PATTERNED LINEAR SYSTEMS

SARAH C. HAMILTON AND MIREILLE E. BROUCKE

Abstract. We introduce and study a new class of linear control systems called patterned systems. Mathematically, this class has the property that the state, input and output transformations of the linear state space model are all functions of a common base transformation. The motivation for studying such systems arises from their interpretation as a collection of identical subsystems with a pattern of interaction between subsystems that is imprinted by the base transformation. From a control perspective, the objective is to provide synthesis methods for feedbacks that preserve the system pattern. Patterned systems may provide a template for the development of a more unified framework for dealing with systems, typically distributed, that consist of subsystems interacting via a fixed pattern.

1. Introduction

Complex systems that are made of a large number of simple subsystems with simple patterns of interaction arise frequently in natural and engineered systems. Such systems arise particularly out of models which are lumped approximations of partial differential equations (PDE's). One such application concerns ring systems, which can be modeled as circulant or block-circulant systems [19]. In [4] circulant systems arising from control of systems modeled by discretized PDE's are studied from a control perspective. The key insight is that all circulant (or block circulant) matrices are diagonalized (or block diagonalized) by a common matrix.

The starting point of the current investigation is a hypothesis that circulant systems have deeper structural properties beyond diagonalization. Circulant matrices have a wealth of interesting relationships with the class of subspaces invariant under the *shift operator* [6, Ch.3]. Important subspaces like the controllable subspace and the unobservable subspace fall within this class. This greatly simplifies the study of control problems like pole placement and stabilization when it is desired that the controller be circulant as well. The fundamental property of circulant matrices that creates this relationship with a class of subspaces is that circulant matrices all share a common set of eigenvectors, which are the eigenvectors of the shift operator. Also well-known is that every circulant matrix is a polynomial function of the shift operator. Any matrix that can be represented as a polynomial function of another matrix shares the eigenvectors of the latter. In this way, results on circulant systems can be extended to a broader family that includes all systems with state, input and output transformations that are functions of a common *base transformation*. We call the members of this family *patterned linear systems*. The extension is relevant because it includes not only ring systems, but also other physically meaningful systems, such as unidirectional chains and trees.

Applications involving ring, chain, or tree patterns include lumped approximation of PDE's modeling smart materials or Micro-Electro-Mechanical Systems (MEMS) [1]; cross-directional

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control of sheet and film processes in paper-making, steel rolling and plastic extrusion [11]; evenly spaced convoys [20, 5], geometric pattern formation [17, 15], and hierarchical command in multi-agent systems [16]; transit lines; serpentine manipulators; multimachine power systems; parallel networks of units in a plant, such as pumps or reactors; and biological systems [19].

The present paper builds on the work of [4], which studies synthesis of circulant or block-circulant feedbacks for circulant or block-circulant systems. A complete pole placement theorem is, however, not obtained in [4]. The main difficulty to obtain a pole placement theorem is to recognize a suitable controllable subspace for circulant systems. Our work provides a resolution to this obstacle in the form of the patterned controllable subspace, and we do it in the more general setting of patterned systems, not only circulant systems. In turn, we are then able to develop a fairly complete geometric theory for patterned systems, which we hope will provide the foundation for a theory on block-patterned systems. Implicit in our work is a confrontation of the delicate tradeoff between the algebraic structure imposed by patterns and the geometric structure that is required to elaborate a geometric control theory for patterned systems.

The paper is organized as follows. We present the required background in Section 2 (see also [2, 21]). In Section 3 patterned linear matrices and maps are introduced and their relationship to certain invariant subspaces is developed. Using this foundation, in Section 4 patterned linear systems are introduced, and we follow classical developments to build system theoretic properties for patterned linear systems; particularly, controllability, pole placement, observability, stabilizability, and decomposition. Next, in Section 5 we study the classic control synthesis problems [21] (roughly Chapters 0-6). These include stabilization, stabilization by measurement feedback, output stabilization, disturbance decoupling, and the regulator problem. A significant outcome of our study is that if a general controller exists to solve any of the studied control synthesis problems, then a patterned feedback exists. Finally, examples are given illustrating the synthesis problems, and the results are related to certain well-studied problems in multiagent systems.

2. Background

We assume that the reader is already familiar with the tools of geometric control theory [2, 21]. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional vector spaces. We consider *linear maps* from $\mathcal{X}$ to $\mathcal{Y}$, denoted by bold capital letters, such as $\mathbf{T} : \mathcal{X} \to \mathcal{Y}$. The plain capital, T, denotes a matrix representation of the map $\mathbf{T}$. Let $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ be an endomorphism. Let $\mathcal{S}_\lambda(\mathbf{T})$ denote the eigenspace of $\mathbf{T}$ associated with eigenvalue λ. The Jordan subspaces of $\mathbf{T}$ are given by

$$\mathcal{J}_{ij}(\mathbf{T}) = \operatorname{span}\left(v_{ij}, g_{i1}, g_{i2}, \ldots, g_{i(p_{ij}-1)}\right),$$

where each eigenvector v_{ij} spawns the Jordan chain:

$$(\mathbf{T} - \lambda_i \mathbf{I}) v_{ij} = 0 \tag{2.1a}$$

$$(\mathbf{T} - \lambda_i \mathbf{I}) g_{i1} = v_{ij} \tag{2.1b}$$

$$(\mathbf{T} - \lambda_i \mathbf{I}) g_{i2} = g_{i1} \tag{2.1c}$$

$$\vdots$$

$$(\mathbf{T} - \lambda_i \mathbf{I}) g_{i(p_{ij}-1)} = g_{i(p_{ij}-2)}\,. \tag{2.1d}$$

Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$, where symbol $\oplus$ means the direct sum of subspaces. The map $\mathbf{Q}_{\mathcal{V}} : \mathcal{X} \to \mathcal{X}$ denotes the projection on $\mathcal{V}$ along $\mathcal{W}$, $\mathbf{N}_{\mathcal{V}} : \mathcal{X} \to \mathcal{V}$ denotes the natural projection, and $\mathbf{S}_{\mathcal{V}} : \mathcal{V} \to \mathcal{X}$ denotes the insertion of $\mathcal{V}$ in $\mathcal{X}$. Useful relations are that $\mathbf{N}_{\mathcal{V}}\mathbf{S}_{\mathcal{V}} = \mathbf{I}_{\mathcal{V}}$ and $\mathbf{N}_{\mathcal{W}}\mathbf{S}_{\mathcal{V}} = \mathbf{0}$. Given a $\mathbf{T}$-invariant subspace $\mathcal{V} \subset \mathcal{X}$, if there also exists a subspace $\mathcal{W} \subset \mathcal{X}$ such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$ and $\mathcal{W}$ is $\mathbf{T}$-invariant, then we call $\mathcal{V}$ a $\mathbf{T}$-*decoupling* subspace. The restriction of $\mathbf{T}$ to $\mathcal{V}$ is denoted by $\mathbf{T}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ and is given by $\mathbf{T}_{\mathcal{V}} = \mathbf{N}_{\mathcal{V}}\mathbf{T}\mathbf{S}_{\mathcal{V}}$. Similarly, define $\mathbf{T}_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$, the restriction of $\mathbf{T}$ to $\mathcal{W}$, by $\mathbf{T}_{\mathcal{W}} = \mathbf{N}_{\mathcal{W}}\mathbf{T}\mathbf{S}_{\mathcal{W}}$.

Let $\mathcal{V}, \mathcal{W}$ be $\mathbf{T}$-decoupling subspaces such that $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$. Suppose the minimal polynomial (m.p.) of $\mathbf{T}$, $\psi(s)$, has been factored as $\psi(s) = \psi^-(s)\psi^+(s)$ such that $\mathcal{X}^-(\mathbf{T}) = \operatorname{Ker}\psi^-(\mathbf{T})$, $\mathcal{X}^+(\mathbf{T}) = \operatorname{Ker}\psi^+(\mathbf{T})$, and $\mathcal{X}^-(\mathbf{T})$ and $\mathcal{X}^+(\mathbf{T})$ are the stable and unstable subspaces of $\mathbf{T}$. Similarly, let $\psi_{\mathcal{V}}(s)$ be the m.p. of $\mathbf{T}_{\mathcal{V}}$ and suppose it has also been factored as $\psi_{\mathcal{V}}(s) = \psi_{\mathcal{V}}^-(s)\psi_{\mathcal{V}}^+(s)$ such that $\mathcal{V}^-(\mathbf{T}_{\mathcal{V}}) = \operatorname{Ker}\psi^-(\mathbf{T}_{\mathcal{V}})$, $\mathcal{V}^+(\mathbf{T}_{\mathcal{V}}) = \operatorname{Ker}\psi^+(\mathbf{T}_{\mathcal{V}})$. Since $\psi_{\mathcal{V}}^+$ and $\psi_{\mathcal{V}}^-$ are coprime, $\mathcal{V} = \mathcal{V}^-(\mathbf{T}_{\mathcal{V}}) \oplus \mathcal{V}^+(\mathbf{T}_{\mathcal{V}})$ [7, Ch.VII]. The following result summarizes useful properties of stable and unstable subspaces. The symbol $\mathbb{C}^-$ denotes the open left half complex plane.

Lemma 2.1 ([21, p.94])**.** *Let* $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ *be a linear map and let* $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ *be* $\mathbf{T}$-*decoupling subspaces such that* $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$. *Then we have (i)* $\mathbf{N}_{\mathcal{V}}\mathcal{X}^+(\mathbf{T}) = \mathcal{V}^+(\mathbf{T}_{\mathcal{V}})$*; and (ii)* $\mathcal{X}^+(\mathbf{T}) \subset \mathcal{V}$ *if and only if* $\sigma(\mathbf{T}_{\mathcal{W}}) \subset \mathbb{C}^-$.

We denote the set of all $\mathbf{T}$-decoupling subspaces in $\mathcal{X}$ by $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{X})$. Similarly, for any $\mathcal{V} \subset \mathcal{X}$, not necessarily a $\mathbf{T}$-invariant subspace, we denote the set of all $\mathbf{T}$-decoupling subspaces contained in $\mathcal{V}$ by $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$; that is, $\mathcal{Y} \in \mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ if $\mathcal{Y} \subset \mathcal{V}$, $\mathcal{Y}$ is $\mathbf{T}$-invariant, and $\mathcal{Y}$ has an $\mathbf{T}$-invariant complement in $\mathcal{X}$. (Note that the complement need not be in $\mathcal{V}$.) We also denote the set of all $\mathbf{T}$-decoupling subspaces in $\mathcal{X}$ containing $\mathcal{V}$ by $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$. Decoupling subspaces are closely linked to Jordan subspaces. The following results can be deduced from the development in Chapter 2 of [8].

Lemma 2.2. *Every Jordan subspace of* $\mathbf{T}$ *is a* $\mathbf{T}$-*decoupling subspace, and every* $\mathbf{T}$-*decoupling subspace is the sum of Jordan subspaces of* $\mathbf{T}$.

Lemma 2.3. *Let* $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{X}$ *be* $\mathbf{T}$-*invariant subspaces, and let* $\mathcal{J} \subset \mathcal{V}_1 + \mathcal{V}_2$ *be a Jordan subspace of* $\mathbf{T}$. *Then* $\mathcal{J} \subset \mathcal{V}_1$ *or* $\mathcal{J} \subset \mathcal{V}_2$.

We say that a subspace $\mathcal{V}^\diamond$ is the *supremum* of $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$, denoted $\mathcal{V}^\diamond = \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$, if $\mathcal{V}^\diamond \in \mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and given $\mathcal{V}' \in \mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$, then $\mathcal{V}' \subset \mathcal{V}^\diamond$. Analogously, we say that a subspace $\mathcal{V}_\diamond$ is the *infimum* of $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$, denoted $\mathcal{V}_\diamond = \inf\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$, if $\mathcal{V}_\diamond \in \mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ and given $\mathcal{V}' \in \mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$, then $\mathcal{V}_\diamond \subset \mathcal{V}'$. Existence and uniqueness of a supremal element of $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and an infimal element of $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ relies on the fact that $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ have the structure of a lattice under the operations of subspace addition and subspace intersection; see for instance [21, Lemma 4.4].

Lemma 2.4. *Given* $\mathcal{V} \subset \mathcal{X}$, *the sets* $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ *and* $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ *are each closed under the operations of subspace addition and subspace intersection.*

Lemma 2.5. *Let* $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{X}$ *be* $\mathbf{T}$-*invariant subspaces, and let* $\mathcal{V}_1^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V}_1)$, $\mathcal{V}_2^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V}_2)$, *and* $(\mathcal{V}_1 + \mathcal{V}_2)^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V}_1 + \mathcal{V}_2)$. *Then* $(\mathcal{V}_1 + \mathcal{V}_2)^\diamond = \mathcal{V}_1^\diamond + \mathcal{V}_2^\diamond$.

3. Patterned Linear Maps

Let $t_0, t_1, \ldots, t_k \in \mathbb{R}$ and consider the polynomial $\rho(s) = t_0 + t_1 s + t_2 s^2 + t_3 s^3 + \ldots + t_k s^k$. Let M be an $n \times n$ real matrix. Then $\rho(\mathrm{M})$ is defined by $\rho(\mathrm{M}) := t_0\mathrm{I} + t_1\mathrm{M} + t_2\mathrm{M}^2 + t_3\mathrm{M}^3 + \ldots + t_k\mathrm{M}^k$. Given $\mathrm{T} = \rho(\mathrm{M})$, then $\rho(s)$ is called a *representer of* T *with respect to* M, and it is generally not unique. By Cayley-Hamilton theorem, our discussion will be confined to $\rho(\mathrm{M})$ of order less than or equal to $n-1$. We define the set of all matrices that are polynomial functions of a given *base matrix* $\mathrm{M} \in \mathbb{R}^{n\times n}$ by

$$\mathfrak{F}(\mathrm{M}) := \left\{ \mathrm{T} \mid (\exists\, t_0, \ldots, t_{n-1} \in \mathbb{R})\ \mathrm{T} = t_0\mathrm{I} + t_1\mathrm{M} + t_2\mathrm{M}^2 + \ldots + t_{n-1}\mathrm{M}^{n-1} \right\}.$$

We call a matrix $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$ an M-*patterned* matrix.

Given $\mathrm{M} \in \mathbb{R}^{n\times n}$, let the n eigenvalues of M be denoted by $\sigma(\mathrm{M}) = \{\delta_1, \delta_2 \ldots, \delta_n\}$. Note that the spectrum is symmetric with respect to the real axis since M is real. Define a symmetric subset

$$\{\mu_1, \ldots, \mu_m\} \subset \sigma(\mathrm{M}) \tag{3.1}$$

such that each distinct eigenvalue is repeated only m_i times in the subset, where m_i is the geometric multiplicity of the eigenvalue. Then, associated with each eigenvalue μ_i is the partial multiplicity p_i. There exists a Jordan transformation Ω such that $\Omega^{-1}\mathrm{M}\Omega = \mathrm{J}$, where J is the Jordan form of M given by

$$\mathrm{J} := \mathrm{diag}\,\left(\mathrm{J}_{p_1}(\mu_1), \mathrm{J}_{p_2}(\mu_2), \ldots, \mathrm{J}_{p_m}(\mu_m)\right), \qquad \mathrm{J}_p(\mu) := \begin{bmatrix} \mu & 1 & & 0 \\ 0 & \mu & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{bmatrix},$$

where $\mathrm{J}_p(\mu) \in \mathbb{C}^{p\times p}$. Let's consider the structure of a polynomial of M, when this same Jordan transformation is applied.

Lemma 3.1 ([8, Sec.2.10])**.** *Let $\rho(s)$ be a real polynomial. Given $\mathrm{T} = \rho(\mathrm{M})$, then*

(1) $\Omega^{-1}\mathrm{T}\Omega = \mathrm{diag}\,(\Gamma_{p_1}(\mu_1), \Gamma_{p_2}(\mu_2) \ldots, \Gamma_{p_m}(\mu_m))$, *where*

$$\Gamma_p(\mu) = \begin{bmatrix} \rho(\mu) & \frac{1}{1!}\rho'(\mu) & \frac{1}{2!}\rho''(\mu) & \cdots & \frac{1}{(k-1)!}\rho^{(p-1)}(\mu) \\ 0 & \rho(\mu) & \frac{1}{1!}\rho'(\mu) & \cdots & \frac{1}{(k-2)!}\rho^{(p-2)}(\mu) \\ & & \rho(\mu) & \ddots & \vdots \\ 0 & 0 & & \ddots & \frac{1}{1!}\rho'(\mu) \\ 0 & 0 & & & \rho(\mu) \end{bmatrix}, \textit{ and}$$

(2) $$\sigma(\mathrm{T}) = \left\{ \underbrace{\rho(\mu_1), \ldots, \rho(\mu_1)}_{\times p_1}, \underbrace{\rho(\mu_2), \ldots, \rho(\mu_2)}_{\times p_2} \ldots\ldots, \underbrace{\rho(\mu_m), \ldots, \rho(\mu_m)}_{\times p_m} \right\}.$$

Some useful observations can be made regarding the structure of each block $\Gamma_p(\mu)$, which is generally not a Jordan block structure. First, generalized eigenvectors of M are not necessarily generalized eigenvectors of T. In fact, if the derivatives $\rho'(\mu), \ldots, \rho^{(p-1)}(\mu)$ evaluate to zero for a given μ, then the generalized eigenvectors associated with μ are actually true eigenvectors of T. A second observation is that if $\mu_i = \mu_j$ then $\rho(\mu_i) = \rho(\mu_j)$. Thus, it follows immediately from property (2) that repeated eigenvalues in M remain repeated in T. (See [8, Sec.2.11] for further discussion).

