Stability of Discrete-Time Switched Systems with Multiple Equilibria using a Common Quadratic Lyapunov Function

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***Abstract*—The paper studies stability of a class of discrete-time switched systems under arbitrary switching in which each subsystem has a different equilibrium. We extend the concept of a common quadratic Lyapunov function in order to characterize asymptotic stability of a set that contains all the equilibria. The results are applied to a model of visuomotor adaptation to analyze boundedness of solutions when the visual error measurement is intermittent.**

***Index Terms*—Switched systems, hybrid systems, Lyapunov methods**

I. Introduction

A switched system consists of a family of dynamic subsystems and a switching signal that activates one system at each (discrete or continuous) time. Switched systems have been extensively studied under the assumption that there is a single equilibrium shared by all the subsystems [9]. More recently researchers have begun investigating switched systems with multiple equilibria due to their relevance in applications such as walking robots [16], [21], aerial robots [4], planning multiple robotic arms [5], power management in wireless networks [1], and modeling of non-spiking neurons [12]. Our interest arises from the application of regulator theory to certain problems of neuroscience [3], [6], [7] in which the error measurement driving disturbance rejection arrives from the environment and can be intermittent. Intermittency of the error signal results in switching between subsystems with different equilibria.

Stability of switched systems with multiple equilibria appears to have been first formulated in [2]. They developed a Lyapunov-based analysis method for continuous-time nonlinear systems to show that trajectories globally converge to a set containing all equilibria, so long as the switching signal satisfies a dwell-time constraint. The method of [2] was extended to switched systems with multiple invariant sets in [4]. Using the same set construction as in [2], the authors in [16] showed that for discrete-time systems with a dwell-time constraint, there exists a set containing all equilibria such that if initial conditions start in this set, then solutions remain in a larger set. Continuous and discrete-time switched systems with disturbances were studied in [21]. The authors show the ultimate boundedness of solutions under bounded disturbances assuming each subsystem is globally input-to-state stable. They also provide a practical stability result when the equilibria of the subsystems are locally exponentially stable. In both cases, an average dwell-time constraint is imposed on the switching signal. Finally, several related studies examine particular classes of switched systems. Continuous-time nonlinear systems in which the equilibrium, but not the vector field, is switched were considered in [13], while continuous and discrete-time positive linear switched systems were analyzed in [10], [11] respectively.

Most prior work on systems with multiple equilibria assumes the switching signal satisfies a dwell-time constraint. While this assumption is reasonable in applications such as walking robots where time must pass between transitions between gaits, the assumption is less relevant in neuroscience applications in which switching arises from exogenous, intermittent measurements. This paper provides the first stability results, to our knowledge, for discrete-time nonlinear systems with multiple equilibria without imposing a dwell-time constraint. The concept of a common quadratic Lyapunov function is extended to systems with multiple equilibria. Starting from the same set construction as in [2], we show the existence of a positively invariant set containing all equilibria. Then we use a slightly different construction to obtain a (possibly larger) positively invariant set that is also globally asymptotically stable. Both results do not assume any dwell-time constraints. The results are applied to a model of visuomotor adaptation, showing that solutions remain bounded under intermittent measurements [7].

II. Problem Formulation

Let N be a positive integer, and define the index set $\mathcal{P} := \{1, \dots, N\}$. Consider the family of discrete-time systems

$$x(k+1) = f_p(x(k)) \qquad p \in \mathcal{P}\,. \tag{1}$$

where $x \in \mathbb{R}^n$ and $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We assume for each $p \in \mathcal{P}$ there exists a unique fixed point $x_p^\star \in \mathbb{R}^n$ such that $x_p^\star = f(x_p^\star)$. In the sequel we associate the system $x(k+1) = f_p(x(k))$ with its index p. A map $\sigma : \mathbb{Z}_0^+ \to \mathcal{P}$ is called a *switching signal* for (1). We denote by $\{s_1, s_2, \dots : s_i > 0\}$ the *switching times*; that is, $\sigma(s_i) \neq \sigma(s_i - 1)$ for all $i \in \mathbb{Z}^+$, and $\sigma(k) = \sigma(k-1)$, otherwise. Associated with a

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switching signal σ is an integer $\tau_d \geq 1$ called the *dwell-time*, corresponding to the minimum number of steps between two successive switches in σ; that is, $\sigma(s_i + k) = \sigma(s_i)$ for all $k < \tau_d$. A *switched system* is given by

$$x(k+1) = f_{\sigma(k)}(x(k))\,, \tag{2}$$

where $\sigma : \mathbb{Z}_0^+ \to \mathcal{P}$ is any switching signal for (1).

Definition 1: We say the continuous function $V : \mathbb{R}^n \to \mathbb{R}$ with the form $V(x) = x^\intercal P x$ is a *common quadratic Lyapunov function* of (1) if $P \in \mathbb{R}^{n\times n}$ is symmetric, positive definite, and for each $p \in \mathcal{P}$, $V_p(x) := V(x - x_p^\star)$ is an *exponential Lyapunov function* for system $p \in \mathcal{P}$; that is, there exist two class $\mathcal{K}_\infty$ maps $\chi_{p,1}$ and $\chi_{p,2}$ and $0 < \epsilon_p < 1$ such that for all $x \in \mathbb{R}^n$

$$\chi_{p,1}(\|x - x_p^\star\|) \leq V_p(x) \leq \chi_{p,2}(\|x - x_p^\star\|) \tag{3a}$$

$$V_p(f_p(x)) \leq \epsilon_p V_p(x)\,. \tag{3b}$$

Problem 1: Consider the switched system (2) with common quadratic Lyapunov function. We want to show there exists a compact set Ω containing all equilibria $\{x_p^\star\}$ such that Ω is positively invariant and globally asymptotically stable under the switched dynamics (2) with any switching signal σ.

