Patterned Linear Systems: Rings, Chains, and Trees

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***Abstract*— In a first paper we studied system theoretic properties of patterned systems and solved classical control synthesis problems with the added requirement to preserve the system pattern. In this second paper we study canonical patterns: rings, chains, and trees, and we give examples drawn from multiagent systems, cellular chemistry, and control of diffusion processes.**

I. Introduction

In the first paper [4] of a two-part series we introduced and studied a new class of linear control systems called patterned systems. Mathematically, this class has the property that the state, input and output transformations of the linear state space model are all functions of a common base transformation. The motivation for studying such systems is their interpretation as a collection of identical subsystems with a pattern of interaction between subsystems that is imprinted by the base transformation.

The control of systems made up of identical subsystems connected in a pattern appears and reappears in the control literature as researchers have come across real applications with notable structural features. Most commonly the pattern is spatial in nature, consequently these systems are often referred to as *spatially interconnected*. A rich source of applications is systems described by a lumped approximation of partial differential equations such as smart materials. As [1] observed, advancements in the design of Micro-Electro-Mechanical (MEM) parts suggest that controlling such systems by means of an array of identical miniature sensors and actuators is an increasingly realistic model. A mature application of a lumped approximation of a PDE's is the cross-directional control of sheet and film processes, such as paper-making, steel rolling and plastic extrusion. In the multi-agent area, applications include evenly spaced convoys and geometric pattern formation. In the field of large-scale systems, decentralized control of systems with symmetrically interconnected identical subsystems has applications in multimachine power systems and parallel networks of units in a plant, such as pumps or reactors. Finally, building up complex systems by repetition of simple components has useful parallels to biological systems.

Many of the examples listed above are complex systems that are made of a large number of simple subsystems with simple patterns of interaction. This suggests that a useful starting point is to examine the most elementary patterns. Examples of some elementary patterns of identical subsystems are depicted in Fig. 1, where each circle represents a subsystem, and arrows represent interactions between subsystems. These patterns are notable because they have physical interpretations. In this paper we consider ring, chain, and tree patterns useful in engineering design. Then we give examples of control of such systems, using the theory developed in the companion paper [4]. These examples include the most typical application areas in multiagent systems, cellular chemistry, and control of PDE's.

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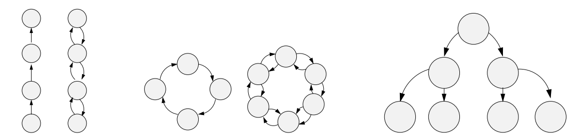

(a) (b) (c)

Fig. 1. Patterns: (a) chains, (b) rings, and (c) trees.

II. Patterns

A. Rings

The topology of a ring system consists of a closed chain of identical subsystems that interact in a repeated pattern. It is a common, simple pattern found in natural and man-made systems. Mathematically, ring systems can be referred to as circulant systems, because the matrices in a state space model of a ring have a circulant, or more generally block circulant, form.

1) Circulants: Circulant matrices are square matrices of the form

$$\mathrm{C} = \mathrm{circ}\,(c_1, c_2, \ldots, c_n) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & & c_{n-1} \\ \vdots & & \ddots & \\ c_2 & c_3 & & c_1 \end{bmatrix}. \tag{1}$$

The set of circulant $n \times n$ matrices is denoted by $\mathfrak{C}_n$. We now explore some additional properties of circulant matrices. First, we define the *shift operator* as a circulant permutation matrix given by $\Pi := \mathrm{circ}\,(0, 1, 0, \ldots, 0)$. Every circulant matrix can be expressed as a function of Π. From the general form of a circulant matrix shown in (1), it is easily seen that C is given by

$$\mathrm{C} = c_1 \mathrm{I} + c_2 \Pi + \cdots + c_n \Pi^{n-1}.$$

Thus, circulant matrices form the patterned class $\mathfrak{F}(\Pi)$, and we can alternately refer to them as Π-patterned matrices.