Suppose we are given an arbitrary matrix T and a base matrix M. We can determine whether or not the matrix is M-patterned. The proof is a direct application of Lemma 3.1.

Theorem 3.2. *Given* $\mathrm{T} \in \mathbb{R}^{n\times n}$, *then* $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$ *if and only if*

(1) $\Omega^{-1}\mathrm{T}\Omega = \mathrm{diag}\,(\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_m)$, *where* $\mathrm{H}_i = \begin{bmatrix} h_{i1} & h_{i2} & \cdots & h_{ip_i} \\ & h_{i1} & \ddots & \vdots \\ 0 & & \ddots & h_{i2} \\ 0 & 0 & & h_{i1} \end{bmatrix}$, $h_{ij} \in \mathbb{C}$,

(2) $\forall\,\{i_1, i_2\} \in \{1, \ldots, m\}$ *if* $\mu_{i_1} = \bar{\mu}_{i_2}$ *then* $h_{i_1 j} = \bar{h}_{i_2 j}\,, \forall j = 1, \ldots, \min(p_{i_1}, p_{i_2})$ *and*

(3) $\forall\,\{i_1, i_2\} \in \{1, \ldots, m\}$ *if* $\mu_{i_1} = \mu_{i_2}$ *then* $h_{i_1 j} = h_{i_2 j}\,, \forall j = 1, \ldots, \min(p_{i_1}, p_{i_2})$.

Suppose we are given an arbitrary symmetric spectrum of n values and an objective to construct an M-patterned matrix with the given spectrum. The next result presents the conditions under which this is possible.

Lemma 3.3. *Let* $\mathfrak{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\,, \lambda_i \in \mathbb{C}$ *be a symmetric spectrum. Suppose the elements of* $\mathfrak{L}$ *can be reordered so that if* $\delta_i = \bar{\delta}_j$ *then* $\lambda_i = \bar{\lambda}_j$, *and if* $\delta_i = \delta_j$ *then* $\lambda_i = \lambda_j$. *Then there exists* $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$, *such that* $\sigma(\mathrm{T}) = \mathfrak{L}$.

Proof. Reorder the elements of $\mathfrak{L}$ accordingly. Consider the symmetric subset (3.1). If $\lambda_i = \lambda_j$ whenever $\delta_i = \delta_j$ then it is possible to define a subset $\{\eta_1, \eta_2, \ldots, \eta_m\} \subset \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\eta_i = \eta_j$ whenever $\mu_i = \mu_j$. And if $\lambda_i = \bar{\lambda}_j$ whenever $\delta_i = \bar{\delta}_j$, then $\eta_i = \bar{\eta}_j$ whenever $\mu_i = \bar{\mu}_j$.

Next define $\mathrm{H} := \mathrm{diag}\,(\mathrm{J}_{p_1}(\eta_1), \mathrm{J}_{p_2}(\eta_2), \ldots, \mathrm{J}_{p_m}(\eta_m))$ and $\mathrm{T} := \Omega\mathrm{H}\Omega^{-1}$. Consider the conditions of Theorem 3.2. Observe that H is in the form of condition (1), where $\forall i$ and $\forall j = 3, \ldots, p_i$ we assign $h_{i1} = \eta_i$, $h_{i2} = 1$ and $h_{ij} = 0$. We have shown that conditions (2) and (3) are met. Thus $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$. □

A spectrum that can be reordered in the manner of Lemma 3.3 is an *M-patterned spectrum*. Note that the matrix T constructed in the proof of Lemma 3.3 is not a unique solution to the spectrum assignment problem. Finally, the following results arise from the Spectral Mapping Theorem.

Lemma 3.4. *Given* $\mathrm{T}, \mathrm{R} \in \mathfrak{F}(\mathrm{M})$ *and a scalar* $\alpha \in \mathbb{R}$, *then* $\{\alpha\mathrm{T}, \mathrm{T}+\mathrm{R}, \mathrm{TR}\} \in \mathfrak{F}(\mathrm{M})$, *and* $\mathrm{T}^{-1} \in \mathfrak{F}(\mathrm{M})$ *assuming* T^{-1} *exists. Moreover, given* $\sigma(\mathrm{T}) = \{\tau_1, \ldots, \tau_n\}$ *and* $\sigma(\mathrm{R}) = \{\varrho_1, \ldots, \varrho_n\}$, *both ordered relative to the eigenvalues of M, then* $\sigma(\alpha\mathrm{T}) = \{\alpha\tau_1, \ldots, \alpha\tau_n\}$, $\sigma(\mathrm{T}+\mathrm{R}) = \{\tau_1 + \varrho_1, \ldots, \tau_n + \varrho_n\}$, $\sigma(\mathrm{TR}) = \{\tau_1\varrho_1, \ldots, \tau_n\varrho_n\}$, *and* $\sigma(\mathrm{T}^{-1}) = \{1/\tau_1, \ldots, 1/\tau_n\}$.

Next, consider a linear map $\mathbf{M} : \mathcal{X} \to \mathcal{X}$. We define the set of linear maps

$$\mathfrak{F}(\mathbf{M}) := \left\{\mathbf{T} \mid (\exists\, t_0, \ldots, t_{n-1} \in \mathbb{R})\; \mathbf{T} = t_0\mathbf{I} + t_1\mathbf{M} + t_2\mathbf{M}^2 + \ldots + t_{n-1}\mathbf{M}^{n-1}\right\}.$$

We call a map $\mathbf{T} : \mathcal{X} \to \mathcal{X}$, $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$ an $\mathbf{M}$-*patterned map.* All the properties of M-patterned matrices described above naturally carry over to $\mathbf{M}$-patterned maps. We now present some important relationships between $\mathbf{M}$-patterned maps and $\mathbf{M}$-invariant subspaces.

Fact 3.5. *Let* $\mathcal{V} \subset \mathcal{X}$. *If* $\mathcal{V}$ *is* $\mathbf{M}$-*invariant, then* $\mathcal{V}$ *is* $\mathbf{T}$-*invariant for every* $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$.

Conversely, is a **T**-invariant subspace always **M**-invariant? The answer is not generally. The eigenvectors of **M** are all eigenvectors of **T**; however, **T** may have additional eigenvectors that are not eigenvectors of **M**.

Example 3.1. Consider the transformation represented by

$$\mathrm{M} = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 2 & -2 \\ 1 & 2 & -2 \end{bmatrix}.$$

The matrix has distinct eigenvalues 1, 0 and 3. Define

$$\mathrm{T} := 2\mathrm{I} - 0.5\mathrm{M} + 0.5\mathrm{M}^2 = \begin{bmatrix} 6.5 & 0 & -4.5 \\ 1.5 & 2 & -1.5 \\ 1.5 & 0 & 0.5 \end{bmatrix}.$$

Let $w = (1, 0, 1)$ and consider the subspace $\mathcal{W} = \text{span}\,\{w\}$. We have $\mathrm{T}w = (2, 0, 2) = 2w$, but $\mathrm{M}w = (-1, -1 - 1)$. Thus $\mathcal{W}$ is T-invariant, but not M-invariant.

Fortunately, it is possible to identify certain **T**-invariant subspaces, useful in a control theory context, that are also **M**-invariant.

Lemma 3.6. *Let* $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$ *and let* $\rho(s)$ *be a polynomial. Then* $\operatorname{Ker} \rho(\mathbf{T})$ *and* $\operatorname{Im} \rho(\mathbf{T})$ *are* $\mathbf{M}$*-invariant and* $\mathbf{R}$*-invariant for every* $\mathbf{R} \in \mathfrak{F}(\mathbf{M})$.

Lemma 3.6 can be used to show that several useful subspaces defined with respect to an **M**-patterned map are **M**-invariant.

Lemma 3.7. *Let* $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. *Then the following subspaces are* $\mathbf{M}$*-invariant and* $\mathbf{T}$*-invariant: the stable and unstable subspaces* $\mathcal{X}^-(\mathbf{T})$ *and* $\mathcal{X}^+(\mathbf{T})$*; and the eigenspaces* $\mathcal{S}_\lambda(\mathbf{T})$, $\lambda \in \sigma(\mathbf{T})$. *In addition, the spectral subspaces of* $\mathbf{T}$ *are* $\mathbf{M}$*-decoupling.*

Proof. By definition $\mathcal{X}^-(\mathbf{T}) := \operatorname{Ker} \psi^-(\mathbf{T})$ and $\mathcal{X}^+(\mathbf{T}) := \operatorname{Ker} \psi^+(\mathbf{T})$, where $\psi^-(s)$ and $\psi^+(s)$ are the stable and the unstable polynomial of **T**, respectively. Also, $\mathcal{S}_\lambda(\mathbf{T}) := \operatorname{Ker} \psi_\lambda(\mathbf{T})$, where $\psi_\lambda(s) = (s - \lambda)^m$ and m is the geometric multiplicity of eigenvalue λ. Then by Lemma 3.6, subspaces $\mathcal{X}^-(\mathbf{T})$, $\mathcal{X}^+(\mathbf{T})$, and $\mathcal{S}_\lambda(\mathbf{T})$ are **M**-invariant and **T**-invariant.

Considering the spectral subspaces, by (ii), the eigenspaces of **T** are **M**-invariant. Moreover, they are **M**-decoupling by Lemma 2.2. Spectral subspaces are sums of eigenspaces, by definition. Thus spectral subspaces of **T** are **M**-decoupling by Lemma 2.4. □

Suppose we are given an **M**-decoupling subspace $\mathcal{V}$. Then there exists an **M**-invariant complement $\mathcal{W}$, such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Since $\mathcal{V}$ is **M**-invariant, the restriction of **M** to $\mathcal{V}$, denoted $\mathbf{M}_\mathcal{V} : \mathcal{V} \to \mathcal{V}$, can be defined by $\mathbf{M}_\mathcal{V} := \mathbf{N}_\mathcal{V}\mathbf{M}\mathbf{S}_\mathcal{V}$. Similarly, the restriction of **M** to $\mathcal{W}$ can be defined by $\mathbf{M}_\mathcal{W} := \mathbf{N}_\mathcal{W}\mathbf{M}\mathbf{S}_\mathcal{W}$. The next lemma contains the important result that the restriction of an **M**-patterned map **T** to an **M**-invariant (or **M**-decoupling) subspace is itself patterned, and the pattern is induced by the restriction of **M** to the subspace.

Lemma 3.8. *Let* $\mathcal{V} \subset \mathcal{X}$ *be an* $\mathbf{M}$*-decoupling subspace. Let* $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. *Then the restriction of* $\mathbf{T}$ *to* $\mathcal{V}$ *is given by*

$$\mathbf{T}_\mathcal{V} = \mathbf{N}_\mathcal{V}\mathbf{T}\mathbf{S}_\mathcal{V}$$

and moreover $\mathbf{T}_\mathcal{V} \in \mathfrak{F}(\mathbf{M}_\mathcal{V})$.

Proof. By Fact 3.5, the restriction of $\mathbf{T}$ to $\mathcal{V}$ can be defined by $\mathbf{T}_{\mathcal{V}} := \mathbf{N}_{\mathcal{V}}\mathbf{T}\mathbf{S}_{\mathcal{V}}$. By assumption $\mathbf{T} = t_0\mathbf{I} + t_1\mathbf{M} + t_2\mathbf{M}^2 + \ldots + t_{n-1}\mathbf{M}^{n-1}$ for some $t_i \in \mathbb{R}$. Thus

$$\mathbf{T}_{\mathcal{V}} = \mathbf{N}_{\mathcal{V}}\mathbf{T}\mathbf{S}_{\mathcal{V}} = t_0\mathbf{N}_{\mathcal{V}}\mathbf{I}\mathbf{S}_{\mathcal{V}} + t_1\mathbf{N}_{\mathcal{V}}\mathbf{M}\mathbf{S}_{\mathcal{V}} + t_2\mathbf{N}_{\mathcal{V}}\mathbf{M}^2\mathbf{S}_{\mathcal{V}} + \ldots + t_{n-1}\mathbf{N}_{\mathcal{V}}\mathbf{M}^{n-1}\mathbf{S}_{\mathcal{V}}.$$

Consider a term $\mathbf{N}_{\mathcal{V}}\mathbf{M}^k\mathbf{S}_{\mathcal{V}}$. Since $\mathbf{M}_{\mathcal{V}}\mathbf{N}_{\mathcal{V}} = \mathbf{N}_{\mathcal{V}}\mathbf{M}$, we have

$$\mathbf{N}_{\mathcal{V}}\mathbf{M}^k\mathbf{S}_{\mathcal{V}} = \mathbf{M}_{\mathcal{V}}\mathbf{N}_{\mathcal{V}}\mathbf{M}^{k-1}\mathbf{S}_{\mathcal{V}} = \cdots = \mathbf{M}_{\mathcal{V}}^k\mathbf{N}_{\mathcal{V}}\mathbf{S}_{\mathcal{V}} = \mathbf{M}_{\mathcal{V}}^k.$$

Thus, $\mathbf{T}_{\mathcal{V}} = t_0\mathbf{I} + t_1\mathbf{M}_{\mathcal{V}} + t_2\mathbf{M}_{\mathcal{V}}^2 + \ldots + t_{n-1}\mathbf{M}_{\mathcal{V}}^{n-1}$. By the Cayley-Hamilton Theorem all powers $\mathbf{M}_{\mathcal{V}}^j$, $j \geq k$ can be rewritten as linear combinations of lower powers, so there exist $\tilde{t}_0, \tilde{t}_1, \cdots, \tilde{t}_{k-1} \in \mathbb{R}$ such that $\mathbf{T}_{\mathcal{V}} = \tilde{t}_0\mathbf{I} + \tilde{t}_1\mathbf{M}_{\mathcal{V}} + \tilde{t}_2\mathbf{M}_{\mathcal{V}}^2 + \ldots + \tilde{t}_{k-1}\mathbf{M}_{\mathcal{V}}^{k-1}$. That is, $\mathbf{T}_{\mathcal{V}} \in \mathfrak{F}(\mathbf{M}_{\mathcal{V}})$. □

Given an $\mathbf{M}$-patterned map, it is possible to create a decomposed matrix representation of the map, which splits into the restrictions to $\mathcal{V}$ and to $\mathcal{W}$.

