Geometric Control of Patterned Linear Systems

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***Abstract*— We introduce and study a new class of linear control systems called patterned systems. Mathematically, this class has the property that the state, input and output transformations of the linear state space model are all functions of a common base transformation. The motivation for studying such systems arises from their interpretation as a collection of identical subsystems with a pattern of interaction between subsystems that is imprinted by the base transformation. The significance of patterned systems as a distinct class is that they may provide a template for the development of a more unified framework for dealing with systems, typically distributed, which consist of subsystems interacting via a fixed pattern.**

I. Introduction

We introduce and study a new class of linear control systems called patterned systems. Mathematically, this class has the property that the state, input and output transformations of the linear state space model are all functions of a common base transformation. The motivation for studying such systems is their interpretation as a collection of identical subsystems with a pattern of interaction between subsystems that is imprinted by the base transformation. The significance of patterned systems as a distinct class is that they may provide a template for the development of a more unified framework for dealing with systems, typically distributed, which consist of subsystems interacting via a fixed pattern.

Complex systems that are made of a large number of simple subsystems with simple patterns of interaction arise frequently in natural and engineering systems. Such systems arise particularly out of models which are lumped approximations of partial differential equations (p.d.e.s). One such application concerns ring systems, which can be modeled as circulant or block-circulant systems. These systems are worthy of special mention because they provide a metaphor for patterned systems. In [2] circulant systems arising from control of systems modeled by discretized partial differential equations are studied from a control perspective. The key insight is that all circulant (or block circulant) matrices are diagonalized (or block diagonalized) by a common matrix.

The starting point of our investigation was a study of circulant systems from a geometric approach, based on a hypothesis that circulant systems have deeper structural properties beyond diagonalization. The essence of the geometric approach is to describe properties of the system in terms of subspaces, and then to express conditions for controller synthesis in terms of these subspaces [6]. Circulant matrices have a wealth of interesting relationships with the class of subspaces invariant under the *shift operator*. Important subspaces like the controllable subspace and the observable subspace fall within this class. This greatly simplifies the study of control problems like pole placement and stabilization when it is desired that the controller be circulant as well. In this paper these ideas about circulant systems are extended to a broader family that includes all systems with state, input and output transformations that are functions of a common *base transformation*. We call the members of this family *patterned linear systems*.

In this first paper of a two part series, we study fundamental properties of patterned maps and their relationships to certain invariant subspaces. Using this foundation, we follow classical developments to build system theoretic properties of patterned linear systems. Finally we study the classic control synthesis problems [6]. A significant outcome of this study is that if a general controller exists to solve any of the studied control synthesis problems, then a patterned feedback exists. In the second paper we study specific patterns - rings, chains, and trees - which give relevance to the class of patterned systems.

II. Background

We assume that the reader is already familiar with the tools of geometric control theory [1], [6]. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional vector spaces. We consider *linear maps* from $\mathcal{X}$ to $\mathcal{Y}$, denoted by bold capital letters, such as $\mathbf{T} : \mathcal{X} \to \mathcal{Y}$. The plain capital, T, denotes a matrix representation of the map $\mathbf{T}$. Let $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ be an endomorphism. Let $\mathcal{S}_\lambda(\mathbf{T})$ denote the eigenspace of $\mathbf{T}$ associated with eigenvalue λ. The Jordan subspaces of $\mathbf{T}$ are given by

$$\mathcal{J}_{ij}(\mathbf{T}) = \mathrm{span}\left(v_{ij}, g_{i1}, g_{i2}, \ldots, g_{i(p_{ij}-1)}\right),$$

where each eigenvector v_{ij} spawns the Jordan chain.

Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Map $\mathbf{Q}_\mathcal{V} : \mathcal{X} \to \mathcal{X}$ denotes the projection on $\mathcal{V}$ along $\mathcal{W}$, $\mathbf{N}_\mathcal{V} : \mathcal{X} \to \mathcal{V}$ denotes the natural projection, and $\mathbf{S}_\mathcal{V} : \mathcal{V} \to \mathcal{X}$ denotes the insertion of $\mathcal{V}$ in $\mathcal{X}$. Useful relations are that $\mathbf{N}_\mathcal{V}\mathbf{S}_\mathcal{V} = \mathbf{I}_\mathcal{V}$ and $\mathbf{N}_\mathcal{W}\mathbf{S}_\mathcal{V} = \mathbf{0}$. Given a $\mathbf{T}$-invariant subspace $\mathcal{V} \subset \mathcal{X}$, if there also exists a subspace $\mathcal{W} \subset \mathcal{X}$ such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$ and $\mathcal{W}$ is $\mathbf{T}$-invariant, then we call $\mathcal{V}$ a $\mathbf{T}$-*decoupling* subspace. The restriction of $\mathbf{T}$ to $\mathcal{V}$ is denoted by $\mathbf{T}_\mathcal{V} : \mathcal{V} \to \mathcal{V}$ and is given by $\mathbf{T}_\mathcal{V} = \mathbf{N}_\mathcal{V}\mathbf{T}\mathbf{S}_\mathcal{V}$. Similarly, define $\mathbf{T}_\mathcal{W} : \mathcal{W} \to \mathcal{W}$, the restriction of $\mathbf{T}$ to $\mathcal{W}$, by $\mathbf{T}_\mathcal{W} = \mathbf{N}_\mathcal{W}\mathbf{T}\mathbf{S}_\mathcal{W}$.

Let $\mathcal{V}, \mathcal{W}$ be $\mathbf{T}$-decoupling subspaces such that $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$. Suppose the minimal polynomial (m.p.) of $\mathbf{T}$, $\psi(s)$, has been factored as $\psi(s) = \psi^-(s)\psi^+(s)$ such that

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$\mathcal{X}^-(\mathbf{T}) = \operatorname{Ker}\psi^-(\mathbf{T})$, $\mathcal{X}^+(\mathbf{T}) = \operatorname{Ker}\psi^+(\mathbf{T})$, and $\mathcal{X}^-(\mathbf{T})$ and $\mathcal{X}^+(\mathbf{T})$ are the stable and unstable subspaces of $\mathbf{T}$. Similarly, let $\psi_{\mathcal{V}}(s)$ be the m.p. of $\mathbf{T}_{\mathcal{V}}$ and suppose it has been also been factored as $\psi_{\mathcal{V}}(s) = \psi^-_{\mathcal{V}}(s)\psi^+_{\mathcal{V}}(s)$ such that $\mathcal{V}^-(\mathbf{T}_{\mathcal{V}}) = \operatorname{Ker}\psi^-(\mathbf{T}_{\mathcal{V}})$, $\mathcal{V}^+(\mathbf{T}_{\mathcal{V}}) = \operatorname{Ker}\psi^+(\mathbf{T}_{\mathcal{V}})$. The following result summarizes useful properties of stable and unstable subspaces.