III. Stability with Dwell Time

In this section we review existing results from [2], [16], as these provide a foundation for our new results. Consider the family of discrete-time systems (1) and suppose each system $p \in \mathcal{P}$ has an exponential Lyapunov function. Define

$$\epsilon := \max_{p\in\mathcal{P}}\{\epsilon_i\}\,. \tag{4}$$

Let $c > 0$. For each $p \in \mathcal{P}$, define the sublevel set

$$\Omega_p(c) := \{x \in \mathbb{R}^n \mid V_p(x) \leq c\}\,, \tag{5}$$

and define their union

$$\Omega(c) := \bigcup_{p\in\mathcal{P}} \Omega_p(c)\,. \tag{6}$$

Next we define constants

$$\omega_p(c) := \max_{x\in\Omega(c)} V_p(x) \tag{7a}$$

$$\omega_{max}(c) := \max_{p\in\mathcal{P}} \omega_p(c) \tag{7b}$$

$$\omega_{min}(c) := \min_{p\in\mathcal{P}} \omega_p(c)\,. \tag{7c}$$

Finally, define the sets

$$\mathcal{M}_p(c) := \{x \in \mathbb{R}^n \mid V_p(x) \leq \omega_p(c)\}\,, \tag{8}$$

and

$$\mathcal{M}(c) := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p(c) \quad ,\mathcal{M}^0(c) := \bigcap_{p\in\mathcal{P}} \mathcal{M}_p(c)\,. \tag{9}$$

The set $\mathcal{M}_p(c)$ is the smallest sublevel set of V_p that contains the set $\Omega(c)$. Figure 1 shows an example with two subsystems.

Remark 1: (i) By (3) each V_p is radially unbounded, so the sublevel sets $\Omega_p(c)$ and $\mathcal{M}_p(c)$ are compact. It follows that $\Omega(c)$, $\mathcal{M}(c)$ and $\mathcal{M}^0(c)$ are compact. Moreover, the continuous function V_p attains its maximum on $\Omega(c)$, implying that $\omega_p(c)$ is well-defined.

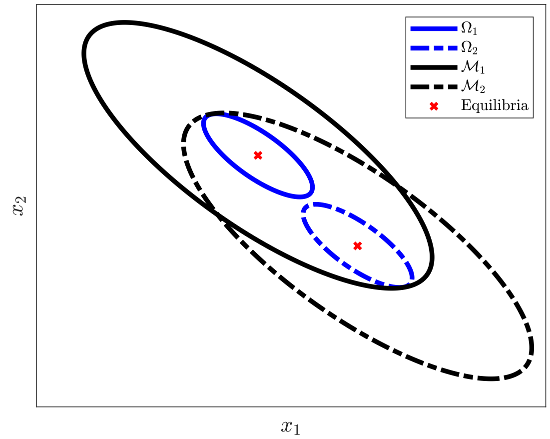

Fig. 1: The sets Ω_p & $\mathcal{M}_p$ for a second order switched system with two subsystems.

(ii) For each $c > 0$, $\Omega(c) \subset \mathcal{M}^0(c)$. For if $y \in \Omega(c)$, then for each $p \in \mathcal{P}$, $V_p(y) \leq \omega_p(c) = \max_{x\in\Omega(c)} V_p(x)$. Thus, $y \in \mathcal{M}_p(c)$ for each p, so $y \in \mathcal{M}^0(c)$.

(iii) By definition of $\Omega(c)$, $\omega_p(c) \geq c$ for each $p \in \mathcal{P}$, and $\omega_{min}(c), \omega_{max}(c) \geq c$.

(iv) By (ii), for all $c > 0$, $\mathcal{M}^0(c) \neq \emptyset$. Also $\mathcal{M}(c)$ is connected since it is the union of connected sets whose intersection $\mathcal{M}^0(c)$ is non-empty. $\triangleleft$

The main result in the literature regards positive invariance of $\mathcal{M}(c)$ under any switching signal that satisfies a lower bound on the dwell-time assuming solutions start in $\mathcal{M}^0(c)$.

Theorem 1 ([16]): Consider the family of discrete-time systems (1), each with a unique fixed point $x_p^\star$, $p \in \mathcal{P}$. Let $V_p : \mathbb{R}^n \to \mathbb{R}$ be an exponential Lyapunov function for system $p \in \mathcal{P}$. Suppose that for every $c > 0$, there exists $\mu(c) > 1$ such that for all $p, q \in \mathcal{P}$ and $x \in \mathbb{R}^n \setminus \Omega(c)$,

$$\frac{V_p(x)}{V_q(x)} \leq \mu(c)\,. \tag{10}$$

Then for any initial condition $x(0) \in \mathcal{M}^0(c)$ and any switching signal σ with dwell-time τ_d satisfying

$$\tau_d \geq \frac{\log(\mu(c)\frac{\omega_{max}(c)}{\omega_{min}(c)})}{\log(\frac{1}{\epsilon})}\,, \tag{11}$$

the solution of (2) starting at $x(0)$ satisfies $x(k) \in \mathcal{M}(c)$ for all $k \geq 0$.