Theorem 3.9 (First Decomposition Theorem)**.** *Let $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ be $\mathbf{M}$-decoupling subspaces such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Let $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$ such that the representation of $\mathbf{T}$ in the new coordinates is given by*

$$\mathrm{R}^{-1}\mathrm{T}\mathrm{R} = \begin{bmatrix} \mathrm{T}_{\mathcal{V}} & 0 \\ 0 & \mathrm{T}_{\mathcal{W}} \end{bmatrix}, \qquad \mathrm{T}_{\mathcal{V}} \in \mathfrak{F}(\mathrm{M}_{\mathcal{V}}),\ \mathrm{T}_{\mathcal{W}} \in \mathfrak{F}(\mathrm{M}_{\mathcal{W}}).$$

The spectrum splits into $\sigma(\mathrm{T}) = \sigma(\mathrm{T}_{\mathcal{V}}) \uplus \sigma(\mathrm{T}_{\mathcal{W}})$.[1]

Proof. Define the coordinate transformation $\mathrm{R} := \begin{bmatrix} \mathrm{S}_{\mathcal{V}} & \mathrm{S}_{\mathcal{W}} \end{bmatrix}$. Then

$$\mathrm{R}^{-1}\mathrm{T}\mathrm{R} = \begin{bmatrix} \mathrm{N}_{\mathcal{V}} \\ \mathrm{N}_{\mathcal{W}} \end{bmatrix} \mathrm{T} \begin{bmatrix} \mathrm{S}_{\mathcal{V}} & \mathrm{S}_{\mathcal{W}} \end{bmatrix} = \begin{bmatrix} \mathrm{N}_{\mathcal{V}}\mathrm{T}\mathrm{S}_{\mathcal{V}} & \mathrm{N}_{\mathcal{V}}\mathrm{T}\mathrm{S}_{\mathcal{W}} \\ \mathrm{N}_{\mathcal{W}}\mathrm{T}\mathrm{S}_{\mathcal{V}} & \mathrm{N}_{\mathcal{W}}\mathrm{T}\mathrm{S}_{\mathcal{W}} \end{bmatrix}.$$

Cleary $\mathrm{N}_{\mathcal{W}}\mathrm{T}\mathrm{S}_{\mathcal{V}} = 0$ and $\mathrm{N}_{\mathcal{V}}\mathrm{T}\mathrm{S}_{\mathcal{W}} = 0$. Define $\mathrm{T}_{\mathcal{V}} := \mathrm{N}_{\mathcal{V}}\mathrm{T}\mathrm{S}_{\mathcal{V}}$ and $\mathrm{T}_{\mathcal{W}} := \mathrm{N}_{\mathcal{W}}\mathrm{T}\mathrm{S}_{\mathcal{W}}$. Then $\mathrm{T}_{\mathcal{V}}$ and $\mathrm{T}_{\mathcal{W}}$ are the restrictions of T to $\mathcal{V}$ and to $\mathcal{W}$, respectively. By Lemma 3.8, $\mathrm{T}_{\mathcal{V}} \in \mathfrak{F}(\mathrm{M}_{\mathcal{V}})$ and $\mathrm{T}_{\mathcal{W}} \in \mathfrak{F}(\mathrm{M}_{\mathcal{W}})$. The spectral decomposition is a simple consequence of the block diagonal structure of $\mathrm{R}^{-1}\mathrm{T}\mathrm{R}$. □

The results above show how an $\mathbf{M}$-patterned map can be decoupled into smaller maps that are each a function of $\mathbf{M}$ restricted to an invariant subspace. Consider now the opposite problem. We are given a map that is a function of $\mathbf{M}$ restricted to a subspace. The map can be lifted into the larger space $\mathcal{X}$, and we give a sufficient condition under which it will be $\mathbf{M}$-patterned.

Lemma 3.10. *Let $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ be $\mathbf{M}$-decoupling subspaces such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Let $\mathbf{T}_1 \in \mathfrak{F}(\mathbf{M}_{\mathcal{V}})$. Define a map $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ by $\mathbf{T} := \mathbf{S}_{\mathcal{V}}\mathbf{T}_1\mathbf{N}_{\mathcal{V}}$. If $\sigma(\mathbf{M}_{\mathcal{V}}) \cap \sigma(\mathbf{M}_{\mathcal{W}}) = \emptyset$, then $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$.*

Proof. Order the subset $\{\mu_1, \ldots, \mu_m\} \subset \sigma(\mathbf{M})$ given in (3.1) such that $\{\mu_1, \ldots, \mu_r\} \subset \sigma(\mathbf{M}_{\mathcal{V}})$ and $\{\mu_{r+1}, \ldots, \mu_m\} \subset \sigma(\mathbf{M}_{\mathcal{W}})$. Then the columns of Ω are ordered as

$$\Omega = \\ \begin{bmatrix} v_1 & \cdots & v_r & g_{r1} & \cdots & g_{r(p_r-1)} & v_{r+1} & g_{(r+1)1} & \cdots & g_{(r+1)(p_r-1)} & \cdots & g_{m(p_m-1)} \end{bmatrix},$$

[1]The symbol $\uplus$ means union with common elements repeated.

where

$$\begin{aligned}\mathcal{V} &= \mathrm{span}\left\{v_1, g_{11}, \ldots, g_{1(p_1-1)}, \ldots, v_r, g_{r1}, \ldots, g_{r(p_r-1)}\right\}\\ \mathcal{W} &= \mathrm{span}\left\{v_{r+1}, g_{(r+1)1}, \ldots, g_{(r+1)(p_r-1)}, \ldots, v_m, g_{m1}, \ldots, g_{m(p_m-1)}\right\}.\end{aligned}$$

Define $\Omega_{\mathcal{V}}$ such that $\mathrm{N}_{\mathcal{V}}\Omega = \begin{bmatrix} \Omega_{\mathcal{V}} & 0 \end{bmatrix}$, and thus $\Omega^{-1}\mathrm{S}_{\mathcal{V}} = \begin{bmatrix} \Omega_{\mathcal{V}}^{-1} \\ 0 \end{bmatrix}$. Then

$$\Omega^{-1}\mathrm{T}\Omega = \Omega^{-1}\mathrm{S}_{\mathcal{V}}\mathrm{T}_1\mathrm{N}_{\mathcal{V}}\Omega = \begin{bmatrix} \Omega_{\mathcal{V}}^{-1} \\ 0 \end{bmatrix} \mathrm{T}_1 \begin{bmatrix} \Omega_{\mathcal{V}} & 0 \end{bmatrix} = \begin{bmatrix} \Omega_{\mathcal{V}}^{-1}\mathrm{T}_1\Omega_{\mathcal{V}} & 0 \\ 0 & 0 \end{bmatrix}.$$

By assumption $\mathrm{T}_1 \in \mathfrak{F}(\mathrm{M}_{\mathcal{V}})$. Thus, by Theorem 3.2 $\Omega_{\mathcal{V}}^{-1}\mathrm{T}_1\Omega_{\mathcal{V}} = \mathrm{diag}\,(\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_r)$. Conditions (2) and (3) are met for all elements of the blocks $\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_r$ with respect to eigenvalues $\mu_1, \ldots, \mu_r$. We have

$$\Omega^{-1}\mathrm{T}\Omega = \mathrm{diag}\,(\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_r, 0_{p_{r+1}}, 0_{p_{r+2}}, \ldots, 0_{p_m}),$$

where 0_{p_i} denotes a zero block of size $p_i \times p_i$. Reapplying Theorem 3.2, we have that $\Omega^{-1}\mathrm{T}\Omega$ meets the form of condition (1). Also, by assumption $\{\mu_1, \ldots, \mu_r\} \cap \{\mu_{r+1}, \ldots, \mu_m\} = \emptyset$, so conditions (2) and (3) are always met for the overall set $\mathrm{H}_1, \ldots, \mathrm{H}_r, 0_{p_{r+1}}, \ldots, 0_{p_m}$. We conclude that $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$. □

4. System Properties

Consider the control system

$$\begin{aligned}\dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t)\\ y(t) &= \mathrm{C}x(t),\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. We denote the state space, input space, and output space by $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$, respectively. If $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C} \in \mathfrak{F}(\mathbf{M})$ with respect to some $\mathbf{M} : \mathcal{X} \to \mathcal{X}$, then $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ is termed an $\mathbf{M}$-*patterned system* or simply a *patterned system*. Observe that for patterned systems, $n = m = p$, thus $\mathcal{X} \simeq \mathcal{U} \simeq \mathcal{Y}$. Also, the open loop poles of the system form an $\mathbf{M}$-patterned spectrum. In this section we examine the system theoretic properties of patterned systems.

4.1. **Controllability.** The *controllable subspace* of a system is denoted by $\mathcal{C}$. Let $\mathcal{B} = \mathrm{Im}\mathbf{B}$. For patterned systems it is immediately observed that $\mathcal{C} = \mathcal{B}$, and $\mathcal{C}$ is $\mathbf{M}$-invariant.

Definition 4.1. The *patterned controllable subspace*, denoted $\mathcal{C}_M$, is the largest $\mathbf{M}$-decoupling subspace contained in $\mathcal{C}$. That is, $\mathcal{C}_M := \sup \mathfrak{D}^{\diamond}(\mathbf{M}; \mathcal{C})$.

Lemma 4.1. *Let* $(\mathbf{A}, \mathbf{B})$ *be an* $\mathbf{M}$*-patterned pair. Then* $\mathcal{C}_M = \{0\} + \sum_{\substack{\lambda \in \sigma(\mathbf{B}),\\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{B})$ *and its* $\mathbf{M}$*-invariant complement is* $\mathcal{S}_0(\mathbf{B})$.

Proof. We split the space $\mathcal{X}$ into a direct sum of Jordan subspaces of $\mathbf{B}$. That is,

$$\mathcal{X} = \mathcal{J}_1(\mathbf{B}) \oplus \mathcal{J}_2(\mathbf{B}) \oplus \cdots \oplus \mathcal{J}_m(\mathbf{B}),$$

corresponding to a (possibly repeated) list of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subset \sigma(\mathbf{B})$ associated with distinct Jordan subspaces $\mathcal{J}_i(\mathbf{B})$. Each $\mathcal{J}_i(\mathbf{B})$ is given by

$$\mathcal{J}_i(\mathbf{B}) = \mathrm{span}\,(v_i, g_{i1}, g_{i2}, \ldots, g_{i(p_i-1)}),$$

where v_i is one of the linearly independent eigenvectors associated with λ_i and the generalized eigenvectors $g_{i1}, g_{i2}, \ldots, g_{i(p_i-1)}$ are generated by (2.1a)-(2.1d). We claim that $\mathcal{J}_i(\mathbf{B}) \subset \mathcal{B} = \mathcal{C}$ for all $\lambda_i \neq 0$. Considering (2.1a)-(2.1d), we have $v_i = \frac{1}{\lambda_i}\mathbf{B}v_i$, $g_{i1} = \frac{1}{\lambda_i}\left(\mathbf{B}g_{i1} - v_i\right)$, $\ldots$, $g_{i(p_i-1)} = \frac{1}{\lambda_i}\left(\mathbf{B}g_{i(p_i-1)} - g_{i(p_i-2)}\right)$. It follows that $v_i \in \mathcal{B}$ and by induction $g_{ij} \in \mathcal{B}$, for all $j = 1, \ldots, p_i - 1$. We conclude $\mathcal{J}_i(\mathbf{B}) \subset \mathcal{B}$. Now $\mathcal{S}_{\lambda_i}(\mathbf{B})$ with $\lambda_i \neq 0$ is the sum of its Jordan subspaces, so $\mathcal{S}_{\lambda_i}(\mathbf{B}) \subset \mathcal{B}$. Moreover, by Lemma 3.7, $\mathcal{S}_{\lambda_i}(\mathbf{B})$ is $\mathbf{M}$-decoupling. Using Lemma 2.4 we conclude $\sum_{\substack{\lambda \in \sigma(\mathbf{B}), \\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{B}) \in \mathfrak{D}^\diamond(\mathbf{M}; \mathcal{B})$.

Conversely, consider a Jordan subspace $\mathcal{J}_i(\mathbf{B})$ for $\lambda_i = 0$. From (2.1a)-(2.1d), we obtain that $\mathbf{B}v_i = 0$, $\mathbf{B}g_{i1} = v_i, \ldots, \mathbf{B}g_{i(p_i-1)} = g_{i(p_i-2)}$. It follows that $v_i \in \operatorname{Ker}\mathbf{B}$, $g_{i1}, \ldots, g_{i(p_i-1)} \notin \operatorname{Ker}\mathbf{B}$, and $v_i, g_{i1}, \ldots, g_{i(p_i-2)} \in \mathcal{B}$. Since $\dim\left(\operatorname{Ker}\left(\mathbf{B} \mid \mathcal{J}_i(\mathbf{B})\right)\right) = 1$, where $\mathbf{B} \mid \mathcal{J}_i(\mathbf{B})$ denotes the restriction of $\mathbf{B}$ to $\mathcal{J}_i(\mathbf{B})$, we must have $\dim\left(\operatorname{Im}\left(\mathbf{B} \mid \mathcal{J}_i(\mathbf{B})\right)\right) = p_i - 1$. Thus, $g_{p_i-1} \notin \mathcal{B}$, and we conclude $\mathcal{J}_i(\mathbf{B}) \not\subset \mathcal{B}$. However, this means $\mathcal{J}_i(\mathbf{B}) \notin \mathfrak{D}^\diamond(\mathbf{M}; \mathcal{B})$, by Lemma 2.2. We conclude $\sum_{\substack{\lambda \in \sigma(\mathbf{B}), \\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{B})$ is the supremum of $\mathfrak{D}^\diamond(\mathbf{M}; \mathcal{B})$, and its $\mathbf{M}$-invariant complement is $\mathcal{S}_0(\mathbf{B})$. □

The following result further clarifies the relationship between $\mathcal{C}$ and $\mathcal{C}_M$.

Lemma 4.2. *The* $\mathbf{M}$*-patterned pair* $(\mathbf{A}, \mathbf{B})$ *is controllable if and only if* $\mathcal{C}_M = \mathcal{X}$.

Proof. If $\mathbf{M}$-patterned pair $(\mathbf{A}, \mathbf{B})$ is controllable, then $\mathcal{B} = \mathcal{X}$. This means $\operatorname{Ker}\mathbf{B} = 0$, so $\mathcal{S}_0(\mathbf{B}) = 0$. By Lemma 4.1, $\mathcal{C}_M = \mathcal{X}$. Conversely, if $\mathcal{C}_M = \mathcal{X}$, then with $\mathcal{C}_M \subset \mathcal{C}$, we get $\mathcal{C} = \mathcal{X}$. □

In addition to the case when $(\mathbf{A}, \mathbf{B})$ is controllable, $\mathcal{C}$ and $\mathcal{C}_M$ also coincide when $\mathcal{S}_0(\mathbf{B}) = \operatorname{Ker}(\mathbf{B})$, which is to say that there are no generalized eigenvectors associated with the zero eigenvalue of $\mathbf{B}$. Instead when $(\mathbf{A}, \mathbf{B})$ is not controllable, then $\mathcal{C}$ and $\mathcal{C}_M$ may differ.

Example 4.1. We are given

$$
\mathrm{M} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 1 & 1 & 0 & -4 & 0 & -4 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 3 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & -4 & 1 & -4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

with the spectrum $\sigma(\mathrm{M}) = \{0, 0, 1, 1, 1, 2+\mathrm{i}, 2-\mathrm{i}\}$. Consider the M-patterned system pair

$$
(\mathrm{A}, \mathrm{B}) = \left(\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & 2 & 0 & 2 \\ -6 & -1 & -5 & 0 & 2 & 0 & 0 \\ -6 & 0 & 0 & 5 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 12 & 0 & 0 & -2 & -10 & 3 & -10 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 3 \\ 6 & 2 & 2 & 0 & -2 & 0 & -2 \\ -4 & -2 & -2 & 0 & 4 & 0 & -2 \\ -4 & 0 & 0 & 4 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & -2 & -2 & 2 & -2 \\ -3 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \right),
$$

where

$$
\begin{aligned}
\mathrm{A} &\doteq -4\mathrm{I} + \mathrm{M} + 3.5\mathrm{M}^2 - 2.7\mathrm{M}^3 - 1.2\mathrm{M}^4 + 1.5\mathrm{M}^5 - 0.44\mathrm{M}^6 \\
\mathrm{B} &\doteq 2\mathrm{M} + 3.7\mathrm{M}^2 - 3.0\mathrm{M}^3 - 1.5\mathrm{M}^4 + 1.7\mathrm{M}^5 - 0.42\mathrm{M}^6.
\end{aligned}
$$

(The notation $\doteq$ means we have rounded to two significant figures.) Since $\mathcal{C}$ and $\mathcal{C}_M$ depend solely on B, let's examine its structure in detail. First, there is a transformation of the form $\Omega = \left[\begin{array}{ccccccc} v_1 & g_{11} & v_2 & g_{21} & v_3 & v_4 & v_5 \end{array}\right]$ (v_2 and v_3 are the eigenvectors associated with eigenvalue 1) that performs a Jordan decomposition of M such that $\Omega^{-1}\mathrm{M}\Omega = \mathrm{J}$. For this particular M, Ω decomposes the space into five Jordan subspaces given by

$$\mathcal{X} = \mathcal{J}_1(\mathrm{M}) \oplus \mathcal{J}_2(\mathrm{M}) \oplus \mathcal{J}_3(\mathrm{M}) \oplus \mathcal{J}_4(\mathrm{M}) \oplus \mathcal{J}_5(\mathrm{M}).$$

By Lemma 3.1, we can apply Ω to B to obtain

$$\Omega^{-1}\mathrm{B}\Omega = \left[\begin{array}{cc|cc|c|c|c} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3+\mathrm{i} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 3-\mathrm{i} \end{array}\right]$$

and $\sigma(\mathrm{B}) = \{0, 0, 1, 1, 1, 3+\mathrm{i}, 3-\mathrm{i}\}$. The patterned controllable subspace is $\mathcal{C}_M = \mathcal{S}_1(\mathrm{B}) + \mathcal{S}_{3+\mathrm{i}}(\mathrm{B}) + \mathcal{S}_{3-\mathrm{i}}(\mathrm{B})$. It is clear from the decomposition above that

$$\mathcal{C}_M = \mathcal{J}_2(\mathrm{M}) \oplus \mathcal{J}_3(\mathrm{M}) \oplus \mathcal{J}_4(\mathrm{M}) \oplus \mathcal{J}_5(\mathrm{M}) = \operatorname{span}\{v_2, g_{21}, v_3, v_4, v_5\}\ .$$

By inspection of $\Omega^{-1}\mathrm{B}\Omega$, however, it is also evident that the standard controllable subspace $\mathcal{C} = \operatorname{Im}\mathrm{B}$ actually spans six dimensions such that $\mathcal{C} = \mathcal{C}_M \oplus \operatorname{span}\{v_1\}$.