Lemma 2.1 ([6]): Let $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ be a linear map and let $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ be $\mathbf{T}$-decoupling subspaces, i.e. $\mathbf{T}\mathcal{V} \subset \mathcal{V}$, $\mathbf{T}\mathcal{W} \subset \mathcal{W}$, and $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$. Then we have (i) $\mathbf{N}_{\mathcal{V}}\mathcal{X}^+(\mathbf{T}) = \mathcal{V}^+(\mathbf{T}_{\mathcal{V}})$; and (ii) $\mathcal{X}^+(\mathbf{T}) \subset \mathcal{V}$ if and only if $\sigma(\mathbf{T}_{\mathcal{W}}) \subset \mathbb{C}^-$.

We denote the set of all $\mathbf{T}$-decoupling subspaces in $\mathcal{X}$ by $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{X})$. Similarly, for any $\mathcal{V} \subset \mathcal{X}$, not necessarily a $\mathbf{T}$-invariant subspace, we denote the set of all $\mathbf{T}$-decoupling subspaces contained in $\mathcal{V}$ by $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$; that is, $\mathcal{Y} \in \mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ if $\mathcal{Y} \subset \mathcal{V}$, $\mathcal{Y}$ is $\mathbf{T}$-invariant, and $\mathcal{Y}$ has an $\mathbf{T}$-invariant complement in $\mathcal{X}$. (Note that the complement need not be in $\mathcal{V}$.) We also denote the set of all $\mathbf{T}$-decoupling subspaces in $\mathcal{X}$ containing $\mathcal{V}$ by $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$. Decoupling subspaces are closely linked to Jordan subspaces.

Lemma 2.2 ([3]): Every Jordan subspace of $\mathbf{T}$ is a $\mathbf{T}$-decoupling subspace, and every $\mathbf{T}$-decoupling subspace is the sum of Jordan subspaces of $\mathbf{T}$.

Lemma 2.3: Let $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{X}$ be $\mathbf{T}$-invariant subspaces, and let $\mathcal{J} \subset \mathcal{V}_1 + \mathcal{V}_2$ be a Jordan subspace of $\mathbf{T}$. Then $\mathcal{J} \subset \mathcal{V}_1$ or $\mathcal{J} \subset \mathcal{V}_2$.

We say that a subspace $\mathcal{V}^\diamond$ is the *supremum* of $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$, denoted $\mathcal{V}^\diamond = \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$, if $\mathcal{V}^\diamond \in \mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and given $\mathcal{V}' \in \mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$, then $\mathcal{V}' \subset \mathcal{V}^\diamond$. Analogously, we say that a subspace $\mathcal{V}_\diamond$ is the *infimum* of $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$, denoted $\mathcal{V}_\diamond = \inf\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$, if $\mathcal{V}_\diamond \in \mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ and given $\mathcal{V}' \in \mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$, then $\mathcal{V}_\diamond \subset \mathcal{V}'$. Existence and uniqueness of a supremal element of $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and an infimal element of $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ relies on the fact that $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ have the structure of a lattice under the operations of subspace addition and subspace intersection.

Lemma 2.4: Given $\mathcal{V} \subset \mathcal{X}$, the sets $\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V})$ and $\mathfrak{D}_\diamond(\mathbf{T};\mathcal{V})$ are each closed under the operations of subspace addition and subspace intersection.

Lemma 2.5: Let $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{X}$ be $\mathbf{T}$-invariant subspaces, and let $\mathcal{V}_1^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V}_1)$, $\mathcal{V}_2^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V}_2)$, and $(\mathcal{V}_1 + \mathcal{V}_2)^\diamond := \sup\mathfrak{D}^\diamond(\mathbf{T};\mathcal{V}_1 + \mathcal{V}_2)$. Then $(\mathcal{V}_1 + \mathcal{V}_2)^\diamond = \mathcal{V}_1^\diamond + \mathcal{V}_2^\diamond$.

III. Patterned Linear Maps

Let $t_0, t_1, \ldots, t_k \in \mathbb{R}$ and consider the polynomial $\rho(s) = t_0 + t_1 s + t_2 s^2 + t_3 s^3 + \ldots + t_k s^k$. The argument of the polynomial can be extended to become a matrix as follows. Let M be an $n \times n$ real matrix. Then $\rho(\mathrm{M})$ is defined by

$$\rho(\mathrm{M}) := t_0 I + t_1\mathrm{M} + t_2\mathrm{M}^2 + t_3\mathrm{M}^3 + \ldots + t_k\mathrm{M}^k.$$

Given $\mathrm{T} = \rho(\mathrm{M})$, then $\rho(s)$ is called a *representer of* T *with respect to* M, and it is generally not unique. By Cayley-Hamilton theorem, our discussion will be confined to $\rho(\mathrm{M})$ of order equal to, or less than, $n-1$. We define the set of all matrices that are polynomial functions of a given *base matrix* $\mathrm{M} \in \mathbb{R}^{n\times n}$ by $\mathfrak{F}(\mathrm{M}) := \left\{\mathrm{T} \mid (\exists\, t_0, \ldots, t_{n-1} \in \mathbb{R})\ \mathrm{T} = t_0\mathrm{I} + t_1\mathrm{M} + \ldots + t_{n-1}\mathrm{M}^{n-1}\right\}$. We call a matrix $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$ an M-*patterned* matrix.

Fact 3.1: Given $\mathrm{T}, \mathrm{R} \in \mathfrak{F}(\mathrm{M})$ then $\mathrm{TR} = \mathrm{RT}$.

Given $\mathrm{M} \in \mathbb{R}^{n\times n}$, let the n eigenvalues of M be denoted by $\sigma(\mathrm{M}) = \{\delta_1, \delta_2 \ldots, \delta_n\}$. Note that the spectrum is symmetric with respect to the real axis since M is real. Define a symmetric subset

$$\{\mu_1, \ldots, \mu_m\} \subset \sigma(\mathrm{M}) \qquad (1)$$

such that each distinct eigenvalue is repeated only m_i times in the subset, where m_i is the geometric multiplicity of the eigenvalue. Then, associated with each eigenvalue μ_i is the partial multiplicity p_i. There exists a Jordan transformation Ω such that $\Omega^{-1}\mathrm{M}\Omega = \mathrm{J}$, where J is the Jordan form of M.