Formula (11) is equivalent to the statement that $\mu(c)\epsilon^{\tau_d}$ is bounded above by $\omega_{min}(c)/\omega_{max}(c)$. Then by (10), it can be shown that at switching times s_i, the state $x(s_i)$ can not be outside the set $\mathcal{M}^0(c)$. Between switching times, the Lyapunov function of the current subsystem will decrease, thus keeping the state $x(k)$ inside the set $\mathcal{M}(c)$.

Example 1: When the dwell time requirement is violated, then the previous result fails. Consider the discrete-time switched system

$$x(k+1) = \begin{cases} A_1 x(k) + b_1 & k \text{ is even} \\ A_2 x(k) + b_2 & k \text{ is odd}\,, \end{cases} \tag{12}$$

where $x(k) \in \mathbb{R}^2$, and

$$A_1 = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9 & 0 \\ 1 & 0.9 \end{bmatrix}, b_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The equilibria are $x_1^\star = (1, 0)$ and $x_2^\star = (0, 1)$. Letting $y(k) = x(2k)$, the switching rule implies y evolves as

$$y(k+1) = A_2 A_1 y(k) + A_2 b_1 + b_2 .$$

Since $\sigma(A_2 A_1) = \{0.28, 2.34\}$, $y(k)$ and $x(k)$ will grow unbounded even though each subsystem is stable. $\triangleleft$

IV. Existence of a Positively Invariant Set

Consider again the switched system (2). Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a common quadratic Lyapunov function of (2). For any $c > 0$, let ϵ, $\omega_p(c)$, $\omega_{max}(c)$ and $\omega_{min}(c)$ as well as the sets $\Omega_p(c)$, $\Omega(c)$, $\mathcal{M}_p(c)$ and $\mathcal{M}^0(c)$ be defined as above with $V_p(x) = V(x - x_p^\star)$, $p \in \mathcal{P}$. To show the existence of a positively invariant set containing all equilibria, we first show that assumption (10) of an upper bound on the ratio $V_i(x)/V_j(x)$ may be removed.

Lemma 2: Consider the switched system (2) with common quadratic Lyapunov function V such that $V_p(x) = V(x - x_p^\star)$ for each $p \in \mathcal{P}$. Let $c > 0$. The following supremum exists:

$$\mu(c) := \sup_{i,j \in \mathcal{P}, x \in \mathbb{R}^n \setminus \Omega(c)} \frac{V_i(x)}{V_j(x)} . \tag{13}$$

Moreover, $\lim_{c\to\infty} \mu(c) = 1$.

Proof: Consider any pair $i, j \in \mathcal{P}$ and any $x \in \mathbb{R}^n$. We have

$$\begin{aligned} V_i(x) &= (x - x_i^\star)^\intercal P (x - x_i^\star) \\ &= (x - x_j^\star + x_j^\star - x_i^\star)^\intercal P (x - x_j^\star + x_j^\star - x_i^\star) \\ &= (x - x_j^\star)^\intercal P (x - x_j^\star) + 2(x_j^\star - x_i^\star)^\intercal P (x - x_j^\star) \\ &\quad + (x_j^\star - x_i^\star)^\intercal P (x_j^\star - x_i^\star) . \end{aligned}$$

Since P is symmetric positive definite, $\|P\| = \sqrt{\lambda_{max}(P^\intercal P)} = \lambda_{max}(P)$, where $\lambda_{max}(P)$ is the maximum eigenvalue of P. Using the Cauchy Schwarz inequality, we have

$$\begin{aligned} (x_j^\star - x_i^\star)^\intercal P (x - x_j^\star) &\le \|x_j^\star - x_i^\star\| \, \|P(x - x_j^\star)\| \\ &\le \lambda_{max}(P) \|x_j^\star - x_i^\star\| \, \|x - x_j^\star\| . \end{aligned}$$

Then we have

$$\begin{aligned} V_i(x) \le V_j(x) &+ 2\lambda_{max}(P) \|x_j^\star - x_i^\star\| \, \|x - x_j^\star\| \\ &+ (x_j^\star - x_i^\star)^\intercal P (x_j^\star - x_i^\star) . \end{aligned}$$