4.2. **Pole Placement.** It is well known that the spectrum of $\sigma(\mathbf{A}+\mathbf{BF})$ can be arbitrarily assigned to any symmetric set of poles by choice of $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable. For a patterned system, the question arises of what possible poles can be achieved by a choice of patterned state feedback.

Theorem 4.3. *The* $\mathbf{M}$*-patterned pair* $(\mathbf{A}, \mathbf{B})$ *is controllable if and only if, for every* $\mathbf{M}$*-patterned spectrum* $\mathfrak{L}$*, there exists a map* $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ *with* $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ *such that* $\sigma(\mathbf{A}+\mathbf{BF}) = \mathfrak{L}$.

Proof. (Necessity) By Lemma 3.3, for any $\mathbf{M}$-patterned spectrum $\mathfrak{L}$, there exists a transformation $\mathbf{T} : \mathcal{X} \to \mathcal{X}$, $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$ such that $\sigma(\mathbf{T}) = \mathfrak{L}$. By assumption $\mathcal{B} = \mathcal{X}$ so $\mathbf{B}^{-1}$ is defined. Let $\mathbf{F} := \mathbf{B}^{-1}(\mathbf{T} - \mathbf{A})$ such that $\mathbf{A} + \mathbf{BF} = \mathbf{T}$. By Lemma 3.4, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$.

(Sufficiency) From the definition of $\mathbf{M}$-patterned spectra it is clear that given any $\mathbf{A} \in \mathfrak{F}(\mathbf{M})$, it is possible to define an $\mathbf{M}$-patterned spectrum $\mathfrak{L}$ such that $\mathfrak{L} \cap \sigma(\mathbf{A}) = \emptyset$. By assumption there exists $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ with $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ such that $\sigma(\mathbf{A}+\mathbf{BF}) = \mathfrak{L}$. Let $\sigma(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$, $\sigma(\mathbf{B}) = \{\beta_1, \ldots, \beta_n\}$, and $\sigma(\mathbf{F}) = \{\phi_1, \ldots, \phi_n\}$. By Lemma 3.4, $\sigma(\mathbf{A}+\mathbf{BF}) = \{\alpha_1 + \beta_1\phi_1, \ldots, \alpha_n + \beta_n\phi_n\}$. Since $\alpha_i \neq \alpha_i + \beta_i\phi_i$ for all i, we have $\beta_i \neq 0$ for all i. This implies the spectral subspace $\mathcal{S}_0(\mathbf{B}) = 0$, so by Lemma 4.1, $\mathcal{C}_M = \mathcal{X}$. By Lemma 4.2 this implies the pair $(\mathbf{A}, \mathbf{B})$ is controllable. □

Corollary 4.4. *Let* $(\mathbf{A}, \mathbf{B})$ *be an* $\mathbf{M}$*-patterned pair and let* $\mathfrak{L}$ *be any symmetric spectrum. If* $\mathfrak{L}$ *is not an* $\mathbf{M}$*-patterned spectrum, then there does not exist* $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ *such that* $\sigma(\mathbf{A}+\mathbf{BF}) = \mathfrak{L}$.

Proof. Suppose by way of contradiction that there exists $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ such that $\sigma(\mathbf{A}+\mathbf{BF}) = \mathfrak{L}$. Then $\mathbf{A}+\mathbf{BF} \in \mathfrak{F}(\mathbf{M})$ by Lemma 3.4 and $\sigma(\mathbf{A}+\mathbf{BF})$ is an $\mathbf{M}$-patterned spectrum, a contradiction. □

We conclude that if we are limited to patterned state feedback, then the poles of an $\mathbf{M}$-patterned system can only be placed in an $\mathbf{M}$-patterned spectrum. This is not a severe limitation on pole placement, since stable $\mathbf{M}$-patterned spectra can be chosen for any $\mathbf{M}$.

4.3. **Controllable Decomposition.** Suppose we have a patterned system that is not fully controllable, i.e. $\mathcal{C} \neq \mathcal{X}$. We show that it is possible to decouple the system into two patterned subsystems, one that is controllable and one that is completely uncontrollable by a patterned state feedback. Since $\mathcal{C}_M$ is $\mathbf{M}$-decoupling there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{C}_M \oplus \mathcal{R} = \mathcal{X}$. Let $\mathbf{S}_{\mathcal{C}_M}$, $\mathbf{N}_{\mathcal{C}_M}$, $\mathbf{S}_{\mathcal{R}}$, and $\mathbf{N}_{\mathcal{R}}$ be the relevant insertion and projection maps, and let the restrictions of $\mathbf{M}$ to $\mathcal{C}_M$ and to $\mathcal{R}$ be denoted by $\mathbf{M}_{\mathcal{C}_M}$ and $\mathbf{M}_{\mathcal{R}}$. Before we present the decomposition, we note the following useful lemma.

Lemma 4.5. *Let $(\mathbf{A}, \mathbf{B})$ be an $\mathbf{M}$-patterned pair. Then $\sigma(\mathbf{M}_{\mathcal{C}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}}) = \emptyset$.*

Proof. Observe that $\sigma(\mathbf{B}_{\mathcal{C}_M}) \cap \sigma(\mathbf{B}_{\mathcal{R}}) = \emptyset$ because by definition of $\mathcal{C}_M$ and $\mathcal{R}$, $\mathbf{B}_{\mathcal{C}_M}$ has all non-zero eigenvalues and $\mathbf{B}_{\mathcal{R}}$ has all zero eigenvalues. This implies $\sigma(\mathbf{M}_{\mathcal{C}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}}) = \emptyset$. For if not, then by Lemma 3.1(2) and the fact that $\mathbf{B}$ has an $\mathbf{M}$-patterned spectrum, an eigenvalue of $\mathbf{M}$ appearing in both $\mathbf{M}_{\mathcal{C}_M}$ and $\mathbf{M}_{\mathcal{R}}$ would have an associated eigenvalue of $\mathbf{B}$ appearing in both $\mathbf{B}_{\mathcal{C}_M}$ and $\mathbf{B}_{\mathcal{R}}$. This is a contradiction. □

Theorem 4.6 (Second Decomposition Theorem)**.** *Let $(\mathbf{A}, \mathbf{B})$ be an $\mathbf{M}$-patterned pair. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$ for the state and input spaces ($\mathcal{U} \simeq \mathcal{X}$), which decouples the system into two subsystems, $(\mathbf{A}_1, \mathbf{B}_1)$ and $(\mathbf{A}_2, \mathbf{B}_2)$, such that*

(1) *pair $(\mathbf{A}_1, \mathbf{B}_1)$ is $\mathbf{M}_{\mathcal{C}_M}$-patterned and controllable,*
(2) *pair $(\mathbf{A}_2, \mathbf{B}_2)$ is $\mathbf{M}_{\mathcal{R}}$-patterned,*
(3) $\sigma(\mathbf{A}) = \sigma(\mathbf{A}_1) \uplus \sigma(\mathbf{A}_2)$,
(4) *$\sigma(\mathbf{A}_2)$ is unaffected by patterned state feedback in the class $\mathfrak{F}(\mathbf{M}_{\mathcal{R}})$,*
(5) *$\mathbf{B}_2 = 0$ if $\mathcal{C}_M = \mathcal{C}$.*

Proof. Since $\mathrm{A}, \mathrm{B} \in \mathfrak{F}(\mathrm{M})$, by Theorem 3.9 there exists a coordinate transformation R given by $\mathrm{R} := \begin{bmatrix} \mathrm{S}_{\mathcal{C}_M} & \mathrm{S}_{\mathcal{R}} \end{bmatrix}$, such that

$$(\mathrm{R}^{-1}\mathrm{AR}, \mathrm{R}^{-1}\mathrm{BR}) = \left(\begin{bmatrix} \mathrm{A}_{\mathcal{C}_M} & 0 \\ 0 & \mathrm{A}_{\mathcal{R}} \end{bmatrix}, \begin{bmatrix} \mathrm{B}_{\mathcal{C}_M} & 0 \\ 0 & \mathrm{B}_{\mathcal{R}} \end{bmatrix} \right),$$

$$\sigma(\mathrm{A}) = \sigma(\mathrm{A}_{\mathcal{C}_M}) \uplus \sigma(\mathrm{A}_{\mathcal{R}}),$$

where $\{\mathrm{A}_{\mathcal{C}_M}, \mathrm{B}_{\mathcal{C}_M}\} \in \mathfrak{F}(\mathrm{M}_{\mathcal{C}_M})$ and $\{\mathrm{A}_{\mathcal{R}}, \mathrm{B}_{\mathcal{R}}\} \in \mathfrak{F}(\mathrm{M}_{\mathcal{R}})$. Define $\mathrm{A}_1 = \mathrm{A}_{\mathcal{C}_M}$, $\mathrm{A}_2 = \mathrm{A}_{\mathcal{R}}$, $\mathrm{B}_1 = \mathrm{B}_{\mathcal{C}_M}$ and $\mathrm{B}_2 = \mathrm{B}_{\mathcal{R}}$. Then the system is decoupled into pairs $(\mathbf{A}_1, \mathbf{B}_1)$ and $(\mathbf{A}_2, \mathbf{B}_2)$, which are $\mathbf{M}_{\mathcal{C}_M}$-patterned and $\mathbf{M}_{\mathcal{R}}$-patterned, respectively. Furthermore, the pair $(\mathbf{A}_1, \mathbf{B}_1)$ is controllable if $\operatorname{Im} \mathbf{B}_1 = \mathcal{C}_M$, which is clearly true given that $\operatorname{Im} \mathbf{B} \supset \mathcal{C}_M$ and $\mathbf{B}_1$ is the restriction of $\mathbf{B}$ to $\mathcal{C}_M$. This proves properties (1), (2) and (3).

We now show that the poles of $\mathbf{A}_2$ are unaffected by any patterned state feedback and are thus completely patterned uncontrollable. First observe that if $\mathcal{C}_M = \mathcal{C}$ then $\mathcal{R} = \operatorname{Ker} \mathbf{B}$. This means $\mathbf{N}_{\mathcal{R}}\mathbf{B} = 0$ and $\mathbf{B}_2 = \mathbf{N}_{\mathcal{R}}\mathbf{B}\mathbf{S}_{\mathcal{R}} = 0$, proving (5). For this case, it is evident that $\sigma(\mathbf{A}_2)$ would be uncontrollable by any feedback. Now consider the possibility that $\mathbf{B}_2 \neq 0$. Since $\mathbf{B}_2$ is the restriction of $\mathbf{B}$ to $\mathcal{R} = \mathcal{S}_0(\mathbf{B})$, we have $\sigma(\mathbf{B}_2) = \{0, \ldots, 0\}$. Then by Lemma 3.4, $\sigma(\mathbf{A}_2 + \mathbf{B}_2\mathbf{F}_2) = \sigma(\mathbf{A}_2)$ for all $\mathbf{F}_2 \in \mathfrak{F}(\mathbf{M}_{\mathcal{R}})$, proving (4). □

4.4. **Stabilizability.** A system, or equivalently the pair $(\mathbf{A}, \mathbf{B})$, is *stabilizable* if there exists $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that $\sigma(\mathbf{A} + \mathbf{BF}) \subset \mathbb{C}^-$. A system is stabilizable if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$. For a patterned system, the question arises of whether the system can be stabilized with a patterned state feedback. We begin with a useful preliminary result.

Lemma 4.7. *Given an* $\mathbf{M}$*-patterned pair* $(\mathbf{A}, \mathbf{B})$*, if* $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$*, then* $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M$*.*

Proof. By Lemma 3.7, $\mathcal{X}^+(\mathbf{A})$ is $\mathbf{M}$-decoupling, so if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$, then $\mathcal{X}^+(\mathbf{A}) \in \mathfrak{D}^\diamond(\mathbf{M}; \mathcal{C})$. This implies $\mathcal{X}^+(\mathbf{A}) \subset \sup \mathfrak{D}^\diamond(\mathbf{M}; \mathcal{C}) = \mathcal{C}_M$. □

Theorem 4.8 (Patterned Stabilizability)**.** *Given an* $\mathbf{M}$*-patterned system* $(\mathbf{A}, \mathbf{B})$*, there exists a patterned state feedback* $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ *with* $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ *such that* $\sigma(\mathbf{A} + \mathbf{BF}) \subset \mathbb{C}^-$ *if and only if* $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$*.*

Proof. (Necessity) The solvability condition is identical to that for general stabilizability. Therefore, it is also necessary for the existence of a feedback that maintains the system pattern.

(Sufficiency) Since $\mathcal{C}_M$ is $\mathbf{M}$-decoupling, there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{C}_M \oplus \mathcal{R} = \mathcal{X}$. By the Second Decomposition Theorem 4.6, the system can be decomposed into an $\mathbf{M}_{\mathcal{C}_M}$-patterned and controllable subsystem $(\mathbf{A}_1, \mathbf{B}_1)$ and an $\mathbf{M}_{\mathcal{R}}$-patterned subsystem $(\mathbf{A}_2, \mathbf{B}_2)$. By Theorem 4.3 there exists a patterned state feedback $\mathbf{F}_1 : \mathcal{C}_M \to \mathcal{U}_1$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M}_{\mathcal{C}_M})$, such that $\sigma(\mathbf{A}_1 + \mathbf{B}_1\mathbf{F}_1) \subset \mathbb{C}^-$. Define $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ as $\mathbf{F} := \mathbf{S}_{\mathcal{C}_M}\mathbf{F}_1\mathbf{N}_{\mathcal{C}_M}$. By Lemma 4.5, $\sigma(\mathbf{M}_{\mathcal{C}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}}) = \emptyset$, so by Lemma 3.10, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$.

Now apply the state feedback $\mathbf{F}$ to obtain the $\mathbf{M}$-patterned closed-loop system map $\mathbf{A} + \mathbf{BF}$. Reapplying Theorem 4.6, the spectrum splits into

$$\sigma(\mathbf{A} + \mathbf{BF}) = \sigma((\mathbf{A} + \mathbf{BF})_{\mathcal{C}_M}) \uplus \sigma((\mathbf{A} + \mathbf{BF})_{\mathcal{R}}). \tag{4.1}$$

Considering $(\mathbf{A} + \mathbf{BF})_{\mathcal{C}_M}$, we have

$$\begin{aligned}(\mathbf{A} + \mathbf{BF})_{\mathcal{C}_M} &= \mathbf{N}_{\mathcal{C}_M}(\mathbf{A} + \mathbf{BF})\,\mathbf{S}_{\mathcal{C}_M} \\ &= \mathbf{N}_{\mathcal{C}_M}\mathbf{A}\mathbf{S}_{\mathcal{C}_M} + \mathbf{N}_{\mathcal{C}_M}\mathbf{B}(\mathbf{S}_{\mathcal{C}_M}\mathbf{F}_1\mathbf{N}_{\mathcal{C}_M})\mathbf{S}_{\mathcal{C}_M} = \mathbf{A}_1 + \mathbf{B}_1\mathbf{F}_1\,,\end{aligned}$$

where we use the fact that $\mathbf{N}_{\mathcal{C}_M}\mathbf{S}_{\mathcal{C}_M} = \mathbf{I}_{\mathcal{C}_M}$. Next, considering $(\mathbf{A} + \mathbf{BF})_{\mathcal{R}}$ we have

$$(\mathbf{A} + \mathbf{BF})_{\mathcal{R}} = \mathbf{N}_{\mathcal{R}}(\mathbf{A} + \mathbf{BF})\,\mathbf{S}_{\mathcal{R}} = \mathbf{A}_2\,,$$

where we use the fact that $\mathbf{N}_{\mathcal{C}_M}\mathbf{S}_{\mathcal{R}} = \mathbf{0}$. Then from (4.1), $\sigma(\mathbf{A} + \mathbf{BF}) = \sigma(\mathbf{A}_1 + \mathbf{B}_1\mathbf{F}_1) \uplus \sigma(\mathbf{A}_2)$. By assumption $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$, which implies by Lemma 4.7 that $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M$. By Lemma 2.1(ii), we get $\sigma(\mathbf{A}_2) \subset \mathbb{C}^-$. In sum, $\sigma(\mathbf{A} + \mathbf{BF}) \subset \mathbb{C}^-$, as desired. □

4.5. **Observability.** The *unobservable subspace* of a system is denoted by $\mathcal{N}$. By a duality argument, for patterned systems $\mathcal{N} = \operatorname{Ker} \mathbf{C}$, and $\mathcal{N}$ is $\mathbf{M}$-invariant.