Suppose we are given an arbitrary matrix and a base pattern M. We can determine whether or not the matrix is M-patterned.

Theorem 3.2: Given $\mathrm{T} \in \mathbb{R}^{n\times n}$, then $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$ if and only if

(1) $\Omega^{-1}\mathrm{T}\Omega = \operatorname{diag}(\mathrm{H}_1, \mathrm{H}_2, \ldots, \mathrm{H}_m)$,

$$\text{where } \mathrm{H}_i = \begin{bmatrix} h_{i1} & h_{i2} & \cdots & h_{ip_i} \\ & h_{i1} & \ddots & \vdots \\ 0 & & \ddots & h_{i2} \\ 0 & 0 & & h_{i1} \end{bmatrix},\ h_{ij} \in \mathbb{C},$$

(2) $\forall\,\{i_1, i_2\} \in \{1, \ldots, m\}$ if $\mu_{i_1} = \bar{\mu}_{i_2}$ then $h_{i_1 j} = \bar{h}_{i_2 j}\,, \forall j = 1, \ldots, \min(p_{i_1}, p_{i_2})$ and

(3) $\forall\,\{i_1, i_2\} \in \{1, \ldots, m\}$ if $\mu_{i_1} = \mu_{i_2}$ then $h_{i_1 j} = h_{i_2 j}\,, \forall j = 1, \ldots, \min(p_{i_1}, p_{i_2})$.

Suppose we are given an arbitrary spectrum of n values and an objective to construct an M-patterned matrix with the given spectrum. The next result presents the conditions under which this is possible.

Lemma 3.3: Let $\mathfrak{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \lambda_i \in \mathbb{C}$. Suppose the elements of $\mathfrak{L}$ can be reordered so that if $\delta_i = \bar{\delta}_j$ then $\lambda_i = \bar{\lambda}_j$, and if $\delta_i = \delta_j$ then $\lambda_i = \lambda_j$. Then there exists $\mathrm{T} \in \mathfrak{F}(\mathrm{M})$, such that $\sigma(\mathrm{T}) = \mathfrak{L}$.

A spectrum that can be reordered in the manner of Lemma 3.3 is an *M-patterned spectrum*.

Lemma 3.4: Given $\mathrm{T}, \mathrm{R} \in \mathfrak{F}(\mathrm{M})$ and a scalar $\alpha \in \mathbb{R}$, then $\{\alpha\mathrm{T}, \mathrm{T}+\mathrm{R}, \mathrm{TR}\} \in \mathfrak{F}(\mathrm{M})$, and $\mathrm{T}^{-1} \in \mathfrak{F}(\mathrm{M})$ assuming T^{-1} exists. Moreover, given $\sigma(\mathrm{T}) = \{\tau_1, \ldots, \tau_n\}$ and $\sigma(\mathrm{R}) = \{\varrho_1, \ldots, \varrho_n\}$, both ordered relative to the eigenvalues of M, then $\sigma(\alpha\mathrm{T}) = \{\alpha\tau_1, \ldots, \alpha\tau_n\}$, $\sigma(\mathrm{T}+\mathrm{R}) = \{\tau_1 + \varrho_1, \ldots, \tau_n + \varrho_n\}$, $\sigma(\mathrm{TR}) = \{\tau_1\varrho_1, \ldots, \tau_n\varrho_n\}$, and $\sigma(\mathrm{T}^{-1}) = \{1/\tau_1, \ldots, 1/\tau_n\}$.

Next, consider a linear map $\mathbf{M} : \mathcal{X} \to \mathcal{X}$. We define the set of linear maps

$$\mathfrak{F}(\mathbf{M}) := \{\mathbf{T} \mid (\exists\, t_0, \ldots, t_{n-1} \in \mathbb{R})\ \mathbf{T} = t_0\mathbf{I} + t_1\mathbf{M} + \ldots + t_{n-1}\mathbf{M}^{n-1}\}.$$

We call a map $\mathbf{T} : \mathcal{X} \to \mathcal{X}$, $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$ an $\mathbf{M}$-*patterned map*. We now present some important relationships between $\mathbf{M}$-patterned maps and $\mathbf{M}$-invariant subspaces.

Fact 3.5: Let $\mathcal{V} \subset \mathcal{X}$. If $\mathcal{V}$ is $\mathbf{M}$-invariant, then $\mathcal{V}$ is $\mathbf{T}$-invariant for every $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$.

Conversely, is a $\mathbf{T}$-invariant subspace always $\mathbf{M}$-invariant? The answer is not generally. The eigenvectors of $\mathbf{M}$ are all eigenvectors of $\mathbf{T}$; however, $\mathbf{T}$ may have additional eigenvectors that are not eigenvectors of $\mathbf{M}$. Fortunately, it is possible to identify certain $\mathbf{T}$-invariant subspaces, useful in a control theory context, that are also $\mathbf{M}$-invariant.

Lemma 3.6: Let $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$ and let $\rho(s)$ be a polynomial. Then $\operatorname{Ker} \rho(\mathbf{T})$ and $\operatorname{Im} \rho(\mathbf{T})$ are $\mathbf{M}$-invariant and $\mathbf{T}'$-invariant for every $\mathbf{T}' \in \mathfrak{F}(\mathbf{M})$.

Lemma 3.7: Let $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. Then the following subspaces are $\mathbf{M}$-invariant and $\mathbf{T}$-invariant: (i) the stable and unstable subspaces: $\mathcal{X}^-(\mathbf{T})$ and $\mathcal{X}^+(\mathbf{T})$, and (ii) the eigenspaces: $\mathcal{S}_\lambda(\mathbf{T})$, $\lambda \in \sigma(\mathbf{T})$. Also, the spectral subspaces of $\mathbf{T}$ are $\mathbf{M}$-decoupling.