Because P is symmetric, $\lambda_{min}(P)\|x - x_j^\star\|^2 \le V_j(x)$, or equivalently $\|x - x_j^\star\| \le \sqrt{V_j(x)/\lambda_{min}(P)}$. Thus, we get

$$V_i(x) \le V_j(x) + \alpha_{ij} \sqrt{V_j(x)} + \beta_{ij} ,$$

where

$$\alpha_{ij} := 2 \frac{\lambda_{max}(P)}{\sqrt{\lambda_{min}(P)}} \|x_j^\star - x_i^\star\| \ge 0 ,$$

and $\beta_{ij} := (x_j^\star - x_i^\star)^\intercal P (x_j^\star - x_i^\star) \ge 0$. Define the constants $\alpha := \max_{i,j \in \mathcal{P}} \alpha_{ij}$ and $\beta := \max_{i,j \in \mathcal{P}} \beta_{ij}$. We conclude that for any pair $i, j \in \mathcal{P}$ and any $x \in \mathbb{R}^n$,

$$V_i(x) \le V_j(x) + \alpha \sqrt{V_j(x)} + \beta . \tag{14}$$

Now for any $x \in \mathbb{R}^n \setminus \Omega(c)$, $V_j(x) > c$. Combining with (14), we have

$$\frac{V_i(x)}{V_j(x)} \le 1 + \frac{\alpha}{\sqrt{c}} + \frac{\beta}{c} . \tag{15}$$

For each $i, j \in \mathcal{P}$, there exists $\bar{x} \in \mathbb{R}^n \setminus \Omega(c)$ such that $\overline{x} - x_i^\star = \kappa(\overline{x} - x_j^\star)$ with $\kappa \ge 1$ ($\overline{x}$ lies on the line joining $x_i^\star$ and $x_j^\star$, extending beyond the bounded set $\Omega(c)$). By the definition of V_p, this implies

$$\frac{V_i(\overline{x})}{V_j(\overline{x})} = \frac{(\overline{x} - x_i^\star)^\intercal P (\overline{x} - x_i^\star)}{(\overline{x} - x_j^\star)^\intercal P (\bar{x} - x_j^\star)} = \kappa^2 \ge 1 . \tag{16}$$

Combining (15) and (16) and taking the supremum over every $x \in \mathbb{R}^n \setminus \Omega(c)$, we obtain

$$1 \le \sup_{x \in \mathbb{R}^n \setminus \Omega(c)} \frac{V_i(x)}{V_j(x)} \le 1 + \frac{\alpha}{\sqrt{c}} + \frac{\beta}{c} . \tag{17}$$

We conclude the supremum $\mu(c)$ exists. Moreover, by taking the limit at $c \to \infty$, we obtain $\lim\limits_{c\to\infty} \mu(c) = 1$. ∎

Lemma 3: Consider the switched system (2) with common quadratic Lyapunov function V such that $V_p(x) = V(x - x_p^\star)$ for each $p \in \mathcal{P}$. Let $c > 0$, and consider $\omega_{max}(c)$ and $\omega_{min}(c)$ defined in (7). We have

$$\lim_{c\to\infty} \frac{\omega_{max}(c)}{\omega_{min}(c)} = 1 .$$

Proof: Let $i \in \mathcal{P}$ and select any $\overline{x}_c \in \operatorname*{argmax}_{x \in \Omega(c)} V_i(x)$; that is $V_i(\overline{x}_c) = \omega_i(c)$. Since $\overline{x}_c \in \Omega(c)$, we know $\overline{x}_c \in \Omega_j(c)$ for some $j \in \mathcal{P}$. Notice that since $\overline{x}_c \in \Omega_j(c)$, $V_j(\overline{x}_c) \le c$. Then applying (14), we find

$$\omega_i(c) = V_i(\overline{x}_c) \le V_j(\overline{x}_c) + \alpha\sqrt{V_j(\overline{x}_c)} + \beta \le c + \alpha\sqrt{c} + \beta , \tag{18}$$

where $\alpha, \beta \ge 0$ are defined in the proof of Lemma 2. By Remark 1(iii), $\omega_{min}(c) \ge c$, which combining with (18) yields

$$1 \le \frac{\omega_i(c)}{\omega_{min}(c)} \le 1 + \frac{\alpha}{\sqrt{c}} + \frac{\beta}{c} .$$

Taking the maximum over $i \in \mathcal{P}$, we have

$$1 \le \frac{\omega_{max}(c)}{\omega_{min}(c)} \le 1 + \frac{\alpha}{\sqrt{c}} + \frac{\beta}{c} . \tag{19}$$

Finally, by evaluating the limit as $c \to \infty$, the result is obtained. ∎

The next result establishes the existence of a positively invariant set containing all equilibria of the switched system.

Theorem 4: Consider the switched system (2) with common quadratic Lyapunov function V. There exists $c^\star > 0$ such that $\mathcal{M}^0(c^\star)$ is positively invariant under the switched dynamics (2) with any switching signal σ.

Proof: By combining Lemmas 2 and 3, we have

$$\lim_{c\to\infty} \mu(c)\frac{\omega_{max}(c)}{\omega_{min}(c)} = 1\,. \tag{20}$$

Recall ϵ defined in (4). By (20), there exists $c^\star > 0$ sufficiently large such that

$$\epsilon\mu(c^\star) < \frac{\omega_{min}(c^\star)}{\omega_{max}(c^\star)}\,. \tag{21}$$

Consider an arbitrary switching signal $\sigma : \mathbb{Z}_0^+ \to \mathcal{P}$. Suppose $x(k) \in \mathcal{M}^0(c^\star)$. We want to show that $x(k+1) \in \mathcal{M}^0(c^\star)$. Suppose by way of contradiction that $x(k+1) \notin \mathcal{M}^0(c^\star)$. By Remark 1(ii), $\Omega(c^\star) \subset \mathcal{M}^0(c^\star)$, so $x(k+1) \notin \Omega(c^\star)$. Then we can apply (13) and (3b) to obtain that for all $p \in \mathcal{P}$,

$$V_p(x(k+1)) \le \mu(c^\star)V_{\sigma(k)}(x(k+1)) \le \epsilon\mu(c^\star)V_{\sigma(k)}(x(k))\,. \tag{22}$$