Definition 4.2. The *patterned unobservable subspace*, denoted $\mathcal{N}_M$, is smallest $\mathbf{M}$-decoupling subspace containing $\mathcal{N}$. That is, $\mathcal{N}_M := \inf \mathfrak{D}_\diamond(\mathbf{M}; \mathcal{N})$.

Lemma 4.9. *Let* $(\mathbf{C}, \mathbf{A})$ *be an* $\mathbf{M}$*-patterned pair. Then* $\mathcal{N}_M = \mathcal{S}_0(\mathbf{C})$ *and its* $\mathbf{M}$*-invariant complement is* $\{0\} + \sum_{\substack{\lambda \in \sigma(\mathbf{C}),\\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{C})$*.*

Lemma 4.10. *The* $\mathbf{M}$*-patterned pair* $(\mathbf{C}, \mathbf{A})$ *is observable if and only if* $\mathcal{N}_M = 0$*.*

In addition to the case when $(\mathbf{C}, \mathbf{A})$ is observable, $\mathcal{N}$ and $\mathcal{N}_M$ also coincide when $\mathcal{S}_0(\mathbf{C}) = \operatorname{Ker}\mathbf{C}$, which is to say that there are no generalized eigenvectors associated with the zero eigenvalue of $\mathbf{C}$. Instead when $(\mathbf{C}, \mathbf{A})$ is not observable, then $\mathcal{N}$ and $\mathcal{N}_M$ may differ.

If a system is observable then it is possible to dynamically estimate the states of the system from the outputs. The construction of an estimate, denoted by $\hat{x}$, for the state of the system from the output is very simple in the case of a patterned observable system. Since $\operatorname{Ker}\mathbf{C} = 0$, the matrix C is invertible, and C^{-1} is M-patterned by Lemma 3.4. Thus, the states can be exactly recovered by the patterned static model $\hat{x} = x = \mathrm{C}^{-1}y$.

4.6. **Observable Decomposition.** Suppose we have a patterned system that is not fully observable, i.e. $\mathcal{N} \neq 0$. We show that it is possible to decouple the system into two patterned subsystems, one that is observable and one that is patterned unobservable, meaning that the poles of the subsystem cannot be moved by any patterned measurement feedback. Since $\mathcal{N}_M$ is $\mathbf{M}$-decoupling, there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{N}_M \oplus \mathcal{R} = \mathcal{X}$. Let $\mathbf{S}_{\mathcal{N}_M}$, $\mathbf{N}_{\mathcal{N}_M}$, $\mathbf{S}_{\mathcal{R}}$, and $\mathbf{N}_{\mathcal{R}}$ be the relevant insertion and projection maps, and let the restrictions of $\mathbf{M}$ to $\mathcal{N}_M$ and to $\mathcal{R}$ be denote by $\mathbf{M}_{\mathcal{N}_M}$ and $\mathbf{M}_{\mathcal{R}}$. We present a supporting lemma, followed by the decomposition. Proofs are omitted since they resemble those for the controllable decomposition.

Lemma 4.11. *Let $(\mathbf{C}, \mathbf{A})$ be an $\mathbf{M}$-patterned pair. Then $\sigma(\mathbf{M}_{\mathcal{N}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}}) = \emptyset$.*

Theorem 4.12 (Third Decomposition Theorem)**.** *Let $(\mathbf{C}, \mathbf{A})$ be an $\mathbf{M}$-patterned pair. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$ for the state and output spaces ($\mathcal{Y} \simeq \mathcal{X}$), which decouples the system into two subsystems, $(\mathbf{C}_1, \mathbf{A}_1)$ and $(\mathbf{C}_2, \mathbf{A}_2)$, such that*

(1) *pair $(\mathbf{C}_1, \mathbf{A}_1)$ is $\mathbf{M}_{\mathcal{R}}$-patterned and observable*
(2) *pair $(\mathbf{C}_2, \mathbf{A}_2)$ is $\mathbf{M}_{\mathcal{N}_M}$-patterned*
(3) $\sigma(\mathbf{A}) = \sigma(\mathbf{A}_1) \uplus \sigma(\mathbf{A}_2)$
(4) $\sigma(\mathbf{A}_2)$ *is unaffected by patterned measurement feedback in the class $\mathfrak{F}(\mathbf{M}_{\mathcal{R}})$*
(5) $\mathbf{C}_2 = 0$ *if $\mathcal{N}_M = \mathcal{N}$.*

4.7. **Detectability.** A system, or equivalently the pair $(\mathbf{C}, \mathbf{A})$, is *detectable* if and only if $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$. If a system is detectable, then it is possible to dynamically estimate any unstable states of the system from the outputs. In the case of a patterned system, we show that the unstable states can be recovered with a patterned static model. First, we prove a useful lemma.

Lemma 4.13. *Given an $\mathbf{M}$-patterned pair $(\mathbf{C}, \mathbf{A})$, if $\mathcal{N} \subset \mathcal{X}^-(\mathbf{A})$, then $\mathcal{N}_M \subset \mathcal{X}^-(\mathbf{A})$.*

By Theorem 4.12 an $\mathbf{M}$-patterned system can be decomposed to separate out an $\mathbf{M}_{\mathcal{R}}$-patterned observable subsystem, denoted by $(\mathbf{C}_1, \mathbf{A}_1)$. Since $\operatorname{Ker}\mathbf{C}_1 = 0$, the matrix C_1 is invertible, and C_1^{-1} is $\mathbf{M}_{\mathcal{R}}$-patterned by Lemma 3.4. Thus, the observable states can be exactly recovered by the patterned static model $x_1 = \mathrm{C}_1^{-1}y_1$. By assumption $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$, which implies $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}_M$ by Lemma 4.13. Equivalently $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{R}$, so by Lemma 2.1(ii), $\sigma(\mathbf{A}_2) \subset \mathbb{C}^-$. Thus, when a patterned system is detectable, all the patterned unobservable states are stable, making it unnecessary to estimate them since they can generally be assumed to be zero.

5. Control Synthesis

With the fundamental patterned system properties established in the previous section, we consider several classic control synthesis questions for patterned systems. The objective is to determine conditions for the existence of a patterned feedback solution. Remarkably, it emerges that the necessary and sufficient conditions for the existence of any feedback solving these synthesis problems are also necessary and sufficient for a patterned feedback.

5.1. **Measurement Feedback.**

We are given a linear system

$$\begin{aligned}\dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t)\\ y(t) &= \mathrm{C}x(t),\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. The measurement feedback problem (MFP) is to find a measurement feedback $u(t) = \mathrm{K}y(t)$ such that $x(t) \to 0$ as $t \to \infty$. A geometric statement of the problem is to find $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\sigma(\mathbf{A} + \mathbf{BKC}) \subset \mathbb{C}^-$. Stabilizing a system using measurement feedback appears to be only a minor variation of stabilization by full state feedback and one anticipates a similarly elegant solution. Unfortunately such an assumption is mistaken, for the problem of stabilization (and more generally pole-placement) by static measurement feedback is very difficult. Finding testable necessary and sufficient conditions for a general solution has been an open problem in control theory for almost forty years despite considerable effort, and remains unsolved today [10, 18]. The dynamic MFP, i.e. the use of an observer, is generally considerably simpler than the static MFP. However, in the context of distributed systems, it is not evident how a single observer can be distributed to multiple subsystems. Thus, the static MFP is of particular interest for distributed systems.

In the geometric framework, the clearest results on the MFP were derived in the seventies. It was shown by Nandi and Herzog [14] that the uncontrollable modes and the unobservable modes of a system are unaffected by static measurement feedback.

Theorem 5.1 ([14])**.** *There exists $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\sigma(\mathbf{A} + \mathbf{BKC}) \subset \mathbb{C}^-$ only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$ and $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$.*

Soon afterwards, Li [12] described a sufficient condition for MFP.

Theorem 5.2 ([12])**.** *Given a controllable and observable triple $(\mathbf{C}, \mathbf{A}, \mathbf{B})$, there exists $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\sigma(\mathbf{A} + \mathbf{BKC}) \subset \mathbb{C}^-$ if*

$$(\mathcal{X}^+(\mathbf{A}) \cap \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle) \cap (\mathcal{X}^+(\mathbf{A}^{\mathrm{T}}) \cap \langle \mathbf{A}^{\mathrm{T}} \mid \operatorname{Ker} \mathbf{B}^{\mathrm{T}} \rangle) = 0. \tag{5.1}$$

The sufficiency of the first part of the condition, $(\mathcal{X}^+(\mathbf{A}) \cap \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle) = 0$, can be derived by reformulating the problem as finding a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ with the restriction $\operatorname{Ker} \mathbf{F} \supset \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$ on the feedback matrix. Observe that $\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$ denotes the smallest $\mathbf{A}$-invariant subspace containing $\operatorname{Ker} \mathbf{C}$. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$ to separate the dynamics on and off $\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$, and in the new coordinates the pair (A, B) becomes

$$(\mathrm{R}^{-1}\mathrm{A}\mathrm{R}, \mathrm{R}^{-1}\mathrm{B}) = \left(\begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_3 \\ 0 & \mathrm{A}_2 \end{bmatrix}, \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} \right).$$

The state feedback has the form $\mathrm{FR} = \begin{bmatrix} 0 & \mathrm{F}_2 \end{bmatrix}$ in new coordinates, giving the closed loop system

$$\dot{x}(t) = \begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_3 + \mathrm{B}_1\mathrm{F}_2 \\ 0 & \mathrm{A}_2 + \mathrm{B}_2\mathrm{F}_2 \end{bmatrix} x(t).$$

Since the pair $(\mathbf{A}_2, \mathbf{B}_2)$ is assumed controllable, there exists $\mathbf{F}_2$ such that $\sigma(\mathbf{A}_2 + \mathbf{B}_2\mathbf{F}_2) \subset \mathbb{C}^-$. The condition that the intersection of $\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$ and $\mathcal{X}^+(\mathbf{A})$ is zero implies that $\sigma(\mathrm{A}_1) \subset \mathbb{C}^-$ by Lemma 2.1(ii). Thus the closed loop system is stable. Because $\operatorname{Ker} \mathbf{F} \supset \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle \supset \operatorname{Ker} \mathbf{C}$, there exists a measurement feedback $u(t) = \mathrm{K}y(t)$ given by $\mathbf{KC} = \mathbf{F}$, which solves the MFP. (The second part of condition (5.1) follows from an appeal to the principle of duality.)

Notice that, in general, the hierarchy of the subspaces is given by

$$\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle \supset \operatorname{Ker} \mathbf{C} \supset \mathcal{N}.$$

In the special case where $\operatorname{Ker} \mathbf{C}$ is $\mathbf{A}$-invariant, however, the subspaces above are all equal. Since Li's sufficient condition requires that the system is observable, it is a given that $\mathcal{X}^+(\mathbf{A}) \cap \mathcal{N} = 0$; therefore, (5.1) is always met for the special case. Patterned systems are one class of system where Li's sufficient condition is always true. We show that the necessary condition of Theorem 5.1 becomes both a necessary and sufficient condition for patterned systems.

Theorem 5.3. *Given an* $\mathbf{M}$*-patterned triple* $(\mathbf{C}, \mathbf{A}, \mathbf{B})$*, there exists a patterned measurement feedback* $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$*,* $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$*, such that* $\sigma(\mathbf{A} + \mathbf{BKC}) \subset \mathbb{C}^-$ *if and only if*

$$\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$$
$$\textit{and } \mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}.$$

Proof. (Necessity) These conditions are exactly the necessary conditions of Theorem 5.1 for the general problem of stabilization by measurement feedback. Therefore, they are also necessary for the existence of a patterned feedback.

(Sufficiency) Since $\mathcal{N}_M$ is $\mathbf{M}$-decoupling, there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{R} \oplus \mathcal{N}_M = \mathcal{X}$. By the Third Decomposition Theorem 4.12, the system can be decomposed into an $\mathbf{M}_\mathcal{R}$-patterned and observable subsystem $(\mathbf{C}_1, \mathbf{A}_1, \mathbf{B}_1)$ and an $\mathbf{M}_{\mathcal{N}_M}$-patterned subsystem $(\mathbf{C}_2, \mathbf{A}_2, \mathbf{B}_2)$. By assumption $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$ and by Lemma 2.1(i)

$$\mathbf{N}_\mathcal{R}\mathcal{X}^+(\mathbf{A}) = \mathcal{R}^+(\mathbf{A}_1) \subset \mathbf{N}_\mathcal{R}\operatorname{Im}(\mathbf{B}) = \operatorname{Im}(\mathbf{B}_1\mathbf{N}_\mathcal{R}) \subset \operatorname{Im}(\mathbf{B}_1).$$

That is, the subsystem $(\mathbf{A}_1, \mathbf{B}_1)$ is stabilizable. By Theorem 4.8 there exists a state feedback $\mathbf{F}_1 : \mathcal{R} \to \mathcal{U}_1$, $\mathbf{F}_1 \in \mathfrak{F}(\mathbf{M}_\mathcal{R})$ such that $\sigma(\mathbf{A}_1 + \mathbf{B}_1\mathbf{F}_1) \subset \mathbb{C}^-$. Since $(\mathbf{C}_1, \mathbf{A}_1)$ is observable, $\operatorname{Ker} \mathbf{C}_1 = 0$, so the inverse map $\mathbf{C}_1^{-1} : \mathcal{Y}_1 \to \mathcal{R}$ exists. Define $\mathbf{K}_1 : \mathcal{Y}_1 \to \mathcal{U}_1$ as $\mathbf{K}_1 := \mathbf{F}_1\mathbf{C}_1^{-1}$. Then by Lemma 3.4, $\mathbf{K}_1 \in \mathfrak{F}(\mathbf{M}_\mathcal{R})$ and $\sigma(\mathbf{A}_1 + \mathbf{B}_1\mathbf{K}_1\mathbf{C}_1) \subset \mathbb{C}^-$. Define $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ by $\mathbf{K} := \mathbf{S}_\mathcal{R}\mathbf{K}_1\mathbf{N}_\mathcal{R}$. By Lemma 4.11, $\sigma(\mathbf{M}_{\mathcal{N}_M}) \cap \sigma(\mathbf{M}_\mathcal{R}) = \emptyset$, so by Lemma 3.10, $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$.

Now apply the measurement feedback $\mathbf{K}$ to obtain the $\mathbf{M}$-patterned closed loop system map $\mathbf{A} + \mathbf{BKC}$. Reapplying Theorem 4.12, the spectrum splits into

$$\sigma(\mathbf{A} + \mathbf{BKC}) = \sigma((\mathbf{A} + \mathbf{BKC})_\mathcal{R}) \uplus \sigma((\mathbf{A} + \mathbf{BKC})_{\mathcal{N}_M}). \tag{5.2}$$

Considering $(\mathbf{A} + \mathbf{BKC})_\mathcal{R}$, we have

$$\begin{aligned}(\mathbf{A} + \mathbf{BKC})_\mathcal{R} &= \mathbf{N}_\mathcal{R}(\mathbf{A} + \mathbf{BKC})\mathbf{S}_\mathcal{R} \\ &= \mathbf{N}_\mathcal{R}\mathbf{A}\mathbf{S}_\mathcal{R} + \mathbf{N}_\mathcal{R}\mathbf{B}(\mathbf{S}_\mathcal{R}\mathbf{K}_1\mathbf{N}_\mathcal{R})\mathbf{C}\mathbf{S}_\mathcal{R} = \mathbf{A}_1 + \mathbf{B}_1\mathbf{K}_1\mathbf{C}_1.\end{aligned}$$

Next consider $(\mathbf{A}+\mathbf{BKC})_{\mathcal{N}_M}$. Note that $\mathbf{C}\mathcal{N}_M \subset \mathcal{N}_M$ by Fact 3.5, so $\mathbf{N}_{\mathcal{R}}\mathbf{C}\mathbf{S}_{\mathcal{N}_M} = 0$. We have

$$\begin{aligned}(\mathbf{A}+\mathbf{BKC})_{\mathcal{N}_M} &= \mathbf{N}_{\mathcal{N}_M}(\mathbf{A}+\mathbf{BKC})\,\mathbf{S}_{\mathcal{N}_M}\\ &= \mathbf{N}_{\mathcal{N}_M}\mathbf{A}\mathbf{S}_{\mathcal{N}_M} = \mathbf{A}_2.\end{aligned}$$

From (5.2), $\sigma(\mathbf{A}+\mathbf{BKC}) = \sigma(\mathbf{A}_1+\mathbf{B}_1\mathbf{K}_1\mathbf{C}_1) \uplus \sigma(\mathbf{A}_2)$. By assumption $\mathcal{N} \subset \mathcal{X}^-(\mathbf{A})$ so by Lemma 4.13, $\mathcal{N}_M \subset \mathcal{X}^-(\mathbf{A})$. Equivalently $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{R}$. By Lemma 2.1(ii), this implies $\sigma(\mathbf{A}_2) \subset \mathbb{C}^-$. We conclude $\sigma(\mathbf{A}+\mathbf{BKC}) \subset \mathbb{C}^-$, as desired. □