Suppose we are given an $\mathbf{M}$-decoupling subspace $\mathcal{V}$. Then there exists an $\mathbf{M}$-invariant complement $\mathcal{W}$, such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Since $\mathcal{V}$ is $\mathbf{M}$-invariant, the restriction of $\mathbf{M}$ to $\mathcal{V}$, denoted $\mathbf{M}_\mathcal{V} : \mathcal{V} \to \mathcal{V}$, can be defined by $\mathbf{M}_\mathcal{V} := \mathbf{N}_\mathcal{V}\mathbf{M}\mathbf{S}_\mathcal{V}$. Similarly, the restriction of $\mathbf{M}$ to $\mathcal{W}$ can be defined by $\mathbf{M}_\mathcal{W} := \mathbf{N}_\mathcal{W}\mathbf{M}\mathbf{S}_\mathcal{W}$. The next lemma contains the important result that the restriction of an $\mathbf{M}$-patterned map $\mathbf{T}$ to an $\mathbf{M}$-invariant (or $\mathbf{M}$-decoupling) subspace is itself patterned, and the pattern is induced by the restriction of $\mathbf{M}$ to the subspace.

Lemma 3.8: Let $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. Then the restriction of $\mathbf{T}$ to $\mathcal{V}$ is given by $\mathbf{T}_\mathcal{V} = \mathbf{N}_\mathcal{V}\mathbf{T}\mathbf{S}_\mathcal{V}$ and moreover $\mathbf{T}_\mathcal{V} \in \mathfrak{F}(\mathbf{M}_\mathcal{V})$.

Given an $\mathbf{M}$-patterned map, it is possible to create a decomposed matrix representation of the map, which splits into the restrictions to $\mathcal{V}$ and to $\mathcal{W}$.

Theorem 3.9 (First Decomposition Theorem): Let $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$ such that the representation of $\mathbf{T}$ in the new coordinates is given by

$$\mathrm{R}^{-1}\mathrm{T}\mathrm{R} = \begin{bmatrix} \mathrm{T}_\mathcal{V} & 0 \\ 0 & \mathrm{T}_\mathcal{W} \end{bmatrix}, \mathrm{T}_\mathcal{V} \in \mathfrak{F}(\mathrm{M}_\mathcal{V}),\, \mathrm{T}_\mathcal{W} \in \mathfrak{F}(\mathrm{M}_\mathcal{W}).$$

The spectrum splits into $\sigma(\mathrm{T}) = \sigma(\mathrm{T}_\mathcal{V}) \uplus \sigma(\mathrm{T}_\mathcal{W})$.

The previous result shows how an $\mathbf{M}$-patterned map can be decoupled into smaller maps that are each a function of $\mathbf{M}$ restricted to an invariant subspace. Now consider the opposite problem. We are given a map that is a function of $\mathbf{M}$ restricted to a subspace. The map can be lifted into the larger space $\mathcal{X}$, and we give a sufficient condition under which it will be $\mathbf{M}$-patterned.

Lemma 3.10: Let $\mathbf{T}_1 \in \mathfrak{F}(\mathbf{M}_\mathcal{V})$. Define a map $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ by $\mathbf{T} := \mathbf{S}_\mathcal{V}\mathbf{T}_1\mathbf{N}_\mathcal{V}$. If $\sigma(\mathbf{M}_\mathcal{V}) \cap \sigma(\mathbf{M}_\mathcal{W}) = \emptyset$, then $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$.

IV. System Properties

Consider the control system

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$
$$y(t) = \mathrm{C}x(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. We denote the state space, input space, and output space by $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$, respectively. If $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C} \in \mathfrak{F}(\mathbf{M})$ with respect to some $\mathbf{M} : \mathcal{X} \to \mathcal{X}$, then $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ is termed an *$\mathbf{M}$-patterned system* or simply a *patterned system*. Observe that for patterned systems, $n = m = p$, thus $\mathcal{X} \simeq \mathcal{U} \simeq \mathcal{Y}$. Also, the open loop poles of the system form an $\mathbf{M}$-patterned spectrum. In this section we examine the system theoretic properties of patterned systems.

A. Controllability

The *controllable subspace* of a system is denoted by $\mathcal{C}$. Let $\mathcal{B} = \mathrm{Im}\mathbf{B}$. For patterned systems it is immediately observed that $\mathcal{C} = \mathcal{B}$, and $\mathcal{C}$ is $\mathbf{M}$-invariant.

Definition 4.1: The *patterned controllable subspace*, denoted $\mathcal{C}_M$, is the largest $\mathbf{M}$-decoupling subspace contained in $\mathcal{C}$. That is, $\mathcal{C}_M := \sup \mathfrak{D}^\diamond(\mathbf{M}; \mathcal{C})$.

Lemma 4.1: Let $(\mathbf{A}, \mathbf{B})$ be an $\mathbf{M}$-patterned pair. Then $\mathcal{C}_M = \{0\} + \sum_{\substack{\lambda \in \sigma(\mathbf{B}), \\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{B})$ and its $\mathbf{M}$-invariant complement is $\mathcal{S}_0(\mathbf{B})$.

Lemma 4.2: The $\mathbf{M}$-patterned pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $\mathcal{C}_M = \mathcal{X}$.

In addition to the case when $(\mathbf{A}, \mathbf{B})$ is controllable, $\mathcal{C}$ and $\mathcal{C}_M$ also coincide when $\mathcal{S}_0(\mathbf{B}) = \operatorname{Ker}(\mathbf{B})$, which is to say that there are no generalized eigenvectors associated with the zero eigenvalue of $\mathbf{B}$. Instead when $(\mathbf{A}, \mathbf{B})$ is not controllable, then $\mathcal{C}$ and $\mathcal{C}_M$ may differ.

B. Pole Placement

It is well known that the spectrum of $\sigma(\mathbf{A} + \mathbf{B}\mathbf{F})$ can be arbitrarily assigned to any symmetric set of poles by choice of $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ if and only if $(\mathbf{A}, \mathbf{B})$ is controllable. For a patterned system, the question arises of what possible poles can be achieved by a choice of patterned state feedback.

Theorem 4.3: The $\mathbf{M}$-patterned pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if, for every $\mathbf{M}$-patterned spectrum $\mathfrak{L}$, there exists a map $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ with $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ such that $\sigma(\mathbf{A} + \mathbf{B}\mathbf{F}) = \mathfrak{L}$.

We conclude that if we are limited to patterned state feedback, then the poles of an $\mathbf{M}$-patterned system can only be placed in an $\mathbf{M}$-patterned spectrum. This is not a severe limitation on pole placement, since stable $\mathbf{M}$-patterned spectra can be chosen for any $\mathbf{M}$.