Because $x(k) \in \mathcal{M}^0(c^\star)$, then $x(k) \in \mathcal{M}_p(c^\star)$ for all $p \in \mathcal{P}$. In particular, $x(k) \in \mathcal{M}_{\sigma(k)}(c^\star)$. By definition of $\mathcal{M}_p(c^\star)$, this implies

$$V_{\sigma(k)}(x(k)) \le \omega_{\sigma(k)}(c^\star)\,. \tag{23}$$

Finally, applying (21) and (23) to (22), we obtain $\forall p \in \mathcal{P}$

$$V_p(x(k+1)) \le \frac{\omega_{min}(c^\star)}{\omega_{max}(c^\star)}\omega_{\sigma(k)}(c^\star) \le \omega_{min}(c^\star) \le \omega_p(c^\star)\,.$$

That is, $x(k+1) \in \mathcal{M}_p(c^\star)$ for all $p \in \mathcal{P}$, so $x(k+1) \in \mathcal{M}^0(c^\star)$, a contradiction. ■

Remark 2: An estimate of $c^\star$ may be obtained from the foregoing derivations. First we select

$$1 + \frac{\alpha}{\sqrt{c^\star}} + \frac{\beta}{c^\star} < \sqrt{\frac{1}{\epsilon}}. \tag{24}$$

Then using (17) and (19), we have $\mu(c^\star)\frac{\omega_{max}(c^\star)}{\omega_{min}(c^\star)} \le \left(1 + \frac{\alpha}{\sqrt{c^\star}} + \frac{\beta}{c^\star}\right)^2 < \frac{1}{\epsilon}$. Hence, (21) is satisfied . ◁

V. Global Asymptotic Stability

The previous section constructed a positively invariant set containing all equilibria. Now we use a somewhat different construction to obtain a (possibly larger) positively invariant set that is also globally asymptotically stable. Consider again the switched system (2) with a common quadratic Lyapunov function V and $V_p(x) = V(x - x_p^\star)$, $p \in \mathcal{P}$. For any $c > 0$, let $\omega_p(c)$, $\omega_{max}(c)$, $\Omega_p(c)$ and $\Omega(c)$ be as above. For each $p \in \mathcal{P}$, define the set

$$\mathcal{N}_p(c) := \{x \in \mathbb{R}^n \mid V_p(x) \le \omega_{max}(c)\}\,, \tag{25}$$

and the intersection

$$\mathcal{N}^0(c) := \bigcap_{p\in\mathcal{P}} \mathcal{N}_p(c)\,. \tag{26}$$

Let $V_{max}(x) := \max_{p\in\mathcal{P}}\{V_p(x)\}$ and define its sublevel set

$$\Omega_{max}(c) := \{x \in \mathbb{R}^n \mid V_{max}(x) \le c\}\,. \tag{27}$$

Remark 3: We note several properties of these new sets.

(i) In contrast to $\mathcal{M}^0(c)$, the set $\mathcal{N}^0(c)$ is a sublevel set of the maximum function V_{max}. That is, for each $c > 0$, $\mathcal{N}^0(c) = \Omega_{max}(\omega_{max}(c))$.

(ii) By (3) each V_p is a continuous radially unbounded function, so $V_{max}(x)$ is also continuous and radially unbounded. Hence, $\Omega_{max}(c)$ and $\mathcal{N}^0(c)$ are compact.

(iii) Analogous to Remark 1(ii), for each $c > 0$, $\Omega(c) \subset \mathcal{N}^0(c)$.

(iv) By Lemma 3, $\omega_{min}(c)$ and hence $\omega_p(c)$, $p \in \mathcal{P}$ approach $\omega_{max}(c)$ as c increases. Hence, the set $\mathcal{M}^0(c)$ approaches $\mathcal{N}^0(c)$ for increasing values of c.

(v) In the case of only two subsystems, namely $N = 2$, we have $\mathcal{N}^0(c) = \mathcal{M}^0(c)$ for each $c > 0$. This follows because $\omega_1(c) = \omega_2(c) = \omega_{max}(c)$, by a symmetry argument. ◁

The following is the main result on existence of a globally asymptotically stable set.

Theorem 5: Consider the switched system (2) with common quadratic Lyapunov function V. Let $c^\star > 0$ be as in Theorem 4. Then $\mathcal{N}^0(c^\star)$ is positively invariant and globally asymptotically stable under (2) with any switching signal σ.