Remark 5.1. An alternative sufficiency proof of the Patterned Measurement Feedback Problem follows from the fact that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{K}$ all commute, because the transformations are all $\mathbf{M}$-patterned. Thus, $\mathbf{A}+\mathbf{BKC} = \mathbf{A}+(\mathbf{BC})\mathbf{K}$. By assumption, $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{B}$ and $\mathcal{X}^-(\mathbf{A}) \supset \operatorname{Ker}\mathbf{C}$. Then $\mathcal{X}^+(\mathbf{A}) \subset \operatorname{Im}\mathbf{BC}$. By the Patterned Stabilizability Theorem 4.8, there exists $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ such that $\mathbf{A}+(\mathbf{BC})\mathbf{F}$ is stable. Take $\mathbf{K} = \mathbf{F}$. ♦

5.2. **Output Stabilization.** We are given a linear system

$$\begin{aligned}\dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t)\\ z(t) &= \mathrm{D}x(t),\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $z(t) \in \mathbb{R}^q$. The output stabilization problem (OSP) is to find a state feedback $u(t) = \mathrm{F}x(t)$ such that $z(t) \to 0$ as $t \to \infty$. The problem can be restated in more geometric terms as finding a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ that makes the unstable subspace unobservable at the output $z(t)$. Equivalently, $\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \operatorname{Ker}\mathbf{D}$. The solution to the OSP requires the notion of controlled invariant subspaces. A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be *controlled invariant* if there exists a map $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that $(\mathbf{A}+\mathbf{BF})\mathcal{V} \subset \mathcal{V}$. Let $\mathfrak{I}(\mathbf{A}, \mathbf{B}; \mathcal{X})$ denote the set of all controlled invariant subspaces in $\mathcal{X}$. Similarly, for any $\mathcal{V} \subset \mathcal{X}$, let $\mathfrak{I}(\mathbf{A}, \mathbf{B}; \mathcal{V})$ denote the set of all controlled invariant subspaces in $\mathcal{V}$. It is well-known that OSP is solvable if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}+\mathcal{V}^\star$ where $\mathcal{V}^\star := \sup\mathfrak{I}(\mathbf{A}, \mathbf{B}; \operatorname{Ker}\mathbf{D})$ [21, Theorem 4.4]. (Note that $\mathcal{V}^\star = \operatorname{Ker}\mathbf{D}$ for patterned systems, by Lemma 3.6.) In order to solve the patterned version of the problem, a new subspace is introduced.

Definition 5.1. We define $\mathcal{V}^\diamond$ to be the largest $\mathbf{M}$-decoupling subspace contained in $\mathcal{V}^\star$. That is, $\mathcal{V}^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{M}; \mathcal{V}^\star)$.

Lemma 5.4. *Given an* $\mathbf{M}$*-patterned triple* $(\mathbf{D}, \mathbf{A}, \mathbf{B})$*, if* $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}+\mathcal{V}^\star$*, then* $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M+\mathcal{V}^\diamond$*.*

Proof. By Lemma 3.7, $\mathcal{X}^+(\mathbf{A})$ is $\mathbf{M}$-decoupling and by assumption $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}+\mathcal{V}^\star$. Thus, $\mathcal{X}^+(\mathbf{A}) \in \mathfrak{D}^\diamond(\mathbf{M}, \mathcal{C}+\mathcal{V}^\star)$ which implies $\mathcal{X}^+(\mathbf{A}) \subset \sup\mathfrak{D}^\diamond(\mathbf{M}, \mathcal{C}+\mathcal{V}^\star) =: (\mathcal{C}+\mathcal{V}^\star)^\diamond$. By Lemma 2.5, this implies $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M+\mathcal{V}^\diamond$, as desired. □

Theorem 5.5. *Given an* $\mathbf{M}$*-patterned triple* $(\mathbf{D}, \mathbf{A}, \mathbf{B})$*, there exists a patterned state feedback* $\mathbf{F} : \mathcal{X} \to \mathcal{U}$*,* $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$*, such that* $\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \operatorname{Ker}\mathbf{D}$ *if and only if* $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}+\mathcal{V}^\star$*.*

Proof. (Necessity) The condition is exactly the necessary condition for a general feedback. Therefore, it is also necessary for the existence of a patterned state feedback.

(Sufficiency) By Lemma 2.4, $\mathcal{C}_M+\mathcal{V}^\diamond$ is $\mathbf{M}$-decoupling, so there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{X} = (\mathcal{C}_M+\mathcal{V}^\diamond) \oplus \mathcal{R}$. Again by Lemma 2.4, $\mathcal{C}_M \cap \mathcal{V}^\diamond$ is $\mathbf{M}$-decoupling, so there exists an $\mathbf{M}$-invariant subspace $\mathcal{W}$ such that $(\mathcal{C}_M \cap \mathcal{V}^\diamond) \oplus \mathcal{W} = \mathcal{X}$. Intersecting all subspaces with $\mathcal{V}^\diamond$ and using the modular distributive rule of subspaces, we have $(\mathcal{V}^\diamond \cap (\mathcal{C}_M \cap \mathcal{V}^\diamond)) \oplus (\mathcal{V}^\diamond \cap \mathcal{W}) = \mathcal{V}^\diamond$. Define $\widehat{\mathcal{V}}^\diamond := \mathcal{V}^\diamond \cap \mathcal{W}$. Then $(\mathcal{C}_M \cap \mathcal{V}^\diamond) \oplus \widehat{\mathcal{V}}^\diamond = \mathcal{V}^\diamond$. This yields $\mathcal{C}_M + \mathcal{V}^\diamond = \mathcal{C}_M \oplus \widehat{\mathcal{V}}^\diamond$. We conclude that the space splits into three $\mathbf{M}$-invariant subspaces given by $\mathcal{X} = \mathcal{C}_M \oplus \widehat{\mathcal{V}}^\diamond \oplus \mathcal{R}$. Let $\mathbf{S}_{\mathcal{C}_M} : \mathcal{C}_M \to \mathcal{X}$ be the insertion of $\mathcal{C}_M$, and let $\mathbf{N}_{\mathcal{C}_M} : \mathcal{X} \to \mathcal{C}_M$ be the natural projection on $\mathcal{C}_M$. The restrictions of $\mathbf{A}$ and of $\mathbf{B}$ to $\mathcal{C}_M$ are defined by $\mathbf{A}_{\mathcal{C}_M} := \mathbf{N}_{\mathcal{C}_M}\mathbf{A}\mathbf{S}_{\mathcal{C}_M}$ and $\mathbf{B}_{\mathcal{C}_M} := \mathbf{N}_{\mathcal{C}_M}\mathbf{B}\mathbf{S}_{\mathcal{C}_M}$. Let $\mathbf{M}_{\mathcal{C}_M}$ denote the restriction of $\mathbf{M}$ to $\mathcal{C}_M$. By Lemma 3.8 we have $\mathbf{A}_{\mathcal{C}_M} \in \mathfrak{F}(\mathbf{M}_{\mathcal{C}_M})$ and $\mathbf{B}_{\mathcal{C}_M} \in \mathfrak{F}(\mathbf{M}_{\mathcal{C}_M})$. By Theorem 4.6 the pair $(\mathbf{A}_{\mathcal{C}_M}, \mathbf{B}_{\mathcal{C}_M})$ is controllable.

By Theorem 4.3 there exists a state feedback $\mathbf{F}_1 : \mathcal{C}_M \to \mathcal{U}_1$, $\mathbf{F}_1 \in \mathfrak{F}(\mathbf{M}_{\mathcal{C}_M})$ such that $\sigma(\mathbf{A}_{\mathcal{C}_M} + \mathbf{B}_{\mathcal{C}_M}\mathbf{F}_1) \subset \mathbb{C}^-$. Define $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ as $\mathbf{F} := \mathbf{S}_{\mathcal{C}_M}\mathbf{F}_1\mathbf{N}_{\mathcal{C}_M}$. By Lemma 4.5, $\sigma(\mathbf{M}_{\mathcal{C}_M}) \cap \sigma(\mathbf{M}_{\widehat{\mathcal{V}}^\diamond \oplus \mathcal{R}}) = \emptyset$, so by Lemma 3.10, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$.

Now apply the state feedback $\mathbf{F}$ to obtain the $\mathbf{M}$-patterned closed loop system map $\mathbf{A} + \mathbf{B}\mathbf{F}$. Applying Theorem 3.9, the spectrum splits into

$$\sigma(\mathbf{A} + \mathbf{B}\mathbf{F}) = \sigma((\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{C}_M}) \uplus \sigma((\mathbf{A} + \mathbf{B}\mathbf{F})_{\widehat{\mathcal{V}}^\diamond}) \uplus \sigma((\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{R}}).$$

Considering $(\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{C}_M}$, we have

$$\begin{aligned} (\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{C}_M} &= \mathbf{N}_{\mathcal{C}_M} (\mathbf{A} + \mathbf{B}\mathbf{F})\, \mathbf{S}_{\mathcal{C}_M} \\ &= \mathbf{N}_{\mathcal{C}_M}\mathbf{A}\mathbf{S}_{\mathcal{C}_M} + \mathbf{N}_{\mathcal{C}_\mathcal{M}}\mathbf{B}(\mathbf{S}_{\mathcal{C}_M}\mathbf{F}_1\mathbf{N}_{\mathcal{C}_M})\mathbf{S}_{\mathcal{C}_M} = \mathbf{A}_{\mathcal{C}_M} + \mathbf{B}_{\mathcal{C}_M}\mathbf{F}_1. \end{aligned}$$

Considering $(\mathbf{A} + \mathbf{B}\mathbf{F})_{\widehat{\mathcal{V}}^\diamond}$, we have

$$\begin{aligned} (\mathbf{A} + \mathbf{B}\mathbf{F})_{\widehat{\mathcal{V}}^\diamond} &= \mathbf{N}_{\widehat{\mathcal{V}}^\diamond} (\mathbf{A} + \mathbf{B}\mathbf{F})\, \mathbf{S}_{\widehat{\mathcal{V}}^\diamond} \\ &= \mathbf{N}_{\widehat{\mathcal{V}}^\diamond}\mathbf{A}\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} + \mathbf{N}_{\widehat{\mathcal{V}}^\diamond}\mathbf{B}(\mathbf{S}_{\mathcal{C}_M}\mathbf{F}_1\mathbf{N}_{\mathcal{C}_M})\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} = \mathbf{A}_{\widehat{\mathcal{V}}^\diamond}, \end{aligned}$$

where we use the fact that $\mathbf{N}_{\mathcal{C}_M}\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} = \mathbf{0}$. Similarly, we obtain that $(\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{R}} = \mathbf{A}_{\mathcal{R}}$. Thus, we have $\sigma(\mathbf{A} + \mathbf{B}\mathbf{F}) = \sigma(\mathbf{A}_{\mathcal{C}_M} + \mathbf{B}_{\mathcal{C}_M}\mathbf{F}_1) \uplus \sigma\left(\mathbf{A}_{\widehat{\mathcal{V}}^\diamond}\right) \uplus \sigma(\mathbf{A}_{\mathcal{R}})$.

By assumption $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$, which by Lemma 5.4 implies $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M + \mathcal{V}^\diamond$. Thus $\sigma(\mathbf{A}_{\mathcal{R}}) \subset \mathbb{C}^-$ by Lemma 2.1(ii). Also by Lemma 2.1(ii), since both $\sigma((\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{C}_M}) \subset \mathbb{C}^-$ and $\sigma((\mathbf{A} + \mathbf{B}\mathbf{F})_{\mathcal{R}}) \subset \mathbb{C}^-$, we obtain $\mathcal{X}^+(\mathbf{A} + \mathbf{B}\mathbf{F}) \subset \widehat{\mathcal{V}}^\diamond \subset \operatorname{Ker}\mathbf{D}$, as desired. □

5.3. **Disturbance Decoupling.** We are given a linear system

$$\begin{aligned} \dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t) + \mathrm{H}w(t) \\ z(t) &= \mathrm{D}x(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^r$ and $z(t) \in \mathbb{R}^q$. The signal $w(t)$ has been introduced to represent a disturbance to the system. Suppose that the disturbance is not directly measured, and furthermore, that we have no information on its characteristics. If the output $z(t)$ is the signal of interest, then one method to compensate for the unknown disturbance is to find a state feedback $u(t) = \mathrm{F}x(t)$ such that $w(t)$ has no influence on $z(t)$ at any time. Then the controlled system is said to be *disturbance decoupled*. Define $\mathcal{H} = \operatorname{Im}\mathrm{H}$. A geometric statement of the disturbance decoupling problem (DDP) is to find a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that $\langle \mathbf{A} + \mathbf{B}\mathbf{F} \mid \mathcal{H} \rangle \subset \operatorname{Ker}\mathbf{D}$.

It is well-known that DDP is solvable if and only if $\mathcal{V}^\star \supset \mathcal{H}$, where $\mathcal{V}^\star := \sup \mathfrak{I}(\mathbf{A}, \mathbf{B}; \operatorname{Ker}\mathbf{D})$ [21, Theorem 4.2]. The necessity of this condition is clear, because for any $\mathbf{F}$ such that $\langle \mathbf{A} + \mathbf{B}\mathbf{F} \mid \mathcal{H} \rangle \subset \operatorname{Ker}\mathbf{D}$ we have $\langle \mathbf{A} + \mathbf{B}\mathbf{F} \mid \mathcal{H} \rangle \in \mathfrak{I}(\mathbf{A}, \mathbf{B}; \operatorname{Ker}\mathbf{D})$ by definition and $\mathcal{H} \subset$

$\langle \mathbf{A} + \mathbf{BF} \mid \mathcal{H} \rangle \subset \mathcal{V}^\star$. The condition is also shown to be sufficient by observing that if $\mathcal{V}^\star \supset \mathcal{H}$ then $\langle \mathbf{A} + \mathbf{BF} \mid \mathcal{H} \rangle \subset \mathcal{V}^\star \subset \operatorname{Ker} \mathbf{D}$.

Theorem 5.6. *Given an* $\mathbf{M}$*-patterned triple* $(\mathbf{D}, \mathbf{A}, \mathbf{B})$ *and a subspace* $\mathcal{H} \subset \mathcal{X}$*, there exists a patterned state feedback* $\mathbf{F} : \mathcal{X} \to \mathcal{U}$*,* $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$*, such that* $\langle \mathbf{A} + \mathbf{BF} \mid \mathcal{H} \rangle \subset \operatorname{Ker} \mathbf{D}$ *if and only if* $\mathcal{V}^\star \supset \mathcal{H}$*.*

Proof. (Necessity) The condition is exactly the necessary condition for a general control. Therefore, it is also necessary for the existence of a patterned state feedback.

(Sufficiency) Choose a patterned state feedback $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ such that $(\mathbf{A} + \mathbf{BF})\mathcal{V}^\star \subset \mathcal{V}^\star$. For instance, the patterned feedback $\mathbf{F} = 0$ can be chosen, because $\mathcal{V}^\star = \operatorname{Ker} \mathbf{D}$ for patterned systems by Lemma 3.6, and $\operatorname{Ker} \mathbf{D}$ is $\mathbf{A}$-invariant. By assumption $\mathcal{H} \subset \mathcal{V}^\star$. Thus $\langle \mathbf{A} + \mathbf{BF} \mid \mathcal{H} \rangle \subset \mathcal{V}^\star = \operatorname{Ker} \mathbf{D}$. □

Given that $\mathcal{V}^\star = \operatorname{Ker} \mathbf{D}$ we deduce that the existence of a solution to the Patterned DDP is independent of the dynamics represented by pair ($\mathbf{A}$, $\mathbf{B}$). All the possible disturbance maps $\mathbf{H}$ that decouple an arbitrary disturbance from the output can be determined from the output map $\mathbf{D}$. Indeed, if the given patterned system is not already disturbance decoupled, then there is no patterned feedback that makes it disturbance decoupled. This last property means that for patterned systems, the problems of disturbance decoupling and closed loop stability are independent, which is not the case for general systems.

5.4. **Regulation.** We are given a linear system

$$\begin{aligned} \dot{x}(t) &= \mathrm{A}x(t) + \mathrm{B}u(t) \\ y(t) &= \mathrm{C}x(t) \\ z(t) &= \mathrm{D}x(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^q$. The output stabilization by measurement feedback problem (OSMFP) is to find a measurement feedback $u(t) = \mathrm{K}y(t)$ such that $z(t) \to 0$ as $t \to \infty$. An equivalent geometric statement of the problem is to find $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\mathcal{X}^+(\mathbf{A} + \mathbf{BKC}) \subset \operatorname{Ker} \mathbf{D}$. Output stabilization by measurement feedback is a regulation problem. The static feedback case presented above is closely related to the Restricted Regulator Problem (RRP), where the latter is formulated as output stabilization by state feedback with a restriction placed on the form of the state feedback in order to capture the condition that only certain states are measurable.