C. Controllable Decomposition

Suppose we have a patterned system that is not fully controllable, i.e. $\mathcal{C} \neq \mathcal{X}$. We show that it is possible to decouple the system into two patterned subsystems, one that is controllable and one that is completely uncontrollable by a patterned state feedback. Since $\mathcal{C}_M$ is $\mathbf{M}$-decoupling there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{C}_M \oplus \mathcal{R} = \mathcal{X}$. Let $\mathbf{S}_{\mathcal{C}_M}$, $\mathbf{N}_{\mathcal{C}_M}$, $\mathbf{S}_\mathcal{R}$, and $\mathbf{N}_\mathcal{R}$ be the relevant insertion and projection maps, and let the restrictions of $\mathbf{M}$ to $\mathcal{C}_M$ and to $\mathcal{R}$ be denoted by $\mathbf{M}_{\mathcal{C}_M}$ and $\mathbf{M}_\mathcal{R}$. Before we present the decomposition, we note the following useful lemma.

Lemma 4.4: Let $(\mathbf{A}, \mathbf{B})$ be an $\mathbf{M}$-patterned pair. Then $\sigma(\mathbf{M}_{\mathcal{C}_M}) \cap \sigma(\mathbf{M}_\mathcal{R}) = \emptyset$.

Theorem 4.5 (Second Decomposition Theorem): Let $(\mathbf{A}, \mathbf{B})$ be an $\mathbf{M}$-patterned pair. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \rightarrow \mathcal{X}$ for the state and input spaces ($\mathcal{U} \simeq \mathcal{X}$), which decouples the system into two subsystems, $(\mathbf{A}_1, \mathbf{B}_1)$ and $(\mathbf{A}_2, \mathbf{B}_2)$, such that

(1) pair $(\mathbf{A}_1, \mathbf{B}_1)$ is $\mathbf{M}_{\mathcal{C}_M}$-patterned and controllable,
(2) pair $(\mathbf{A}_2, \mathbf{B}_2)$ is $\mathbf{M}_{\mathcal{R}}$-patterned,
(3) $\sigma(\mathbf{A}) = \sigma(\mathbf{A}_1) \uplus \sigma(\mathbf{A}_2)$,
(4) $\sigma(\mathbf{A}_2)$ is unaffected by patterned state feedback in the class $\mathfrak{F}(\mathbf{M}_{\mathcal{R}})$,
(5) $\mathbf{B}_2 = 0$ if $\mathcal{C}_M = \mathcal{C}$.

D. Stabilizability

A system, or equivalently the pair $(\mathbf{A}, \mathbf{B})$, is *stabilizable* if there exists $\mathbf{F} : \mathcal{X} \rightarrow \mathcal{U}$ such that $\sigma(\mathbf{A} + \mathbf{BF}) \subset \mathbb{C}^-$. A system is stabilizable if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$. For a patterned system, the question arises of whether the system can be stabilized with a patterned state feedback. We begin with a useful preliminary result.

Lemma 4.6: Given an $\mathbf{M}$-patterned pair $(\mathbf{A}, \mathbf{B})$, if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$, then $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M$.

Theorem 4.7 (Patterned Stabilizability): Given an $\mathbf{M}$-patterned system $(\mathbf{A}, \mathbf{B})$, there exists a patterned state feedback $\mathbf{F} : \mathcal{X} \rightarrow \mathcal{U}$ with $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ such that $\sigma(\mathbf{A} + \mathbf{BF}) \subset \mathbb{C}^-$ if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$.

E. Observability

The *unobservable subspace* of a system is denoted by $\mathcal{N}$. By a duality argument, for patterned systems $\mathcal{N} = \operatorname{Ker} \mathbf{C}$, and $\mathcal{N}$ is $\mathbf{M}$-invariant.

Definition 4.2: The *patterned unobservable subspace*, denoted $\mathcal{N}_M$, is smallest $\mathbf{M}$-decoupling subspace containing $\mathcal{N}$. That is, $\mathcal{N}_M := \inf \mathfrak{D}_\diamond(\mathbf{M}; \mathcal{N})$.

Lemma 4.8: Let $(\mathbf{C}, \mathbf{A})$ be an $\mathbf{M}$-patterned pair. Then $\mathcal{N}_M = \mathcal{S}_0(\mathbf{C})$ and its $\mathbf{M}$-invariant complement is $\{0\} + \sum_{\substack{\lambda \in \sigma(\mathbf{C}),\\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{C})$.

Lemma 4.9: The $\mathbf{M}$-patterned pair $(\mathbf{C}, \mathbf{A})$ is observable if and only if $\mathcal{N}_M = 0$.

In addition to the case when $(\mathbf{C}, \mathbf{A})$ is observable, $\mathcal{N}$ and $\mathcal{N}_M$ also coincide when $\mathcal{S}_0(\mathbf{C}) = \operatorname{Ker} \mathbf{C}$, which is to say that there are no generalized eigenvectors associated with the zero eigenvalue of $\mathbf{C}$. Instead when $(\mathbf{C}, \mathbf{A})$ is not observable, then $\mathcal{N}$ and $\mathcal{N}_M$ may differ.

F. Observable Decomposition

Suppose we have a patterned system that is not fully observable, i.e. $\mathcal{N} \neq 0$. We show that it is possible to decouple the system into two patterned subsystems, one that is observable and one that is patterned unobservable, meaning that the poles of the subsystem cannot be moved by any patterned measurement feedback. Since $\mathcal{N}_M$ is $\mathbf{M}$-decoupling, there exists an $\mathbf{M}$-invariant subspace $\mathcal{R}$ such that $\mathcal{N}_M \oplus \mathcal{R} = \mathcal{X}$. Let $\mathbf{S}_{\mathcal{N}_M}$, $\mathbf{N}_{\mathcal{N}_M}$, $\mathbf{S}_{\mathcal{R}}$, and $\mathbf{N}_{\mathcal{R}}$ be the relevant insertion and projection maps, and let the restrictions of $\mathbf{M}$ to $\mathcal{N}_M$ and to $\mathcal{R}$ be denote by $\mathbf{M}_{\mathcal{N}_M}$ and $\mathbf{M}_{\mathcal{R}}$. We present a supporting lemma, followed by the decomposition.

Lemma 4.10: Let $(\mathbf{C}, \mathbf{A})$ be an $\mathbf{M}$-patterned pair. Then $\sigma(\mathbf{M}_{\mathcal{N}_M}) \cap \sigma(\mathbf{M}_{\mathcal{R}}) = \emptyset$.