2) Symmetric Circulants: Consider now the subclass of *symmetric circulant* matrices. A circulant matrix, C, is termed *symmetric* if $\mathrm{C}^{\mathrm{T}} = \mathrm{C}$. Then C has the form

$$\begin{bmatrix} c_0 & c_1 & & c_2 & c_1 \\ c_1 & c_0 & & & c_2 \\ & & \ddots & & \\ c_2 & & & c_0 & c_1 \\ c_1 & c_2 & & c_1 & c_0 \end{bmatrix}.$$

An appropriate base matrix for the symmetric circulant class is any matrix whose eigenvectors are the Fourier vectors (hence a circulant matrix) and whose spectrum has the form $\{\lambda_1, \lambda_2, \ldots, \lambda_2, \lambda_1\}$, where eigenvalues $\lambda_1, \ldots, \lambda_{\frac{n+1}{2}}$ are distinct for odd n, or $\lambda_1, \ldots, \lambda_{\frac{n}{2}+1}$ are distinct for even n. A base matrix with these properties is

$$\Sigma = \begin{bmatrix} 0 & 1 & & 0 & 1 \\ 1 & 0 & \ddots & & 0 \\ & \ddots & \ddots & 1 & \\ 0 & & 1 & 0 & 1 \\ 1 & 0 & & 1 & 0 \end{bmatrix},$$

where $\Sigma = \Pi + \Pi^{\mathrm{T}}$. Clearly, Σ is itself symmetric circulant, and we find that its eigenvalues are $\{2, \omega + \omega^{n-1}, \omega^2 + \omega^{n-2}, \ldots, \omega^2 + \omega^{n-2}, \omega + \omega^{n-1}\}$, as expected.

3) Factor Circulants: Factor circulants are not a sub-class of circulants; they are a generalization of the circulant form. Consider the matrix

$$\begin{bmatrix} c_0 & c_1 & & c_{n-2} & c_{n-1} \\ \varphi c_{n-1} & c_0 & & & c_{n-2} \\ & & \ddots & & \\ \varphi c_2 & & & c_0 & c_1 \\ \varphi c_1 & \varphi c_2 & & \varphi c_{n-1} & c_0 \end{bmatrix}.$$

It is called a φ*-circulant* or, more broadly, a *factor circulant* matrix. Factor circulants are a sub-class of Toeplitz matrices, and given some φ it is easily observed that every φ-circulant matrix is a function of the base matrix

$$\Pi_{(\varphi)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ \varphi & 0 & 0 & \cdots & 0 \end{bmatrix},$$

so factor circulants are patterned matrices. A notable class of factor circulants are the $\Pi_{(-1)}$-patterned or *skew-circulants* matrices.

4) Hierarchies of Circulants: Block patterned systems, where each subsystem has its own multi-state internal model, generally do not fit directly into the framework that we have presented, and they are an area of future research. However, certain hierarchies of patterns can be modeled as simple patterned systems. One example that has been identified is a hierarchy of circulant systems. For example, consider three subsystems that are connected in a circulant pattern, and where the internal 2-state model of each subsystem is itself circulant. This can be viewed as a two layer hierarchy of circulant systems. The general model of this hierarchy is given by

$$\dot{x} = \mathrm{A}x = \left[\begin{array}{cc|cc|cc} a_0 & a_1 & b_0 & b_1 & c_0 & c_1 \\ a_1 & a_0 & b_1 & b_0 & c_1 & c_0 \\ \hline c_0 & c_1 & a_0 & a_1 & b_0 & b_1 \\ c_1 & c_0 & a_1 & a_0 & b_1 & b_0 \\ \hline b_0 & b_1 & c_0 & c_1 & a_0 & a_1 \\ b_1 & b_0 & c_1 & c_0 & a_1 & a_0 \end{array}\right] x,$$

where A is a block circulant matrix of 2×2 circulant blocks. The base matrix for this class is the Kronecker tensor product of two shift operators given by

$$\mathrm{H} = \Pi_3 \otimes \Pi_2 = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then we have $\mathrm{A} = a_0\mathrm{I} + b_0\mathrm{H} + c_0\mathrm{H}^2 + a_1\mathrm{H}^3 + b_1\mathrm{H}^4 + c_1\mathrm{H}^5$.

