Y}(x_0) \cap T_{\mathcal{P}}(x_0) \neq \emptyset$, $x_0 \in \mathcal{P} \backslash \mathcal{F}_0$. Since $T_{\mathcal{P}}(x_0) = \mathcal{C}(x_0)$, $x_0 \in \mathcal{P} \backslash \mathcal{F}_0$, it follows that $\mathcal{C}(x_0) \cap \mathcal{Y}(x_0) \neq \emptyset$, $x_0 \in \mathcal{P} \setminus \mathcal{F}_0$. In particular, the invariance conditions are solvable at $v_i \in \mathcal{P} \setminus \mathcal{F}_0$.

Now consider $v_i \in \mathcal{F}_0$. If $v_i \in \mathcal{O}$, then the invariance conditions are solvable by selecting $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = 0$. Instead, suppose $v_i \notin \mathcal{O}$. Suppose by the way of contradiction that $\mathcal{Y}(v_i) \cap \mathcal{C}(v_i) = \emptyset$. Then $\mathcal{Y}(v_i)$ and $\mathcal{C}(v_i)$ are non-empty disjoint polyhedral convex sets in $\mathbb{R}^n$. By Corollary 19.3.3 of [12], they are strongly separated. That is, there exists $\epsilon > 0$ such that $\inf_{y \in \mathcal{Y}(v_i), z \in \mathcal{C}(v_i)} \|y - z\| > \epsilon$. By the upper semicontinuity of $x \mapsto \mathcal{Y}(x)$, there exists $\delta > 0$ such that if $\|x - v_i\| < \delta$, then $\mathcal{Y}(x) \subset \mathcal{Y}(v_i) + \frac{\epsilon}{2}\mathscr{B}$. Because $\mathcal{P}$ is a simple polytope, by Remark 4.1(i) $v_i \in \mathcal{F}_0$ is the intersection of exactly n edges, and by Remark 4.1(ii), $n-1$ of these edges are contained in $\mathcal{F}_0$. Let $\overline{v_i v_j}$ be the edge that is not contained in $\mathcal{F}_0$. By definition of a simple polytope, $\overline{v_i v_j}$ is the intersection of exactly $n-1$ facets. Since $\overline{v_i v_j}$ is not contained in $\mathcal{F}_0$ and the vertex $v_i \in \overline{v_i v_j}$ is contained in exactly n facets by definition of a simple polytope, then the $n-1$ facets whose intersection forms $\overline{v_i v_j}$ are the restricted facets of $\mathcal{P}$ at v_i. This implies by the definition of $J(x)$ that $J(x) = J(v_i)$, for each $x \in [v_i, v_j)$. Then by (1), we conclude that for all $x \in [v_i, v_j)$, $\mathcal{C}(x) = \mathcal{C}(v_i)$. Let $\bar{x} \in (v_i, v_j) \cap \{x \in \mathbb{R}^n \mid \|x - v_i\| < \delta\}$. Since $\bar{x} \in (v_i, v_j)$, $\mathcal{C}(\bar{x}) = \mathcal{C}(v_i)$ so $\inf_{y \in \mathcal{Y}(v_i), z \in \mathcal{C}(\bar{x})} \|y - z\| > \epsilon$. However, we also have $\mathcal{Y}(\bar{x}) \subset \mathcal{Y}(v_i) + \frac{\epsilon}{2}\mathscr{B}$ which implies $\mathcal{C}(\bar{x}) \cap \mathcal{Y}(\bar{x}) = \emptyset$, a contradiction to $\mathcal{C}(x) \cap \mathcal{Y}(x) \neq \emptyset$, $x \in \mathcal{P} \setminus \mathcal{F}_0$.

Example 4.1. *We return to the counterexample and identify the defect to be that $\mathcal{P}$ is not simple - vertex v_1 is contained in four facets. Therefore, Theorem 4.1 does not apply. Indeed, in this specific example solvability of the invariance conditions at v_1 is not necessary for solvability of RCP by open-loop controls.*

5. Conclusion

In this note we have shown by way of a counterexample that, in contrast with the case for continuous state feedback, the invariance conditions are not necessary for solvability of RCP on polytopes by open-loop controls. We have identified a suitable class of polytopes, simple polytopes, for which the invariance conditions are necessary. An open problem is to find the largest class of feedbacks needed to solve RCP on simple polytopes assuming it is solvable by open-loop controls.

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