Theorem 4.11 (Third Decomposition Theorem): Let $(\mathbf{C}, \mathbf{A})$ be an $\mathbf{M}$-patterned pair. There exists a coordinate transformation $\mathbf{R} : \mathcal{X} \rightarrow \mathcal{X}$ for the state and output spaces ($\mathcal{Y} \simeq \mathcal{X}$), which decouples the system into two subsystems, $(\mathbf{C}_1, \mathbf{A}_1)$ and $(\mathbf{C}_2, \mathbf{A}_2)$, such that

(1) pair $(\mathbf{C}_1, \mathbf{A}_1)$ is $\mathbf{M}_{\mathcal{R}}$-patterned and observable
(2) pair $(\mathbf{C}_2, \mathbf{A}_2)$ is $\mathbf{M}_{\mathcal{N}_M}$-patterned
(3) $\sigma(\mathbf{A}) = \sigma(\mathbf{A}_1) \uplus \sigma(\mathbf{A}_2)$
(4) $\sigma(\mathbf{A}_2)$ is unaffected by patterned measurement feedback in the class $\mathfrak{F}(\mathbf{M}_{\mathcal{R}})$
(5) $\mathbf{C}_2 = 0$ if $\mathcal{N}_M = \mathcal{N}$.

G. Detectability

A system, or equivalently the pair $(\mathbf{C}, \mathbf{A})$, is *detectable* if and only if $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$. If a system is detectable, then it is possible to dynamically estimate any unstable states of the system from the outputs. In the case of a patterned system, we show that the unstable states can be recovered with a patterned static model. First, we have a useful lemma.

Lemma 4.12: Given an $\mathbf{M}$-patterned pair $(\mathbf{C}, \mathbf{A})$, if $\mathcal{N} \subset \mathcal{X}^-(\mathbf{A})$, then $\mathcal{N}_M \subset \mathcal{X}^-(\mathbf{A})$.

By Theorem 4.11 an $\mathbf{M}$-patterned system can be decomposed to separate out an $\mathbf{M}_{\mathcal{R}}$-patterned observable subsystem, denoted by $(\mathbf{C}_1, \mathbf{A}_1)$. Since $\operatorname{Ker} \mathbf{C}_1 = 0$, the matrix C_1 is invertible, and C_1^{-1} is $\mathbf{M}_{\mathcal{R}}$-patterned by Lemma 3.4. Thus, the observable states can be exactly recovered by the patterned static model $x_1 = \mathrm{C}_1^{-1} y_1$. By assumption $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$, which implies $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}_M$ by Lemma 4.12. Equivalently $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{R}$, so by Lemma 2.1(ii), $\sigma(\mathbf{A}_2) \subset \mathbb{C}^-$. Thus, when a patterned system is detectable, all the patterned unobservable states are stable, making it unnecessary to estimate them since they can generally be assumed to be zero.

V. Control Synthesis

With the fundamental patterned system properties established in the previous section, we consider several classic control synthesis questions for patterned systems. The objective is to determine conditions for the existence of a patterned feedback solution. Remarkably, it emerges that the necessary and sufficient conditions for the existence of any feedback solving these synthesis problems are also necessary and sufficient for a patterned feedback.

A. Measurement Feedback

We are given a linear system

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$
$$y(t) = \mathrm{C}x(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. The measurement feedback problem (MFP) is to find a measurement feedback $u(t) = \mathrm{K}y(t)$ such that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. A geometric statement of the problem is to find $\mathbf{K} : \mathcal{Y} \rightarrow \mathcal{U}$

such that $\sigma(\mathbf{A}+\mathbf{BKC}) \subset \mathbb{C}^-$. Stabilizing a system using measurement feedback appears to be only a minor variation of stabilization by full state feedback and one anticipates a similarly elegant solution. Unfortunately such an assumption is mistaken, for the problem of stabilization (and more generally pole-placement) by static measurement feedback is very difficult. Finding testable necessary and sufficient conditions for a general solution has been an open problem in control theory for almost forty years despite considerable effort, and remains unsolved today. The dynamic MFP, i.e. the use of an observer, is generally considerably simpler than the static MFP. However, in the context of distributed systems, it is not evident how a single observer can be distributed to multiple subsystems. Thus, the static MFP is of particular interest for distributed systems. In the geometric framework, the clearest results on the MFP were derived in the seventies.

Theorem 5.1 ([5]): There exists $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\sigma(\mathbf{A}+\mathbf{BKC}) \subset \mathbb{C}^-$ only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$ and $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$.

Soon afterwards, Li [4] described a sufficient condition for MFP.

Theorem 5.2 ([4]): Given a controllable and observable triple $(\mathbf{C}, \mathbf{A}, \mathbf{B})$, there exists $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\sigma(\mathbf{A}+\mathbf{BKC}) \subset \mathbb{C}^-$ if

$$(\mathcal{X}^+(\mathbf{A}) \cap \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle) \cap (\mathcal{X}^+(\mathbf{A}^{\mathrm{T}}) \cap \langle \mathbf{A}^{\mathrm{T}} \mid \operatorname{Ker} \mathbf{B}^{\mathrm{T}} \rangle) = 0. \tag{2}$$

The sufficiency of the first part of the condition, $(\mathcal{X}^+(\mathbf{A}) \cap \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle) = 0$, can be derived by reformulating the problem as finding a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ with the restriction $\operatorname{Ker} \mathbf{F} \supset \langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$ on the feedback matrix. Observe that $\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle$ denotes the smallest $\mathbf{A}$-invariant subspace containing $\operatorname{Ker} \mathbf{C}$. In general, the hierarchy of the subspaces is given by $\langle \mathbf{A} \mid \operatorname{Ker} \mathbf{C} \rangle \supset \operatorname{Ker} \mathbf{C} \supset \mathcal{N}$. In the special case where $\operatorname{Ker} \mathbf{C}$ is $\mathbf{A}$-invariant, however, the subspaces above are all equal. Since Li's sufficient condition requires that the system is observable, it is a given that $\mathcal{X}^+(\mathbf{A}) \cap \mathcal{N} = 0$; therefore, (2) is always met for the special case. Patterned systems are one class of system where Li's sufficient condition is always true. We show that the necessary condition of Theorem 5.1 becomes both a necessary and sufficient condition for patterned systems.

Theorem 5.3: Given an $\mathbf{M}$-patterned triple $(\mathbf{C}, \mathbf{A}, \mathbf{B})$, there exists a patterned measurement feedback $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$, $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$, such that $\sigma(\mathbf{A}+\mathbf{BKC}) \subset \mathbb{C}^-$ if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}$ and $\mathcal{X}^-(\mathbf{A}) \supset \mathcal{N}$.