B. Chains

Open chains of identical subsystems are modeled by Toeplitz matrices. Numerous applications of open chain systems exist, including vehicle convoys, mass transit lines, serpentine manipulators, cross-directional control of continuous processes such as papermaking, and lumped approximations of PDE's. Despite the simple structure of open chains, proving the existence of Toeplitz controllers to solve control problems for general Toeplitz systems is difficult because Toeplitz matrices do not form a patterned class with a single base pattern. However, if a long chain can be reasonably approximated as having infinite length, then certain control problems are actually simplified. For example, Brockett and Willems [2] showed that the optimal control of infinite Toeplitz systems has an infinite Toeplitz form. While the optimal control of finite-dimensional Toeplitz systems is not generally Toeplitz, there is a method for arriving at the optimal control of symmetric Toeplitz systems through a conversion to a circulant form [7]. An example in the context of pole placement is given below. Although Toeplitz systems do not form a patterned class, the special case of upper (or lower) triangular Toeplitz matrices is patterned, which corresponds to open chains with the property that interconnections between subsystems are in one direction only.

1) Uni-directional Chains: A uni-directional chain describes a pattern where each subsystem only interacts with neigbouring systems ahead of it (or behind it) in a chain. Figure 2 shows an example of a chain of length four and all the possible levels of interaction that might be present in one direction. Assuming subsystems have one state, an arrow from subsystem i to subsytem j denotes that the dynamics of j are impacted by the state of i. When all these levels of interaction are summed together we obtain the general

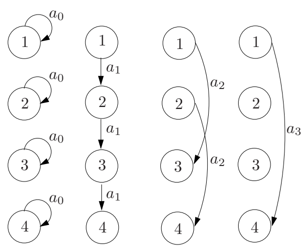

Fig. 2. Levels of interaction for a chain of four identical subsystems

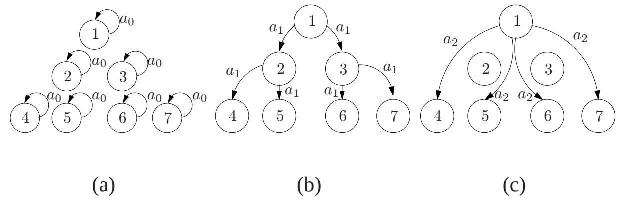

Fig. 3. Levels of interaction for a three level symmetric tree of identical subsystems

matrix form of a triangular Toeplitz matrix

$$\mathrm{Z} = \begin{bmatrix} z_0 & & & 0 & 0 \\ z_1 & z_0 & & 0 & 0 \\ \vdots & \ddots & \ddots & & \\ z_{n-2} & & \ddots & z_0 & \\ z_{n-1} & z_{n-2} & \cdots & z_1 & z_0 \end{bmatrix}, \qquad (2)$$

and by inspection it is easily seen that triangular Toeplitz matrices are actually a sub-class of factor circulants, where the factor is zero.

Now, let $\mathcal{X}$ be an n-dimensional vector space and let $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ be some transformation. If there exists k such that $\mathbf{T}^k = 0$, then $\mathbf{T}$ is called *nilpotent*. Consider the transformation represented in the natural basis by

$$\mathrm{N} = \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \; \mathrm{N} \in \mathbb{R}^{n \times n}.$$

It is easily shown that $\mathrm{N}^n = 0$, so N is nilpotent and we call N the *fundamental nilpotent matrix*. It is easily shown that every lower triangular Toeplitz matrix is a function of N. Examining the general form (2), Z is given by

$$\mathrm{Z} = z_0\mathrm{I} + z_1\mathrm{N} + z_2\mathrm{N}^2 + \cdots + z_{n-1}\mathrm{N}^{n-1}.$$

The matrix N is already in Jordan form (transposed), and its properties reveal some interesting limitations of uni-directional chains. Since N consists of a single Jordan block, its spectrum is a set of n zeros and it has only one eigenvector. Thus, an N-patterned spectrum consists of n identical real eigenvalues, and given a triangular Toeplitz system, all the system poles must be moved together if the pattern is to be preserved. This implies that triangular Toeplitz systems are either completely controllable or not patterned controllable at all.

Consider a slight modification to the triangular Toeplitz model to allow the first subsystem in the chain, referred to as the leader subsystem, to have different local dynamics and interactions from the rest of the chain. We define the following base pattern

$$\mathrm{N_L} = \begin{bmatrix} 1 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \; \mathrm{N_L} \in \mathbb{R}^{5 \times 5}.$$

Let $n = 5$. Then an arbitrary polynomial of $\mathrm{N_L}$ has the form

$$\begin{bmatrix} \star & & & 0 & 0 \\ \star & z_0 & & 0 & 0 \\ \star & z_1 & z_0 & & \\ \star & z_2 & z_1 & z_0 & \\ z_4 & z_3 & z_2 & z_1 & z_0 \end{bmatrix}.$$

The elements z_0 through z_4 can be set arbitrarily, but the elements denoted by $\star$ are dependent on the choice of z_i. The pattern implies a trade-off between the interactions of the leader subsystem and those of its followers. In this way, one can propose a number of base matrices that are variations on the fundamental nilpotent matrix and which impose a uni-directional chain pattern on a system.