Proof: Let $\sigma : \mathbb{Z}_0^+ \to \mathcal{P}$ be an arbitrary switching signal. From (21)

$$\mu(c^\star) < \frac{\omega_{min}(c^\star)}{\epsilon\omega_{max}(c^\star)} \le \frac{1}{\epsilon}\,. \tag{28}$$

First, we show that for all $p \in \mathcal{P}$, if $f_p(x) \notin \mathcal{N}^0(c^\star) = \Omega_{max}(\omega_{max}(c^\star))$, then $V_{max}(f_p(x)) < V_{max}(x)$. To that end, fix $p \in \mathcal{P}$ and $x \in \mathbb{R}^n$ such that $f_p(x) \notin \mathcal{N}^0(c^\star)$. By Remark 3(iii), $\Omega(c^\star) \subset \mathcal{N}^0(c^\star)$, so $f_p(x) \notin \Omega(c^\star)$. By (13), we have

$$V_q(f_p(x)) = V_q(f_p(x))\frac{V_p(f_p(x))}{V_p(f_p(x))} \le \mu(c^\star)V_p(f_p(x))\,,$$

for all $q \in \mathcal{P}$. By (3), $V_p(f_p(x)) \le \epsilon_p V_p(x)$, and by (28), $\mu(c^\star) < \frac{1}{\epsilon}$. Thus,

$$V_q(f_p(x)) \le \mu(c^\star)\epsilon_p V_p(x) < V_p(x)\,,$$

for all $q \in \mathcal{P}$. Then we have

$$\begin{aligned} V_{max}(f_p(x)) &= \max_{q\in\mathcal{P}} V_q(f_p(x)) < V_p(x) \\ &\le \max_{q\in\mathcal{P}} V_q(x) = V_{max}(x)\,. \end{aligned}$$

This proves that for all $p \in \mathcal{P}$,

$$f_p(x) \notin \mathcal{N}^0(c^\star) \implies V_{max}(f_p(x)) < V_{max}(x)\,. \tag{29}$$

Second we show that for every $p \in \mathcal{P}$, if $x \notin \mathcal{N}^0(c^\star)$ then $V_{max}(f_p(x)) < V_{max}(x)$. Fix $p \in \mathcal{P}$ and let $x \notin \mathcal{N}^0(c^\star)$. If $f_p(x) \notin \mathcal{N}^0(c^\star)$, then we are done. If $f_p(x) \in \mathcal{N}^0(c^\star)$, then $V_{max}(f_p(x)) \le \omega_{max}(c^\star)$. Since $x \notin \mathcal{N}^0(c^\star)$, we also know $V_{max}(x) > \omega_{max}(c^\star)$. Therefore, $V_{max}(f_p(x)) < V_{max}(x)$. This proves that for all $p \in \mathcal{P}$,

$$x \notin \mathcal{N}^0(c^\star) \implies V_{max}(f_p(x)) < V_{max}(x)\,. \tag{30}$$

The rest of the proof has three parts: (i) positive invariance of $\mathcal{N}^0(c^\star)$; (ii) global attractivity of $\mathcal{N}^0(c^\star)$; and (iii) stability of $\mathcal{N}^0(c^\star)$.

(i) We show that for each $\omega \geq \omega_{max}(c^\star)$, $\Omega_{max}(\omega)$ is positively invariant. Let $x(k) \in \Omega_{max}(\omega)$ and $x(k+1) \notin \Omega_{max}(\omega)$. Then $V_{max}(x(k)) \leq \omega$. Since $\Omega_{max}(\omega_{max}(c^\star)) \subseteq \Omega_{max}(\omega)$, then $x(k+1) \notin \Omega_{max}(\omega_{max}(c^\star))$. By Remark 3(i) and (29), $V_{max}(x(k+1)) < V_{max}(x(k)) \leq \omega$, a contradiction. This proves the positive invariance of the set $\Omega_{max}(\omega)$ and in particular $\mathcal{N}^0(c^\star)$.

(ii) To prove attractivity of $\mathcal{N}^0(c^\star)$, we show that for all $x(0) \in \mathbb{R}^n$, the point-to-set distance $\mathrm{d}(x(k), \mathcal{N}^0(c^\star)) \to 0$ as $k \to \infty$. Let $x_0 \in \mathbb{R}^n$ and $x(0) = x_0$. If there exists $k_0 \geq 0$ such that $x(k_0) \in \mathcal{N}^0(c^\star)$, then by part (i), $\mathrm{d}(x(k), \mathcal{N}^0(c^\star)) = 0$ for all $k \geq k_0$. Suppose no such $k_0 \geq 0$ exists. Using (30), $x(k) \notin \mathcal{N}^0(c^\star)$ implies $V_{max}(x(k+1)) < V_{max}(x(k))$ for all $k \geq 0$. Therefore, $V_{max}(x(k))$ is monotonically decreasing and bounded from below. Therefore $V_{max}(x(k))$ converges to some ω as $k \to \infty$. We claim $\omega = \omega_{max}(c^\star)$. Suppose not. Let $\omega_0 = V_{max}(x_0)$. Since $x_0 \notin \mathcal{N}^0(c^\star)$, then $\omega_0 > \omega_{max}(c^\star)$. By Remark 3(ii), $V_{max}(x)$ is continuous and radially unbounded, so $\Omega_{max}(w_0)$ is compact. By part (i), $\Omega_{max}(\omega_0)$ is positively invariant. Defining the compact set $\mathcal{W} := \{x \in \mathbb{R}^n \mid \omega \leq V_{max}(x) \leq \omega_0\}$, we have $x(k) \in \mathcal{W}$ for all $k \geq 0$. Since $\Delta V_{max,p}(x) := V_{max}(f_p(x)) - V_{max}(x)$ is a continuous function, it attains its maximum on $\mathcal{W}$. Let $\alpha_p := \max_{x \in \mathcal{W}} \Delta V_{max,p}(x)$ and $\alpha := \max_{p \in \mathcal{P}} \alpha_p$. Since $x \in \mathcal{W}$ implies $x \notin \mathcal{N}^0(c^\star)$, then by (30) we know that $\alpha < 0$. Then we compute $V_{max}(x(k+1)) = \omega_0 + \sum_{j=0}^{k} V_{max}(x(j+1)) - V_{max}(x(j)) \leq \omega_0 + (k+1)\alpha$. This calculation implies Then $V_{max}(x(k))$ eventually decreases below $\omega_{max}(c^\star)$, a contradiction. Thus, $\omega = \omega_{max}(c^\star)$ and $V_{max}(x(k)) \to \omega_{max}(c^\star)$ as $k \to \infty$. We conclude $\mathrm{d}(x(k), \mathcal{N}^0(c^\star)) \to 0$ as $k \to \infty$.