B. Output Stabilization

We are given a linear system

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$
$$z(t) = \mathrm{D}x(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $z(t) \in \mathbb{R}^q$. The output stabilization problem (OSP) is to find a state feedback $u(t) = \mathrm{F}x(t)$ such that $z(t) \to 0$ as $t \to \infty$. The problem can be restated in more geometric terms as finding a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ that makes the unstable subspace unobservable at the output $z(t)$. Equivalently, $\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \operatorname{Ker} \mathbf{D}$. The solution to the OSP requires the notion of controlled invariant subspaces. A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be *controlled invariant* if there exists a map $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that $(\mathbf{A}+\mathbf{BF})\mathcal{V} \subset \mathcal{V}$. Let $\mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \mathcal{X})$ denote the set of all controlled invariant subspaces in $\mathcal{X}$. Similarly, for any $\mathcal{V} \subset \mathcal{X}$, let $\mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \mathcal{V})$ denote the set of all controlled invariant subspaces in $\mathcal{V}$. It is well-known that OSP is solvable if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$ where $\mathcal{V}^\star := \sup \mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \operatorname{Ker} \mathbf{D})$ [6]. In order to solve the patterned version of the problem, a new subspace is introduced.

Definition 5.1: We define $\mathcal{V}^\diamond$ to be the largest $\mathbf{M}$-decoupling subspace contained in $\mathcal{V}^\star$. That is, $\mathcal{V}^\diamond := \sup \mathfrak{D}^\diamond(\mathbf{M};\, \mathcal{V}^\star)$.

Lemma 5.4: Given an $\mathbf{M}$-patterned triple $(\mathbf{D}, \mathbf{A}, \mathbf{B})$, if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$, then $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M + \mathcal{V}^\diamond$.

Theorem 5.5: Given an $\mathbf{M}$-patterned triple $(\mathbf{D}, \mathbf{A}, \mathbf{B})$, there exists a patterned state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$, such that $\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \operatorname{Ker} \mathbf{D}$ if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$.

C. Disturbance Decoupling

We are given a linear system

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) + \mathrm{H}w(t)$$
$$z(t) = \mathrm{D}x(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^r$ and $z(t) \in \mathbb{R}^q$. The signal $w(t)$ has been introduced to represent a disturbance to the system. Suppose that the disturbance is not directly measured, and furthermore, that we have no information on its characteristics. If the output $z(t)$ is the signal of interest, then one method to compensate for the unknown disturbance is to find a state feedback $u(t) = \mathrm{F}x(t)$ such that $w(t)$ has no influence on $z(t)$ at any time. Then the controlled system is said to be *disturbance decoupled*. Define $\mathcal{H} = \operatorname{Im} \mathrm{H}$. A geometric statement of the disturbance decoupling problem (DDP) is to find a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that $\langle \mathbf{A}+\mathbf{BF} \mid \mathcal{H} \rangle \subset \operatorname{Ker} \mathbf{D}$. It is well-known that DDP is solvable if and only if $\mathcal{V}^\star \supset \mathcal{H}$, where $\mathcal{V}^\star := \sup \mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \operatorname{Ker} \mathbf{D})$. The necessity of this condition is clear, because for any $\mathbf{F}$ such that $\langle \mathbf{A}+\mathbf{BF} \mid \mathcal{H} \rangle \subset \operatorname{Ker} \mathbf{D}$ we have $\langle \mathbf{A}+\mathbf{BF} \mid \mathcal{H} \rangle \in \mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \operatorname{Ker} \mathbf{D})$ by definition and $\mathcal{H} \subset \langle \mathbf{A}+\mathbf{BF} \mid \mathcal{H} \rangle \subset \mathcal{V}^\star$. The condition is also shown to be sufficient by observing that if $\mathcal{V}^\star \supset \mathcal{H}$ then $\langle \mathbf{A}+\mathbf{BF} \mid \mathcal{H} \rangle \subset \mathcal{V}^\star \subset \operatorname{Ker} \mathbf{D}$.

Theorem 5.6: Given an $\mathbf{M}$-patterned triple $(\mathbf{D}, \mathbf{A}, \mathbf{B})$ and a subspace $\mathcal{H} \subset \mathcal{X}$, there exists a patterned state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$, such that $\langle \mathbf{A}+\mathbf{BF} \mid \mathcal{H} \rangle \subset \operatorname{Ker} \mathbf{D}$ if and only if $\mathcal{V}^\star \supset \mathcal{H}$.

D. Regulation

We are given a linear system

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$
$$y(t) = \mathrm{C}x(t)$$
$$z(t) = \mathrm{D}x(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^q$. The output stabilization by measurement feedback problem (OSMFP) is to find a measurement feedback $u(t) = \mathrm{K}y(t)$ such that $z(t) \to 0$ as $t \to \infty$. An equivalent geometric statement of the problem is to find $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ such that $\mathcal{X}^+(\mathbf{A}+\mathbf{BKC}) \subset \operatorname{Ker}\mathbf{D}$. Output stabilization by measurement feedback is a regulation problem. The static feedback case presented above is closely related to the Restricted Regulator Problem (RRP), where the latter is formulated as output stabilization by state feedback with a restriction placed on the form of the state feedback in order to capture the condition that only certain states are measurable.

Problem 5.1 (Restricted Regulator Problem (RRP)): Given a subspace $\mathcal{L} \subset \mathcal{X}$ with $\mathbf{A}\mathcal{L} \subset \mathcal{L}$, find a state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ such that

$$\operatorname{Ker}\mathbf{F} \supset \mathcal{L}$$
$$\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \operatorname{Ker}\mathbf{D}.$$

The subspace $\mathcal{L}$ provides a geometric way to capture the information structure in the problem. This is because the condition $\operatorname{Ker}\mathbf{F} \supset \mathcal{L}$ effectively characterizes which states can be employed by the state feedback. A key condition in the statement of the RRP is that $\mathcal{L}$ must be an $\mathbf{A}$-invariant subspace; this condition makes the problem tractable. The choice of $\mathcal{L}$ can be understood a little better by decomposing the dynamics of the system. Since $\mathcal{L}$ is $\mathbf{A}$-invariant there exists a coordinate transformation $\mathbf{R} : \mathcal{X} \to \mathcal{X}$, such that in the new coordinates the matrix pair (A, B) becomes

$$\left(\mathrm{R}^{-1}\mathrm{A}\mathrm{R}, \mathrm{R}^{-1}\mathrm{B}\right) = \left(\begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 \\ 0 & \mathrm{A}_3 \end{bmatrix}, \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} \right).$$