Problem 5.1 (Restricted Regulator Problem (RRP)). Given a subspace $\mathcal{L} \subset \mathcal{X}$ with $\mathbf{A}\mathcal{L} \subset \mathcal{L}$, find a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that

$$\operatorname{Ker} \mathbf{F} \supset \mathcal{L}$$
$$\mathcal{X}^+(\mathbf{A} + \mathbf{BF}) \subset \operatorname{Ker} \mathbf{D}.$$

The subspace $\mathcal{L}$ provides a geometric way to capture the information structure in the problem. This is because the condition $\operatorname{Ker} \mathbf{F} \supset \mathcal{L}$ effectively characterizes which states can be employed by the state feedback. A key condition in the statement of the RRP is that $\mathcal{L}$ must be an $\mathbf{A}$-invariant subspace; this condition makes the problem tractable. The choice of $\mathcal{L}$ can be understood a little better by decomposing the dynamics of the system. Since $\mathcal{L}$

is $\mathbf{A}$-invariant there exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$, such that in the new coordinates the matrix pair (A, B) becomes

$$(\mathrm{R}^{-1}\mathrm{AR}, \mathrm{R}^{-1}\mathrm{B}) = \left(\begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 \\ 0 & \mathrm{A}_3 \end{bmatrix}, \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} \right).$$

This separates the dynamics on and off $\mathcal{L}$. The condition $\operatorname{Ker} \mathbf{F} \supset \mathcal{L}$ implies that in new coordinates $\tilde{\mathrm{F}} = \begin{bmatrix} 0 & \mathrm{F}_2 \end{bmatrix}$, and

$$\tilde{\mathrm{A}} + \tilde{\mathrm{B}}\tilde{\mathrm{F}} = \begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 + \mathrm{B}_1\mathrm{F}_2 \\ 0 & \mathrm{A}_3 + \mathrm{B}_2\mathrm{F}_2 \end{bmatrix}.$$

The idea is to choose $\mathcal{L}$ such that all the states off $\mathcal{L}$, or at least estimates of them, are available to be used as feedback. Then the dynamics of the available states can be controlled separately from those on $\mathcal{L}$. If an observer is employed, one could use $\mathcal{N}$, the unobservable subspace, as $\mathcal{L}$ since it is always $\mathbf{A}$-invariant. However, OSMFP calls for only static measurement feedback, rather than an observer. To obtain a solution, a necessary criterion is $\mathcal{L} \supset \operatorname{Ker} \mathbf{C}$.

There is a special case, $\operatorname{Ker} \mathbf{C} = \mathcal{N}$, corresponding to all the observable states being recoverable by a simple transformation of the measurements. Then $\operatorname{Ker} \mathbf{C}$ is $\mathbf{A}$-invariant and could be used as $\mathcal{L}$, which implies that the RRP is exactly equivalent to the original Output Stabilization by Measurement Feedback Problem. In the case where $\operatorname{Ker} \mathbf{C} \neq \mathcal{N}$, $\operatorname{Ker} \mathbf{C}$ is not $\mathbf{A}$-invariant and a larger subspace must be chosen for $\mathcal{L}$, generally the smallest $\mathbf{A}$-invariant subspace containing $\operatorname{Ker} \mathbf{C}$, which is $\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$. The subtle difficulty is that now the RRP is more stringent than the original problem, and the solution to the RRP represents only sufficient, but not necessary, conditions for output stabilization by measurement feedback. To find sufficient and necessary conditions is not generally solved at this time. Ultimately it is the same static Measurement Feedback Problem described previously, and it is a longstanding open problem in control.

The general solution to the RRP relies on finding a maximal element, denoted by $\mathcal{V}^{\mathrm{M}}$, of a rather structurally complex family of subspaces [21, p.138]. There exists a simpler condition that applies under the sufficient condition that $\mathcal{V}^{\mathrm{M}} = \mathcal{V}^{\star}$, where $\mathcal{V}^{\star} := \sup \mathfrak{I}(\mathbf{A}, \mathbf{B}; \operatorname{Ker} \mathbf{D})$.

Corollary 5.7 ([21])**.** *Suppose* $\mathbf{A}(\mathcal{L} \cap \operatorname{Ker} \mathbf{D}) \subset \operatorname{Ker} \mathbf{D}$. *Then the RRP is solvable if and only if* $\mathcal{X}^{+}(\mathbf{A}) \cap \mathcal{L} \subset \operatorname{Ker} \mathbf{D}$ *and* $\mathcal{X}^{+}(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^{\star}$.

Now we return to the problem for patterned systems. Given an $\mathbf{M}$-patterned triple $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ and an output map $\mathbf{D} : \mathcal{X} \to \mathcal{Z}$, $\mathbf{D} \in \mathfrak{F}(\mathbf{M})$, the OSMFP problem is to find a patterned measurement feedback $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$, $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$, such that $\mathcal{X}^{+}(\mathbf{A} + \mathbf{BKC}) \subset \operatorname{Ker} \mathbf{D}$. For patterned systems, the appropriate $\mathcal{L}$ to choose is the patterned unobservable subspace $\mathcal{N}_M$. It is $\mathbf{A}$-invariant by Fact 3.5, so we can show that solving the patterned OSMFP is exactly equivalent to solving the following restricted regulator problem.

Theorem 5.8. *Given an* $\mathbf{M}$*-patterned pair* $(\mathbf{A}, \mathbf{B})$*, and an output map* $\mathbf{D} : \mathcal{X} \to \mathcal{Z}$, $\mathbf{D} \in \mathfrak{F}(\mathbf{M})$*, there exists a patterned state feedback* $\mathbf{F} : \mathcal{X} \to \mathcal{U}$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$*, such that*

$$\operatorname{Ker} \mathbf{F} \supset \mathcal{N}_M$$
$$\mathcal{X}^{+}(\mathbf{A} + \mathbf{BF}) \subset \operatorname{Ker} \mathbf{D}$$

if and only if

$$\mathcal{X}^+(\mathbf{A}) \cap \mathcal{N}_M \subset \operatorname{Ker}\mathbf{D}$$
$$\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$$

where $\mathcal{V}^\star = \sup \mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \operatorname{Ker}\mathbf{D})$.

Proof. First we show that Corollary 5.7 applies for patterned systems. Since $\mathcal{N}_M$ is $\mathbf{M}$-invariant by definition and $\operatorname{Ker}\mathbf{D}$ is $\mathbf{M}$-invariant by Lemma 3.6, then $\mathcal{N}_M \cap \operatorname{Ker}\mathbf{D}$ is $\mathbf{M}$-invariant. By Fact 3.5 this implies $\mathbf{A}(\mathcal{N}_M \cap \operatorname{Ker}\mathbf{D}) \subset (\mathcal{N}_M \cap \operatorname{Ker}\mathbf{D}) \subset \operatorname{Ker}\mathbf{D}$. Therefore, Corollary 5.7 applies with $\mathcal{L} = \mathcal{N}_M$.

(Necessity) Observe that the solvability conditions for the Patterned RRP are exactly the necessary conditions for Corollary 5.7. Since they are necessary for the existence of a general state feedback, they are also necessary for the existence of state feedback that maintains the system pattern.

(Sufficiency) By Lemma 2.4, $\mathcal{C}_M + \mathcal{V}^\diamond + \mathcal{N}_M$ is an $\mathbf{M}$-decoupling subspace, so there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{X} = (\mathcal{C}_M + \mathcal{V}^\diamond + \mathcal{N}_M) \oplus \mathcal{R}$. Divide $\mathcal{C}_M$, $\mathcal{V}^\diamond$ and $\mathcal{N}_M$ such that $\mathcal{C}_M = \hat{\mathcal{C}}_M \oplus (\mathcal{C}_M \cap \mathcal{N}_M)$, $\mathcal{V}^\diamond = (\mathcal{V}^\diamond \cap \widehat{\mathcal{C}}_M) \oplus \widehat{\mathcal{V}}^\diamond$ and $\mathcal{N}_M = (\mathcal{N}_M \cap \mathcal{V}^\diamond) \oplus \widehat{\mathcal{N}}_M$. Again by Lemma 2.4 all these subspaces are $\mathbf{M}$-invariant. We conclude that the space splits into four $\mathbf{M}$-invariant subspaces given by $\mathcal{X} = \widehat{\mathcal{C}}_M \oplus \widehat{\mathcal{V}}^\diamond \oplus \widehat{\mathcal{N}}_M \oplus \mathcal{R}$. Let $\mathbf{S}_{\widehat{\mathcal{C}}_M} : \widehat{\mathcal{C}}_M \to \mathcal{X}$ be the insertion of $\widehat{\mathcal{C}}_M$, and let $\mathbf{N}_{\widehat{\mathcal{C}}_M} : \mathcal{X} \to \widehat{\mathcal{C}}_M$ be the natural projection on $\widehat{\mathcal{C}}_M$. The restrictions of $\mathbf{A}$ and of $\mathbf{B}$ to $\widehat{\mathcal{C}}_M$ are defined by $\mathbf{A}_{\widehat{\mathcal{C}}_M} := \mathbf{N}_{\widehat{\mathcal{C}}_M}\mathbf{A}\mathbf{S}_{\widehat{\mathcal{C}}_M}$ and $\mathbf{B}_{\widehat{\mathcal{C}}_M} := \mathbf{N}_{\widehat{\mathcal{C}}_M}\mathbf{B}\mathbf{S}_{\widehat{\mathcal{C}}_M}$. Let $\mathbf{M}_{\widehat{\mathcal{C}}_M}$ denote the restriction of $\mathbf{M}$ to $\widehat{\mathcal{C}}_M$. By Lemma 3.8 we have $\mathbf{A}_{\widehat{\mathcal{C}}_M} \in \mathfrak{F}(\mathbf{M}_{\widehat{\mathcal{C}}_M})$ and $\mathbf{B}_{\widehat{\mathcal{C}}_M} \in \mathfrak{F}(\mathbf{M}_{\widehat{\mathcal{C}}_M})$. The pair $(\mathbf{A}_{\mathcal{C}_M}, \mathbf{B}_{\mathcal{C}_M})$ is controllable by Theorem 4.6. By Fact 3.5, $\widehat{\mathcal{C}}_M$ is $\mathbf{A}$-invariant, so by Proposition 1.3 of [21], the pair $(\mathbf{A}_{\widehat{\mathcal{C}}_M}, \mathbf{B}_{\widehat{\mathcal{C}}_M})$ is controllable.

By Theorem 4.3, there exists a state feedback $\mathbf{F}_1 : \widehat{\mathcal{C}}_M \to \mathcal{U}_1$, $\mathbf{F}_1 \in \mathfrak{F}(\mathbf{M}_{\widehat{\mathcal{C}}_M})$, such that $\sigma(\mathbf{A}_{\widehat{\mathcal{C}}_M} + \mathbf{B}_{\widehat{\mathcal{C}}_M}\mathbf{F}_1) \subset \mathbb{C}^-$. Define $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ as $\mathbf{F} := \mathbf{S}_{\widehat{\mathcal{C}}_M}\mathbf{F}_1\mathbf{N}_{\widehat{\mathcal{C}}_M}$. Let $\mathcal{R}_1$ be an $\mathcal{M}$-decoupling subspace such that $\mathcal{X} = \mathcal{C}_M \oplus \mathcal{R}_1$. From Lemma 4.5, $\sigma(\mathbf{M}_{\mathcal{C}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}_1}) = \emptyset$, so $\sigma(\mathbf{M}_{\widehat{\mathcal{C}}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}_1}) = \emptyset$. Let $\mathcal{R}_2$ be an $\mathcal{M}$-decoupling subspace such that $\mathcal{X} = \mathcal{N}_M \oplus \mathcal{R}_2$. From Lemma 4.11, $\sigma(\mathbf{M}_{\mathcal{N}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}_2}) = \emptyset$, but $\mathcal{C}_M \cap \mathcal{N}_M \subset \mathcal{N}_M$ and $\widehat{\mathcal{C}}_M \subset \mathcal{R}_2$, so $\sigma(\mathbf{M}_{\widehat{\mathcal{C}}_M}) \cap \sigma(\mathbf{M}_{\mathcal{C}_M \cap \mathcal{N}_M}) = \emptyset$. Now $\mathcal{X} = \widehat{\mathcal{C}}_M \oplus (\mathcal{C}_M \cap \mathcal{N}_M) \oplus \mathcal{R}_1$. We conclude $\sigma(\mathbf{M}_{\widehat{\mathcal{C}}_M}) \cap \sigma(\mathbf{M}_{\widehat{\mathcal{V}}^\diamond \oplus \widehat{\mathcal{N}}_M \oplus \mathcal{R}}) = \emptyset$, so by Lemma 3.10, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$.

By construction, $\mathbf{F}\widehat{\mathcal{V}}^\diamond = 0$, $\mathbf{F}\widehat{\mathcal{N}}_M = 0$ and $\mathbf{F}\mathcal{R} = 0$. Thus, $\operatorname{Ker}\mathbf{F} \supset \mathcal{N}_M$, as desired.

Now apply the state feedback $\mathbf{F}$ to obtain the $\mathbf{M}$-patterned closed loop system map $\mathbf{A} + \mathbf{B}\mathbf{F}$. Applying Theorem 3.9, the spectrum splits into

$$\sigma(\mathbf{A}+\mathbf{B}\mathbf{F}) = \sigma((\mathbf{A}+\mathbf{B}\mathbf{F})_{\widehat{\mathcal{C}}_M}) \uplus \sigma((\mathbf{A}+\mathbf{B}\mathbf{F})_{\widehat{\mathcal{V}}^\diamond}) \uplus \sigma((\mathbf{A}+\mathbf{B}\mathbf{F})_{\widehat{\mathcal{N}}_M}) \uplus \sigma((\mathbf{A}+\mathbf{B}\mathbf{F})_{\mathcal{R}})\,.$$

Considering $(\mathbf{A}+\mathbf{B}\mathbf{F})_{\widehat{\mathcal{C}}_M}$, we have

$$\begin{aligned}(\mathbf{A}+\mathbf{B}\mathbf{F})_{\widehat{\mathcal{C}}_M} &= \mathbf{N}_{\widehat{\mathcal{C}}_M}\,(\mathbf{A}+\mathbf{B}\mathbf{F})\,\mathbf{S}_{\widehat{\mathcal{C}}_M} \\ &= \mathbf{N}_{\widehat{\mathcal{C}}_M}\mathbf{A}\mathbf{S}_{\widehat{\mathcal{C}}_M} + \mathbf{N}_{\widehat{\mathcal{C}}_M}\mathbf{B}(\mathbf{S}_{\widehat{\mathcal{C}}_M}\mathbf{F}_1\mathbf{N}_{\widehat{\mathcal{C}}_M})\mathbf{S}_{\widehat{\mathcal{C}}_M} = \mathbf{A}_{\widehat{\mathcal{C}}_M} + \mathbf{B}_{\widehat{\mathcal{C}}_M}\mathbf{F}_1.\end{aligned}$$

Considering $(\mathbf{A}+\mathbf{BF})_{\widehat{\mathcal{V}}^\diamond}$, we have

$$\begin{aligned}(\mathbf{A}+\mathbf{BF})_{\widehat{\mathcal{V}}^\diamond} &= \mathbf{N}_{\widehat{\mathcal{V}}^\diamond}(\mathbf{A}+\mathbf{BF})\,\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} \\ &= \mathbf{N}_{\widehat{\mathcal{V}}^\diamond}\mathbf{A}\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} + \mathbf{N}_{\widehat{\mathcal{V}}^\diamond}\mathbf{B}(\mathbf{S}_{\mathcal{C}_M}\mathbf{F}_1\mathbf{N}_{\mathcal{C}_M})\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} = \mathbf{A}_{\widehat{\mathcal{V}}^\diamond},\end{aligned}$$

where we use the fact that $\mathbf{N}_{\mathcal{C}_M}\mathbf{S}_{\widehat{\mathcal{V}}^\diamond} = \mathbf{0}$. Similarly, we obtain that $(\mathbf{A}+\mathbf{BF})_{\mathcal{R}} = \mathbf{A}_{\mathcal{R}}$ and $(\mathbf{A}+\mathbf{BF})_{\widehat{\mathcal{N}}_M} = \mathbf{A}_{\widehat{\mathcal{N}}_M}$. Thus, we have

$$\sigma(\mathbf{A}+\mathbf{BF}) = \sigma(\mathbf{A}_{\widehat{\mathcal{C}}_M} + \mathbf{B}_{\widehat{\mathcal{C}}_M}\mathbf{F}_1) \uplus \sigma(\mathbf{A}_{\widehat{\mathcal{V}}^\diamond}) \uplus \sigma(\mathbf{A}_{\widehat{\mathcal{N}}_M}) \uplus \sigma(\mathbf{A}_{\mathcal{R}}).$$

By Lemma 2.4, $\mathcal{X}^+(\mathbf{A})\cap\mathcal{N}_M$ is $\mathbf{M}$-decoupling and by assumption $\mathcal{X}^+(\mathbf{A})\cap\mathcal{N}_M \subset \mathcal{V}^\star$. Thus, $\mathcal{X}^+(\mathbf{A})\cap\mathcal{N}_M \subset \sup\mathfrak{D}^\diamond(\mathbf{M},\mathcal{V}^\star) = \mathcal{V}^\diamond$. In turn, this implies $\mathcal{X}^+(\mathbf{A}) \subset \widehat{\mathcal{C}}_M + \widehat{\mathcal{V}}^\diamond + \mathcal{R}$. Also by assumption, $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$, which implies that $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M + \mathcal{V}^\diamond$, by Lemma 5.4. By Lemma 2.1(ii), $\sigma(\mathbf{A}_{\widehat{\mathcal{N}}_M}) \subset \mathbb{C}^-$ and $\sigma(\mathbf{A}_{\mathcal{R}}) \subset \mathbb{C}^-$.