C. Trees

Consider the typical structure of an organizational hierarchy, such as a military chain of command. At the top is a single individual leader, and the leader has some number of direct reports. Then each of these sub-leaders in turn has a number of individuals reporting to them, and so forth down the chain of command. Graphically such a structure resembles a tree. Tree structures appear to be less studied in the control literature than rings or chains, thus the class seems particularly fruitful for further investigation.

1) Uni-directional Trees: A uni-directional tree describes a pattern where each subsystem only interacts with the layers above (or below it) in a hierarchy. Two examples of trees with identical subsystems are depicted in Figures 3 and 4, along with all the possible levels of interaction that might be present in one direction. Consider first the system in Figure 3. Assuming subsystems each have one state, when the levels are summed together we obtain the system model

$$\dot{x} = \mathrm{A}_1 x = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_0 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & a_0 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 & a_0 & 0 & 0 \\ a_2 & 0 & a_1 & 0 & 0 & a_0 & 0 \\ a_2 & 0 & a_1 & 0 & 0 & 0 & a_0 \end{bmatrix} x.$$

The matrix A_1 is in the family of triangular matrices, but we are not aware of an established name for this particular structure. A suggestion is to call it the 1-2-2 tree class. All 1-2-2 trees can be generated by the base matrix

$$\mathrm{H}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

such that

$$\mathrm{A}_1 = a_0\mathrm{I} + a_1\mathrm{H}_1 + a_2\mathrm{H}_1^2 .$$

Next we consider the system in Figure 4, which models a more lopsided tree. When the levels are summed together we obtain the system model

$$\dot{x} = \mathrm{A}_2 x = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 & a_0 & 0 & 0 & 0 \\ a_2 & 0 & a_1 & 0 & 0 & a_0 & 0 & 0 \\ a_3 & 0 & a_2 & 0 & 0 & a_1 & a_0 & 0 \\ a_3 & 0 & a_2 & 0 & 0 & a_1 & 0 & a_0 \end{bmatrix} x.$$

Despite the lack of symmetry in the layers of the tree, there exists a base matrix that generates this class of matrices. Define

$$\mathrm{H}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

then

$$\mathrm{A}_2 = a_0\mathrm{I} + a_1\mathrm{H}_2 + a_2\mathrm{H}_2^2 + a_3\mathrm{H}_2^3 .$$

As with uni-directional chains, variations on these tree structures can be experimented with to allow some subsystems to be different from others. Bi-directional trees are not generally patterned, consequently they present a more difficult problem, in analogy with bi-directional chains.

III. Illustrative Examples

We present several examples of patterned systems with associated stabilization problems. These basic examples are not at the level of true applications; rather they are intended to convey the breadth of research areas that touch on patterned systems and to illustrate the meaning of our theoretical results.

A. Multi-agent Consensus

A multi-agent system consists of several subsystems that act autonomously, and an extensively studied multi-agent

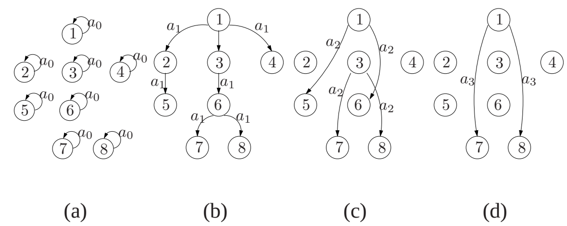

Fig. 4. Levels of interaction for a four level asymmetric tree of identical subsystems

objective is consensus. The consensus problem is an output stabilization by measurement feedback problem. Consensus can be achieved if there exists a measurement feedback controller $u = \mathrm{K}y$, such that $z \to 0$ as $t \to \infty$, where variable z defines the global consensus objective. A general K assumes full communication between agents. It is desirable to impose structural constraints on K to limit communication. We illustrate with an example.