This separates the dynamics on and off $\mathcal{L}$. The condition $\operatorname{Ker}\mathbf{F} \supset \mathcal{L}$ implies that in new coordinates $\tilde{\mathrm{F}} = \begin{bmatrix} 0 & \mathrm{F}_2 \end{bmatrix}$, and

$$\tilde{\mathrm{A}} + \tilde{\mathrm{B}}\tilde{\mathrm{F}} = \begin{bmatrix} \mathrm{A}_1 & \mathrm{A}_2 + \mathrm{B}_1\mathrm{F}_2 \\ 0 & \mathrm{A}_3 + \mathrm{B}_2\mathrm{F}_2 \end{bmatrix}.$$

The idea is to choose $\mathcal{L}$ such that all the states off $\mathcal{L}$, or at least estimates of them, are available to be used as feedback. Then the dynamics of the available states can be controlled separately from those on $\mathcal{L}$. If an observer is employed, one could use $\mathcal{N}$, the unobservable subspace, as $\mathcal{L}$ since it is always $\mathbf{A}$-invariant. However, OSMFP calls for only static measurement feedback, rather than an observer. To obtain a solution, a necessary criterion is $\mathcal{L} \supset \operatorname{Ker}\mathbf{C}$.

There is a special case, $\operatorname{Ker}\mathbf{C} = \mathcal{N}$, corresponding to all the observable states being recoverable by a simple transformation of the measurements. Then $\operatorname{Ker}\mathbf{C}$ is $\mathbf{A}$-invariant and could be used as $\mathcal{L}$, which implies that the RRP is exactly equivalent to the original Output Stabilization by Measurement Feedback Problem. In the case where $\operatorname{Ker}\mathbf{C} \neq \mathcal{N}$, $\operatorname{Ker}\mathbf{C}$ is not $\mathbf{A}$-invariant and a larger subspace must be chosen for $\mathcal{L}$, generally the smallest $\mathbf{A}$-invariant subspace containing $\operatorname{Ker}\mathbf{C}$, which is $\langle \mathbf{A} \mid \operatorname{Ker}\mathbf{C} \rangle$. The subtle difficulty is that now the RRP is more stringent than the original problem, and the solution to the RRP represents only sufficient, but not necessary, conditions for output stabilization by measurement feedback. To find sufficient and necessary conditions is not generally solved at this time. Ultimately it is the same static Measurement Feedback Problem described previously, and it is a longstanding open problem in control.

The general solution to the RRP relies on finding a maximal element, denoted by $\mathcal{V}^{\mathrm{M}}$, of a rather structurally complex family of subspaces (refer to [6]). There exists a simpler condition that applies under the sufficient condition that $\mathcal{V}^{\mathrm{M}} = \mathcal{V}^\star$, where $\mathcal{V}^\star := \sup \mathfrak{I}(\mathbf{A}, \mathbf{B}; \operatorname{Ker}\mathbf{D})$.

Corollary 5.7 ([6]): Suppose $\mathbf{A}(\mathcal{L} \cap \operatorname{Ker}\mathbf{D}) \subset \operatorname{Ker}\mathbf{D}$. Then the RRP is solvable if and only if $\mathcal{X}^+(\mathbf{A}) \cap \mathcal{L} \subset \operatorname{Ker}\mathbf{D}$ and $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$.

Now we return to the problem for patterned systems. Given an $\mathbf{M}$-patterned triple $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ and an output map $\mathbf{D} : \mathcal{X} \to \mathcal{Z}$, $\mathbf{D} \in \mathfrak{F}(\mathbf{M})$, the OSMP problem is to find a patterned measurement feedback $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$, $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$, such that $\mathcal{X}^+(\mathbf{A}+\mathbf{BKC}) \subset \operatorname{Ker}\mathbf{D}$. For patterned systems, the appropriate $\mathcal{L}$ to choose is the patterned unobservable subspace $\mathcal{N}_M$. It is $\mathbf{A}$-invariant by Fact 3.5, so we can show that solving the patterned OSMF is exactly equivalent to solving the following restricted regulator problem.

Theorem 5.8: Given an $\mathbf{M}$-patterned pair $(\mathbf{A}, \mathbf{B})$, and an output map $\mathbf{D} : \mathcal{X} \to \mathcal{Z}$, $\mathbf{D} \in \mathfrak{F}(\mathbf{M})$, there exists a patterned state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$, such that $\operatorname{Ker}\mathbf{F} \supset \mathcal{N}_M$ and $\mathcal{X}^+(\mathbf{A}+\mathbf{BF}) \subset \operatorname{Ker}\mathbf{D}$ if and only if $\mathcal{X}^+(\mathbf{A}) \cap \mathcal{N}_M \subset \operatorname{Ker}\mathbf{D}$ and $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$, where $\mathcal{V}^\star = \sup \mathfrak{I}(\mathbf{A}, \mathbf{B}; \operatorname{Ker}\mathbf{D})$.

Assume that the conditions to solve the Patterned RRP are met for a given system. Then there exists a patterned state feedback $\mathbf{F} : \mathcal{X} \to \mathcal{U}$, $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$, such that $\operatorname{Ker}\mathbf{F} \supset \mathcal{N}_M \supset \operatorname{Ker}\mathbf{C}$. It follows that there exists a measurement feedback $\mathbf{K} : \mathcal{Y} \to \mathcal{U}$ that solves the equation $\mathbf{KC} = \mathbf{F}$. Furthermore, $\mathbf{K} \in \mathfrak{F}(\mathbf{M})$, and we have that $\mathcal{X}^+(\mathbf{A}+\mathbf{BKC}) \subset \operatorname{Ker}\mathbf{D}$. Conversely, if $\operatorname{Ker}\mathbf{F} \not\supseteq \operatorname{Ker}\mathbf{C}$ then there exists no solution $\mathbf{K}$ to $\mathbf{KC} = \mathbf{F}$. And, if $\mathbf{F} \notin \mathfrak{F}(\mathbf{M})$ then any solution $\mathbf{K}$ would not be a member of $\mathfrak{F}(\mathbf{M})$. We draw the following conclusion.

Corollary 5.9: There exists a solution to the Patterned OSMFP if and only if there exists a solution to the Patterned RRP.

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