We conclude that $\sigma((\mathbf{A}+\mathbf{BF})_{\widehat{\mathcal{N}}_M}) \subset \mathbb{C}^-$, $\sigma((\mathbf{A}+\mathbf{BF})_{\widehat{\mathcal{C}}_M}) \subset \mathbb{C}^-$, and $\sigma((\mathbf{A}+\mathbf{BF})_{\mathcal{R}}) \subset \mathbb{C}^-$. Therefore applying Lemma 2.1(ii) again, we obtain $\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \widehat{\mathcal{V}}^\diamond \subset \operatorname{Ker}\mathbf{D}$, as desired. □

Assume that the conditions to solve the Patterned RRP are met for a given system. Then there exists a patterned state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$, such that $\operatorname{Ker}\mathbf{F} \supset \mathcal{N}_M \supset \operatorname{Ker}\mathbf{C}$. It follows that there exists a measurement feedback $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ that solves the equation $\mathbf{KC} = \mathbf{F}$. Furthermore, $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$, and we have that $\mathcal{X}^+(\mathbf{A}+\mathbf{BKC}) \subset \operatorname{Ker}\mathbf{D}$. Conversely, if $\operatorname{Ker}\mathbf{F} \not\supseteq \operatorname{Ker}\mathbf{C}$ then there exists no solution $\mathbf{K}$ to $\mathbf{KC} = \mathbf{F}$. And, if $\mathbf{F} \notin \mathfrak{F}(\mathbf{M})$ then any solution $\mathbf{K}$ would not be a member of $\mathfrak{F}(\mathbf{M})$. We draw the following conclusion showing that static measurement feedback and regulation are inextricably intertwined for patterned systems.

Corollary 5.9. *There exists a solution to the Patterned OSMFP if and only if there exists a solution to the Patterned RRP.*

6. Illustrative Examples

We present several examples of patterned systems with associated stabilization problems. These basic examples are intended to convey the breadth of research areas that touch on patterned systems and to illustrate the meaning of our theoretical results. A more detailed discussion of specific patterns with physically meaningful interpretations can be found in [9] including circulants, symmetric circulants, factor circulants, hierarchies of circulants, uni-directional chains (triangular Toeplitz systems) and uni-directional trees.

6.1. **Multi-agent Consensus.** A multi-agent system consists of several subsystems that act autonomously, and an extensively studied multi-agent objective is consensus. The consensus problem is an output stabilization by measurement feedback problem. Consensus can be achieved if there exists a measurement feedback controller $u = \mathrm{K}y$, such that $z \to 0$ as $t \to \infty$, where variable z defines the global consensus objective. A general K assumes full communication between agents. It is desirable to impose structural constraints on K to limit communication. We illustrate with an example.

We are given n identical robots and the global objective of rendezvous. Suppose the measurements taken by each robot must be identical up to indices, and identical local controllers (up to indices) must be distributed. What measurements are required for local controllers to exist? The robots are modeled as integrators: $\dot{x}_i = u_i$ for $i = 1, \ldots, n$. Combine the n robot subsystems together to obtain

$$\dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} u.$$

We restrict the measurement matrix C to take on a circulant pattern, so that each robot takes the same measurements up to indices, giving

$$y = \mathrm{C}x = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & & c_{n-2} \\ \vdots & & \ddots & \\ c_1 & c_2 & & c_0 \end{bmatrix} x.$$

but we do not specify C up front. Rendezvous is achieved when all the robots converge to a common position, which can also be expressed as the relative positions of all robots stabilizing to zero. A suitable global objective model is

$$z = \mathrm{D}x = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x.$$

Thus we have a circulant system where $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \in \mathfrak{F}(\Pi)$. The control problem is to find $u = \mathrm{K}y$, $\mathrm{K} \in \mathfrak{F}(\Pi)$ such that $z \to 0$ as $t \to \infty$. By the results of Section 5.4, there exists a solution to this Patterned Output Stabilization by Measurement Feedback Problem if and only if there exists a solution to the Patterned Restricted Regulator Problem. A solution to the Patterned RRP exists if and only if $\mathcal{X}^+(\mathrm{A}) \cap \mathcal{N}_M \subset \mathrm{Ker}\, \mathrm{D}$ and $\mathcal{X}^+(\mathrm{A}) \subset \mathcal{C} + \mathcal{V}^\star$. For the given system, we have $\mathcal{X}^+(\mathrm{A}) = \mathbb{R}^n$, $\mathcal{N}_M = \mathrm{Ker}\, \mathrm{C}$, (C is still undefined), $\mathrm{Ker}\, \mathrm{D} = \mathrm{span}\, \{(1, 1, \ldots, 1)\}$, $\mathcal{C} = \mathrm{Im}\, \mathrm{B} = \mathbb{R}^n$, and $\mathcal{V}^* = \mathrm{Ker}\, \mathrm{D}$. Then a suitable controller will exist provided that

$$\begin{aligned} &\mathbb{R}^n \cap \mathcal{N}_M \subset \mathrm{span}\, \{(1, 1, \ldots, 1)\}, \\ \text{and} \qquad &\mathbb{R}^n \subset \mathbb{R}^n + \mathrm{span}\, \{(1, 1, \ldots, 1)\}. \end{aligned}$$

Clearly, the second condition holds. The first condition imposes constraints on $\mathcal{N}_M$. If we choose the measurement model

$$y = \mathrm{C}x = \begin{bmatrix} -1 & 1 & & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x,$$

then $\mathcal{N}_M = (1, 1, \ldots, 1)$ and the first condition also holds. In this case, we conclude that a circulant controller to achieve consensus exists. One solution would be the decentralized controller $u = y$.

Suppose we choose instead a measurement model where a robot measures its relative distance to the robot two places ahead, given by

$$y = \mathrm{C}x = \begin{bmatrix} -1 & 0 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 0 & -1 & \cdots & 0 & 0 \\ & & \vdots & & & \vdots & \\ 0 & 1 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} x.$$

Then $\mathcal{N}_M = (1, 1, \dots, 1)$ if n is odd, but $\mathcal{N}_M = \text{span}\,\{(1, 0, 1, 0, \dots, 1, 0), (0, 1, 0, 1, \dots, 0, 1)\}$ if n is even. In the first scenario, a controller exists; whereas, in the second, the conditions of the Patterned RRP are not met. The conclusions from this example can also be interpreted in terms of graph theory results on consensus.

6.2. **Cellular Chemistry.** Turing [19] proposed that, for the purposes of studying cellular chemical reactions, one simple and illustrative arrangement of cells is a ring. Given a ring of n identical cells, let x_i denote the concentration of chemical X in cell i. Turing's model is given by

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \alpha x_i(t) + \beta u_i(t) + \kappa\left(x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)\right) \\ &= \kappa\left(x_{i+1}(t) + (\frac{\alpha}{\kappa} - 2)x_i(t) + x_{i-1}(t)\right) + \beta u_i(t), \end{aligned}$$

for $i = 1, \dots, n$. Let $\alpha = 2$, $\beta = -1$ and $\kappa = 0.5$. Consider the concentration of chemical U to be a controlled input in each cell. Then the cellular ring system has the circulant state space model

$$\dot{x}_i(t) = \begin{bmatrix} 1 & 0.5 & 0 & \cdots & 0.5 \\ 0.5 & 1 & 0.5 & & 0 \\ & \vdots & & & \\ 0.5 & 0 & 0 & \cdots & 1 \end{bmatrix} x_i(t) - I u_i(t).$$

Observe that this system is unstable. We assume that the cell concentrations are measurable, and the objective is to find a state feedback controller $u(t) = \mathrm{F}x(t)$ that brings the concentrations into equilibrium. We can express this objective as $z(t) \to 0$ as $t \to \infty$, where

$$z(t) = \mathrm{D}x(t) = \begin{bmatrix} -1 & 1 & & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x(t),\ \mathrm{D} \in \mathfrak{C}.$$

This is the Patterned OSP, which, by the results of Section 5.2, is solvable if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$. Since $\mathcal{C} = \operatorname{Im}\mathrm{B} = \mathbb{R}^n$, the problem has a solution.

6.3. **Discretized Partial Differential Equations.** We consider a simple example of how a symmetric Toeplitz system can be converted to a patterned circulant system for the purposes of computing a controller. Let $x(t, d)$ be a continuous function of two variables, defined over an interval $0 < d < l$. A lumped approximation to the multi-dimensional function is a set of $n + 1$ continuous functions $x_0(t), x_1(t), \dots, x_n(t)$ that sample $x(t, d)$ at regular spacings along the interval d. Let the space between sample functions be $h := \frac{l}{n}$, then $x_i(t) = x(t, ih)$. If the partial derivatives in time and space are appropriately approximated (using finite differences), one obtains a discretization of the PDE. For example, consider the diffusion process

$$\frac{\partial x(t,d)}{\partial t} = k\frac{\partial^2 x(t,d)}{\partial d^2}, \tag{6.1}$$

where x is the process variable and d is a spatial variable. When the process variable is temperature, this PDE is called the heat equation. Assume the model holds over an interval $0 < d < l$, and assume boundary conditions on the process of $x(t,0) = x(t,l) = 0$ for all time. Suppose we control the diffusion process by adding $n-1$ control inputs that act on the derivative of the process variable and that are spaced evenly along the spatial extent. There are also sensors of the process variable at each controller location. Then, using the finite difference method, we obtained the discretized model

$$\frac{dx_i(t)}{dt} = \frac{k}{h^2}\left(x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)\right) + u_i(t),$$

$i = 1,\dots,n-1$ and the boundary conditions $x_0(t) = 0$ and $x_n(t) = 0$ for all time.

In matrix form this system has A and B matrices which are symmetric and Toeplitz. Brockett and Willems [4] showed that one way to find a near Toeplitz state feedback F such that $u(t) = Fx(t)$ achieves a desired trajectory $x(t)$ is to model the Toeplitz system by a larger circulant system. A circulant solution can be easily reduced to a solution for the Toeplitz system. The expanded circulant system is constructed by creating a mirror image of the original system and then connecting it to the original system at the boundary points. Consider the expanded circulant system

$$\dot{x} = \frac{k}{h^2}\mathrm{circ}\,(-2,1,0,\cdots,0,1)x + Iu$$

where $x \in \mathbb{R}^{2n-1}$ and $u \in \mathbb{R}^{2n-1}$. Let the initial states $x_i(0)$ in the extended system equal the initial states $x_i(0)$ in the original Toeplitz system for $i = 1,\dots,n-1$. Further, assume that $x_{2n-i}(0) = -x_i(0)$ for $i = 1,\dots,n-1$. Then we have the following result.

Proposition 6.1 ([4]). *If (a) $u_0(t) = u_n(t) = 0$; (b) $u_i(t)$ in the extended system is applied as $u_i(t)$ in the original system for $i = 0,1,\dots,n$; and (c) $u_{2n-i}(t) = -u_i(t)$ for $i = 1,\dots,n-1$, then $x_0(t) = x_n(t) = 0$, $x_i(t)$ is the same for both systems for $i = 1,\dots,n-1$, and $x_{2n-i}(t) = -x_i(t)$, for $i = 1,\dots,n-1$.*

Now we apply the method to a pole placement problem for the diffusion process (6.1) with $k = 2$ over the interval $0 < d < 4$. Let the spacing between lumped approximations along the interval be 1, then $n = 4$. This discretizes the PDE into three differential equations. Assuming that 3 discrete controllers are spaced evenly along the interval, the equations are given by

$$\dot{x}(t) = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t),$$

with assumed boundary conditions $x_0(t) = x_4(t) = 0$ for all time. The poles of the system, $\{-1.17, -4, -6.83\}$, are already stable but it is desirable to place the poles further into the left half plane in order to increase the speed at which the process variable converges to the boundary conditions. Suppose our objective is to find a feedback $u(t) = \mathrm{F}x(t)$ to place the poles at $\{-8, -10, -10\}$. Using the state space extension method, we create the symmetric circulant 8×8 system

$$\dot{x}_e = \mathrm{circ}\,(-4,2,0,0,0,0,0,2)x_e + Iu_e = \mathrm{A}_e x_e + \mathrm{B}_e u_e\,.$$

Note that the poles of the extended systems are $\{0, -1.17, -1.17, -4, -4, -6.83, -6.83, -8\}$, which consists of the spectrum of the original system, duplicated once, and two additional poles at 0 and -8. These additional poles are immaterial, because they will disappear when we convert back to the original system.

It is known that symmetric circulant systems are Σ-patterned systems where $\Sigma = \Pi + \Pi^{\mathsf{T}}$ and Π is the shift operator. Therefore, by Theorem 4.3 there exists a symmetric circulant feedback $\mathrm{F}_e \in \mathfrak{F}(\Sigma)$ to place the poles in any Σ-patterned spectrum if and only if $\mathcal{X}^+(\mathrm{A}_e) \subset \mathcal{C}$. The controllable subspace of the patterned system is $\operatorname{Im}\mathrm{B}_e$, so clearly $\mathcal{C} = \mathcal{X}$ and the condition $\mathcal{X}^+(\mathrm{A}_e) \subset \mathcal{C}$ holds. Let

$$\mathrm{F}_e := \operatorname{circ}(-4, -0.65, 1.5, 0.65, 1, 0.65, 1.5, -0.65).$$

It can be shown that

$$\begin{aligned}\mathrm{F}_e = &-10\mathrm{I} + 0.058\Sigma + 0.055\Sigma^2 + 0.098\Sigma^3 + 0.084\Sigma^4 \\ &+ 0.13\Sigma^5 + 0.069\Sigma^6 - 0.0082\Sigma^7,\end{aligned}$$

confirming that $\mathrm{F}_e \in \mathfrak{F}(\Sigma)$. We obtain $\sigma(\mathrm{A}_e + \mathrm{B}_e\mathrm{F}_e) = \{0, -8, -8, -10, -10, -10, -10, -8\}$, which meets our pole placement criteria. Since we have found a symmetric circulant solution to the extended problem, we will meet the conditions of Proposition 6.1. The corresponding solution F to the original Toeplitz system is

$$\mathrm{F} = \begin{bmatrix} -5.5 & -1.29 & 0.5 \\ -1.29 & -5 & -1.29 \\ 0.5 & -1.29 & -5.5 \end{bmatrix}.$$

Then the closed loop system becomes

$$\dot{x}(t) = \begin{bmatrix} -9.5 & 0.71 & 0.5 \\ 0.71 & -9 & 0.71 \\ 0.5 & 0.71 & -9.5 \end{bmatrix} x(t),$$

where $\sigma(\mathrm{A} + \mathrm{BF}) = \{-8, -10, -10\}$, as desired. Notice that the solution F that we have found is not exactly Toeplitz, but near Toeplitz, as desired.

7. Conclusion

We have introduced a new class of linear control systems called patterned linear systems. The contribution is in uniting multiple patterns under the umbrella of a general theory, and we place emphasis on some common patterns in applications: certain classes of rings, chains, and trees. A significant outcome is that if a general controller exists to solve any of the studied control synthesis problems, then a patterned feedback exists. If there is parameter uncertainty so that system matrices are "almost patterned", then known results on robustness in feedback systems can be applied. Examination of applications is left to future research; however, the range of practical applications can be significantly enlarged if the theory is extended to block patterned systems. Other future research directions include the patterned robust regulator problem, infinite dimensional patterned systems, linear combinations of two or more base patterns, and the pattern identification problem.

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Dept. of Electrical and Computer Engineering, University of Toronto, Toronto ON Canada M5S 3G4

E-mail address: broucke@control.utoronto.ca