We are given n identical robots and the global objective of rendezvous. Suppose the measurements taken by each robot must be identical up to indices, and identical local controllers (up to indices) must be distributed. What measurements are required for local controllers to exist? The robots are modeled as integrators: $\dot{x}_i = u_i$ for $i = 1, \dots, n$. Combine the n robot subsystems together to obtain

$$\dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} u.$$

We restrict the measurement matrix C to take on a circulant pattern, so that each robot takes the same measurements up to indices, giving

$$y = \mathrm{C}x = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & & c_{n-2} \\ \vdots & & \ddots & \\ c_1 & c_2 & & c_0 \end{bmatrix} x.$$

but we do not specify C up front. Rendezvous is achieved when all the robots converge to a common position, which can also be expressed as the relative positions of all robots stabilizing to zero. A suitable global objective model is

$$z = \mathrm{D}x = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x.$$

Thus we have a circulant system where $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\} \in \mathfrak{F}(\Pi)$. The control problem is to find $u = \mathrm{K}y$, $\mathrm{K} \in \mathfrak{F}(\Pi)$ such that $z \to 0$ as $t \to \infty$. By the results of [4], there exists a solution to this Patterned Output Stabilization by Measurement Feedback Problem if and only if there exists a solution to the Patterned Restricted Regulator Problem.

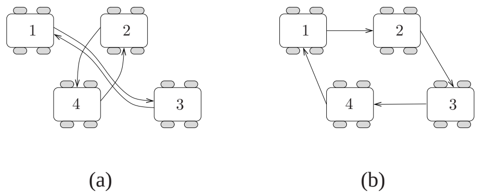

Fig. 5. Graphs of different sensor formations for multi-robot consensus: (a) unconnected graph, and (b) connected graph

A solution to the Patterned RRP exists if and only if $\mathcal{X}^+(\mathrm{A}) \cap \mathcal{N}_M \subset \mathrm{Ker}\,\mathrm{D}$ and $\mathcal{X}^+(\mathrm{A}) \subset \mathcal{C} + \mathcal{V}^\star$. For the given system, we have $\mathcal{X}^+(\mathrm{A}) = \mathbb{R}^n$, $\mathcal{N}_M = \mathrm{Ker}\,\mathrm{C}$, (C is still undefined), $\mathrm{Ker}\,\mathrm{D} = \mathrm{span}\,\{(1,1,\ldots,1)\}$, $\mathcal{C} = \mathrm{Im}\,\mathrm{B} = \mathbb{R}^n$, and $\mathcal{V}^* = \mathrm{Ker}\,\mathrm{D}$. Then a suitable controller will exist provided that

$$\begin{aligned} &\mathbb{R}^n \cap \mathcal{N}_M \subset \mathrm{span}\,\{(1,1,\ldots,1)\}, \\ \text{and} \quad &\mathbb{R}^n \subset \mathbb{R}^n + \mathrm{span}\,\{(1,1,\ldots,1)\}. \end{aligned}$$

Clearly, the second condition holds. The first condition imposes constraints on $\mathcal{N}_M$. If we choose the measurement model

$$y = \mathrm{C}x = \begin{bmatrix} -1 & 1 & & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x,$$

then $\mathcal{N}_M = (1,1,\ldots,1)$ and the first condition also holds. In this case, we conclude that a circulant controller to achieve consensus exists. One solution would be the decentralized controller $u = y$.

Suppose we choose instead a measurement model where a robot measures its relative distance to the robot two places ahead, given by

$$y = \mathrm{C}x = \begin{bmatrix} -1 & 0 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 0 & -1 & \cdots & 0 & 0 \\ & & \vdots & & & \vdots & \\ 0 & 1 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} x.$$

Then $\mathcal{N}_M = (1,1,\ldots,1)$ if n is odd, but $\mathcal{N}_M = \mathrm{span}\,\{(1,0,1,0,\ldots,1,0),(0,1,0,1,\ldots,0,1)\}$ if n is even. In the first scenario, a controller exists; whereas, in the second, the conditions of the Patterned RRP are not met. The conclusions from this example can also be interpreted in terms of graph theory results on consensus.