(iii) To prove stability of $\mathcal{N}^0(c^\star)$, we show that for every $r > 0$ there exists $\delta > 0$ such that if $x(0) \in \mathcal{B}_\delta(\mathcal{N}^0(c^\star))$, the δ-neighborhood of $\mathcal{N}^0(c^\star)$, then $x(k) \in \mathcal{B}_r(\mathcal{N}^0(c^\star))$ for all $k \geq 0$. Fix $r > 0$ and let $\omega_0 = \min_{x \in \partial\mathcal{B}_r(\mathcal{N}^0(c^\star))} V_{max}(x)$. Choose $\omega \in (\omega_{max}(c^\star), \omega_0)$. It follows that $\mathcal{N}^0(c^\star) \subset \Omega_{max}(\omega) \subset \mathcal{B}_r(\mathcal{N}^0(c^\star))$. Moreover, the set $\Omega_{max}(\omega)$ is positively invariant by part (i). By the continuity of $V_{max}(x)$ and compactness of $\mathcal{N}^0(c^\star)$, there exists $\delta > 0$ such that $\mathcal{B}_\delta(\mathcal{N}^0(c^\star)) \subset \Omega_{max}(\omega)$. Then we have $x(0) \in \mathcal{B}_\delta(\mathcal{N}^0(c^\star)) \subset \Omega_{max}(\omega)$ implies $x(k) \in \Omega_{max}(\omega) \subset \mathcal{B}_r(\mathcal{N}^0(c^\star))$ for all $k \geq 0$, as desired. ∎

VI. Applications

A. Numerical Example

First, we apply the results to a first-order nonlinear switched system. Consider (2) with $x \in \mathbb{R}$, $\mathcal{P} = \{1, 2\}$, $f_1(x) = (1 + x^2)^{-1}$ and $f_2(x) = x^2(1+x^2)^{-1}$. Each subsystem has a single equilibrium point $x_1^\star = 0.6823$, and $x_2^\star = 0$. Let $V(x) = x^2$ and $V_i(x) = V(x - x_i^\star)$, $i \in \mathcal{P}$. It can be shown that $V_1(x)$ and $V_2(x)$ satisfy (3) with $\epsilon_1 = 0.5$ and $\epsilon_2 = 0.3$. Hence, $V(x)$ is a common quadratic Lyapunov function and by Theorem 5 and Remark 3(v) there exists $c^\star > 0$ such that the compact set $\mathcal{M}^0(c^\star)$ is positively invariant and globally asymptotically stable under (2) with any switching signal σ. By Remark 2, we can choose $c^\star = 13.5$ to satisfy (24). Then, we have $\Omega_1(c^\star) = [-2.99 \quad 4.36]$, $\Omega_2(c^\star) = [-3.67 \quad 3.67]$, $\omega_1(c^\star) = \omega_2(c^\star) = 18.98$, $\mathcal{M}_1(c^\star) = [-3.67 \quad 5.04]$, $\mathcal{M}_2(c^\star) = [-4.36 \quad 4.36]$ and $\mathcal{M}^0(c^\star) = [-3.67 \quad 4.36]$.

B. Visuomotor Adaptation

Visuomotor adaptation is a subconscious brain process taking place over repetitive trials and elicited by a visual error closely following the execution of a movement. It is intended to calibrate over a lifetime the mapping between what is seen and how to move. In [6] we showed that the dynamic properties of visuomotor adaptation are consistent with a computational model based on disturbance rejection of constant disturbances. The computational model was further developed in [7], where we considered new neuroscience experiments whose theme is to study the effect of removal of the visual error during sets of trials of a particular body movement such as fast arm reaches [8], [20], [14].