B. Cellular Chemistry

Turing [6] proposed that, for the purposes of studying cellular chemical reactions, one simple and illustrative arrangement of cells is a ring. Given a ring of n identical cells, let x_i denote the concentration of chemical X in cell i. Turing's model is given by

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \alpha x_i(t) + \beta u_i(t) + \kappa\left(x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)\right) \\ &= \kappa\left(x_{i+1}(t) + (\frac{\alpha}{\kappa} - 2)x_i(t) + x_{i-1}(t)\right) + \beta u_i(t), \end{aligned}$$

for $i = 1,\ldots,n$. Let $\alpha = 2$, $\beta = -1$ and $\kappa = 0.5$. Consider the concentration of chemical U to be a controlled input in each cell. Then the cellular ring system has the circulant state space model

$$\dot{x}_i(t) = \begin{bmatrix} 1 & 0.5 & 0 & \cdots & 0.5 \\ 0.5 & 1 & 0.5 & & 0 \\ & & \vdots & & \\ 0.5 & 0 & 0 & \cdots & 1 \end{bmatrix} x_i(t) - Iu_i(t).$$

Observe that this system is unstable. We assume that the cell concentrations are measurable, and the objective is to find a state feedback controller $u(t) = \mathrm{F}x(t)$ that brings the concentrations into equilibrium. We can express this objective as $z(t) \to 0$ as $t \to \infty$, where

$$z(t) = \mathrm{D}x(t) = \begin{bmatrix} -1 & 1 & & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x(t), \ \mathrm{D} \in \mathfrak{C}.$$

This is the Patterned OSP, which, by the results of [4], is solvable if and only if $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} + \mathcal{V}^\star$. Since $\mathcal{C} = \mathrm{Im}\,\mathrm{B} = \mathbb{R}^n$, the problem has a solution.

C. Discretized Partial Differential Equations

We consider a simple example of how a symmetric Toeplitz system can be converted to a patterned circulant system for the purposes of computing a controller. Let $x(t,d)$ be a continuous function of two variables, defined over an interval $0 < d < l$. A lumped approximation to the multi-dimensional function is a set of $n+1$ continuous functions $x_0(t), x_1(t), \ldots, x_n(t)$ that sample $x(t,d)$ at regular spacings along the interval d. Let the space between sample functions be $h := \frac{l}{n}$, then $x_i(t) = x(t, ih)$. If the partial derivatives in time and space are appropriately approximated (using finite differences), one obtains a discretization of the PDE. For example, consider the diffusion process

$$\frac{\partial x(t,d)}{\partial t} = k \frac{\partial^2 x(t,d)}{\partial d^2}, \tag{3}$$

where x is the process variable and d is a spatial variable. When the process variable is temperature, this PDE is called the heat equation. Assume the model holds over an interval $0 < d < l$, and assume boundary conditions on the process of $x(t,0) = x(t,l) = 0$ for all time. Suppose we control the diffusion process by adding $n-1$ control inputs that act on the derivative of the process variable and that are spaced evenly along the spatial extent. There are also sensors of the process variable at each controller location. Then, using the finite difference method, we obtained the discretized model

$$\frac{dx_i(t)}{dt} = \frac{k}{h^2}\left(x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)\right) + u_i(t),$$

$i = 1, \ldots, n-1$ and the boundary conditions $x_0(t) = 0$ and $x_n(t) = 0$ for all time.

In matrix form this system has A and B matrices which are symmetric and Toeplitz. Brockett and Willems [3] showed that one way to find a near Toeplitz state feedback F such that $u(t) = Fx(t)$ achieves a desired trajectory $x(t)$ is to model the Toeplitz system by a larger circulant system. A circulant solution can be easily reduced to a solution for the Toeplitz system. The expanded circulant system is constructed by creating a mirror image of the original system and then connecting it to the original system at the boundary points. Consider the expanded circulant system

$$\dot{x} = \frac{k}{h^2}\mathrm{circ}\,(-2, 1, 0, \cdots, 0, 1)x + Iu$$

where $x \in \mathbb{R}^{2n-1}$ and $u \in \mathbb{R}^{2n-1}$. Let the initial states $x_i(0)$ in the extended system equal the initial states $x_i(0)$ in the original Toeplitz system for $i = 1, \ldots, n-1$. Further, assume that $x_{2n-i}(0) = -x_i(0)$ for $i = 1, \ldots, n-1$. Then we have the following result.

Proposition 3.1 ([3]): If (a) $u_0(t) = u_n(t) = 0$; (b) $u_i(t)$ in the extended system is applied as $u_i(t)$ in the original system for $i = 0, 1, \ldots, n$; and (c) $u_{2n-i}(t) = -u_i(t)$ for $i = 1, \ldots, n-1$, then $x_0(t) = x_n(t) = 0$, $x_i(t)$ is the same for both systems for $i = 1, \ldots, n-1$, and $x_{2n-i}(t) = -x_i(t)$, for $i = 1, \ldots, n-1$.

Now we apply the method to a pole placement problem for the diffusion process (3) with $k = 2$ over the interval $0 < d < 4$. Let the spacing between lumped approximations along the interval be 1, then $n = 4$. This discretizes the PDE into three differential equations. Assuming that 3 discrete controllers are spaced evenly along the interval, the equations are given by

$$\dot{x}(t) = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t),$$

with assumed boundary conditions $x_0(t) = x_4(t) = 0$ for all time. The poles of the system, $\{-1.17, -4, -6.83\}$, are already stable but it is desirable to place the poles further into the left half plane in order to increase the speed at which the process variable converges to the boundary conditions. Suppose our objective is to find a feedback $u(t) = \mathrm{F}x(t)$ to place the poles at $\{-8, -10, -10\}$. Using the state space extension method, we create the symmetric circulant 8×8 system

$$\dot{x}_e(t) = \mathrm{circ}\,(-4, 2, 0, 0, 0, 0, 0, 2)x_e(t) + Iu_e(t).$$

Note that the poles of the extended systems are $\{0, -1.17, -1.17, -4, -4, -6.83, -6.83, -8\}$, which consists of the spectrum of the original system, duplicated once, and two additional poles at 0 and -8. These additional poles are immaterial, because they will disappear when we convert back to the original system.

It is known that symmetric circulant systems are Σ-patterned systems; therefore, by the results of [4], there exists a symmetric circulant feedback $\mathrm{F}_e \in \mathfrak{F}(\Sigma)$ to place the poles in any Σ-patterned spectrum if and only if $\mathcal{X}^+(\mathrm{A}_e) \subset \mathcal{C}$. The controllable subspace of the patterned system is $\mathrm{Im}\,\mathrm{B}_e$, so clearly $\mathcal{C} = \mathcal{X}$ and the condition $\mathcal{X}^+(\mathrm{A}_e) \subset \mathcal{C}$ holds. Let

$$\mathrm{F}_e := \mathrm{circ}\,(-4, -0.65, 1.5, 0.65, 1, 0.65, 1.5, -0.65).$$

It can be shown that

$$\begin{aligned}\mathrm{F}_e = &-10\mathrm{I} + 0.058\Sigma + 0.055\Sigma^2 + 0.098\Sigma^3 + 0.084\Sigma^4 \\ &+ 0.13\Sigma^5 + 0.069\Sigma^6 - 0.0082\Sigma^7,\end{aligned}$$

confirming that $\mathrm{F}_e \in \mathfrak{F}(\Sigma)$. We obtain $\sigma(\mathrm{A}_e + \mathrm{B}_e\mathrm{F}_e) = \{0, -8, -8, -10, -10, -10, -10, -8\}$, which meets our pole placement criteria. Since we have found a symmetric circulant solution to the extended problem, we will meet the conditions of Proposition 3.1. The corresponding solution F to the original Toeplitz system is

$$\mathrm{F} = \begin{bmatrix} -5.5 & -1.29 & 0.5 \\ -1.29 & -5 & -1.29 \\ 0.5 & -1.29 & -5.5 \end{bmatrix}.$$

Then the closed loop system becomes

$$\dot{x}(t) = \begin{bmatrix} -9.5 & 0.71 & 0.5 \\ 0.71 & -9 & 0.71 \\ 0.5 & 0.71 & -9.5 \end{bmatrix} x(t),$$

where $\sigma(\mathrm{A}+\mathrm{BF}) = \{-8, -10, -10\}$, as desired. Notice that the solution F that we have found is not exactly Toeplitz, but near Toeplitz, as desired.

IV. Conclusion

We have introduced a new class of linear control systems called patterned linear systems. The contribution is in uniting multiple patterns under the umbrella of a general theory. Examination of applications is left to future research; however, the range of practical applications can be significantly enlarged if the theory is extended to block patterned systems. Other future research ideas include linear combinations of patterned systems and infinite-dimensional systems.

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