The new experiments show that after steady-state is reached with a non-zero disturbance in an experiment such as fast arm reaches, the continued observation of a zero or small visual error is a flag to the brain to "keep the internal model charged up". Instead, if after reaching steady-state with a non-zero disturbance, the visual error is removed, then the subject makes movements that suggest an internal model is dissipating its estimate of the disturbance at an exponential rate. Consider the model of visuomotor adaptation proposed in [7]:

$$x(k+1) = Ax(k) + Bu(k) \tag{31a}$$
$$e(k) = -x(k) - \overline{d} \tag{31b}$$
$$w_0(k+1) = Fw_0(k) + FGe(k) \tag{31c}$$
$$w_1(k+1) = Fw_1(k) - Ge(k) \tag{31d}$$
$$w_2(k+1) = Fw_2(k) - Gu(k) \tag{31e}$$
$$\hat{w}(k) = w_0(k) + Ge(k) + Aw_1(k) - Bw_2(k) \tag{31f}$$
$$u(k) = Ke(k) + \frac{\psi}{B}\hat{w}(k)\,, \tag{31g}$$

where $x(k) \in \mathbb{R}$ is the position of a single degree of freedom of the body at the end of the k-th trial; $e(k)$ is a visual error at the end of the k-th trial between the observed body position and a reference position at zero; $\overline{d} \in \mathbb{R}$ is an unknown constant disturbance imposed on the subject's visual perception, and $w_i \in \mathbb{R}$, $i \in \{1, 2, 3\}$ are the filter states of the internal model. The controller u consists of a stabilizing error feedback $Ke(k)$ and a term $\frac{\psi}{B}\hat{w}(k)$ to cancel the disturbance (filtered through the plant). Nominal parameters values to recover the dynamic properties of visuomotor adaptation are $A = 0$, $B = 1$, $F = 0.9$, $G = 0.1$, $\psi = \frac{1-F}{G}$, $K = 0.25$; see [7] for details.

When there is no error signal, then the filters (31c) and (31d) are stable, but the filter (31e) (that accounts for the internal model principle) is not. To make this filter and thus the overall system stable when there is no error measurement, we proposed a switching mechanism in [7] to remove $u(k)$ from (31e) when the error is not available at the end of trial k. The result is a switched system consisting of two exponentially stable subsystems each with a different equilibrium. The switched system is given by

$$y(k+1) = A_{\sigma(k)}y(k) + b_{\sigma(k)}\,, \tag{32}$$

where $y(k) = (x(k), w_0(k), w_1(k), w_2(k))$,

$$A_1 = \begin{bmatrix} A - BK - \psi G & \psi & \psi A & -\psi B \\ -FG & F & 0 & 0 \\ G & 0 & F & 0 \\ G(K + G\psi B^{-1}) & -G\psi B^{-1} & -GA\psi B^{-1} & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} A & \psi & \psi A & -\psi B \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix}, b_1 = \begin{bmatrix} -(BK + \psi G) \\ -FG \\ G \\ G(K + G\psi B^{-1}) \end{bmatrix} \overline{d},$$

$b_2 = \mathbf{0}$, and $\sigma : \mathbb{Z}_0^+ \to \{1, 2\}$.

This switching model accounts for experiments in which a dwell time constraint is present; for example, Figure 2 in [8]. We could not make any deductions about boundedness of solutions with arbitrary switching due to the lack of available theoretical results, despite the fact that experiments clearly show that the brain is able to handle rapid on and off switching of the internal model; see Figure 2 of [20] where the imposition of a certain noise distribution on the error measurement appears to cause rapid changes in the subject's response.

To apply our new stability results, we must identify a common quadratic Lyapunov function. Standard conditions for existence of a common quadratic Lyapunov function include that A_1 and A_2 commute [17]; or that A_1 and A_2 are simultaneously triangularizable [15]. Neither of these sufficient conditions applies in our problem, so instead we seek a numerical solution. Using nominal parameter values we have

$$A_1 = \begin{bmatrix} -0.35 & 1 & 0 & -1 \\ -0.09 & 0.9 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 \\ 0.035 & -0.1 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix}.$$

A common quadratic Lyapunov function can be efficiently found by solving the LMIs: $P - A_i^\intercal P A_i > 0$, $i \in \{1, 2\}$ for some positive definite $P \in \mathbb{R}^4$. A solution is given by

$$P = \begin{bmatrix} 2.2 & -1.2 & 0.1 & 1.6 \\ -1.2 & 10.3 & 0.6 & -3.1 \\ 0.1 & 0.6 & 7.3 & 0.7 \\ 1.6 & -3.1 & 0.7 & 9.2 \end{bmatrix}.$$

Hence, the function $V(y) := y^\intercal P y$ is a common quadratic Lyapunov function of (32). By Theorem 5 and Remark 3(v) there exists $c^\star > 0$ such that the compact set $\mathcal{M}^0(c^\star)$ is positively invariant and globally asymptotically stable under (32) with any switching signal σ. Finally, a robustness analysis was carried out to account for plant parameter uncertainty, but the details are omitted.

VII. Conclusion

We presented two stability results for discrete-time switched systems with multiple equilibria, extending the available methods in the literature [2], [16]. The first result uses a common quadratic Lyapunov function to find a compact, positively invariant set containing all equilibria under arbitrary switching signals. The second result provides a possibly larger compact superset that is both positively invariant and globally asymptotically stable. The results were applied to a model of visuomotor adaptation to show boundedness of solutions despite intermittent measurements.

Immediate extensions are to study the case of a not necessarily quadratic, common Lyapunov function and to relax the assumption of a unique equilibrium for each subsystem by allowing for multiple invariant sets. Construction of common Lyapunov functions for switched nonlinear systems whose subsystems share certain structure is an area that warrants further development; see [9], [18], [19]. Finally, a more comprehensive theory of stability of regulators with rapid on/off switching of internal models is needed.

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