Reach Control Problem with Disturbance Rejection*

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***Abstract*— This paper begins by presenting an extension of classical linear regulator theory to affine systems and exosystems. The extension also allows for stable exosystem dynamics. This extension then provides the basis to develop a framework for the reach control problem with disturbances.**

I. INTRODUCTION

A disturbance rejection problem for a linear affine system defined on a full-dimensional polytope is studied. The objective is to design a controller that asymptotically rejects a disturbance while also steering trajectories of the affine system to a prespecified facet of the polytope. This work is mainly motivated by previous results in [7], [8], [10] that address the Reach Control Problem (RCP) without disturbances. The method for addressing disturbance rejection from the view of RCP is motivated by [5], [6].

The main issue that RCP addresses that a traditional control design does not is safety constraints. The classical view is to first design a stabilizing controller and then check afterwards to see if safety constraints are met. RCP turns this process on it's head. An RCP-based design puts safety at the front end of the control design, while still being able to achieve other control objectives. Much has already been written about RCP [7], [8], [10], [3], [1], [2], [4], [9], [11] and a number of control classes having been discovered to solve the problem [7], [3], [1], [4]. It is time to explore some capabilities of RCP outside of it's standard formulation.

In this paper we introduce for the first time the notion of a measurement for RCP and the presence of a disturbance. The main contribution is resolving how to incorporate these changes in the standard RCP formulation. To help address these additions we draw from existing literature in the area of disturbance rejection by measurement feedback, more commonly known as *regulator theory* [12], [13], [5], [14]. The main challenge is that regulator theory builds up from stabilization, whereas RCP comprises a radically different control objective regarding non-steady-state behaviour. Finally, we remark that some proofs of the paper are suppressed due to space limitations.

II. REACH CONTROL PROBLEM

In this section we give the highlights of RCP; the reader is referred to the extensive literature for further background, examples, and motivation.

*This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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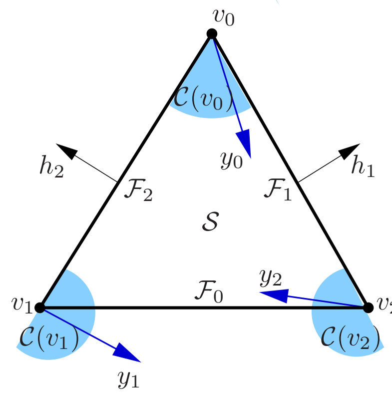

Fig. 1. Notation for the reach control problem.

Suppose we are given a state space which is a full-dimensional polyhedron $\mathcal{P}$ and suppose we have a triangulation $\mathbb{T}$ of $\mathcal{P}$. Consider a specific full-dimensional simplex $\mathcal{S} := \text{co}\{v_0, \ldots, v_n\}$ of $\mathbb{T}$. Let it's vertex set be $V := \{v_0, \ldots, v_n\}$ and its facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$. The facet will be indexed by the vertex it does not contain. Let $h_j,\ j \in \{0, \ldots, n\}$ be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the exit facet. Define the index set $I := \{1, \ldots, n\}$ and define $I(x)$ to be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \text{co}\{v_i | i \in I(x)\}$. For $x \in \mathcal{S}$ define the closed, convex $\mathcal{C}(x) := \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in I \backslash I(x)\}$. See Figure 1 where the notation is illustrated.

We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a, \qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n \times m}$. Let $\mathcal{B} = \text{Im}\ (B)$, the image of B. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at x_0 under input u.

Problem 2.1 (Reach Control Problem): Consider system (1) defined on a simplex $\mathcal{S}$. Find a state feedback $u(x)$ such that for each initial condition $x_0 \in \mathcal{S}$, there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

RCP is typically solved by affine feedback $u = Kx + g$, and the main computation involves solving *invariance conditions* that guarantee that closed-loop trajectories do not cross the non-exit facets $\mathcal{F}_i$, $i \in I$ [7]. The procedure is to find control values $u_0, \ldots, u_n \in \mathbb{R}^m$ such that

$$Av_i + Bu_i + a \in \mathcal{C}(v_i), \qquad i \in \{i, \ldots, n\}. \tag{2}$$

Their solution can be computed by hand or via a linear program. If we set $y_i := Av_i + Bu_i + a$, then Figure 1 illustrates the invariance conditions, $y_i \in \mathcal{C}(v_i)$.

III. Preliminaries

This section provides some preliminary definitions and results which will be used later in proving our main results. Particularly, we extend what is meant by stabilizability and detectability for a shifted complex half plane. This allows us to extend current results in regulator theory to stable exosystems. Let $\alpha > 0$. The α-shifted open left-half plane is $\mathbb{C}^-_\alpha := \{x \in \mathbb{C} : \mathrm{Re}\ x < -\alpha\}$, and the α-shifted closed right-half plane is $\bar{\mathbb{C}}^+_\alpha := \{x \in \mathbb{C} : \mathrm{Re}\ x \geq -\alpha\}$.

Definition 3.1: Let $\alpha > 0$. The pair (A, B) is α-stabilizable if there exists a K such that $\sigma(A+BK) \subset \mathbb{C}^-_\alpha$. The pair (C, A) is α-detectable if there exists an L such that $\sigma(A + LC) \subset \mathbb{C}^-_\alpha$.

The following lemma allows us to approximate the rate of decay of a matrix exponential by the rate of decay of it's largest eigenvalue.

Lemma 3.2: Let $A \in \mathbb{R}^{n\times n}$ and let $\sigma(A)$ denote the spectrum of A. Then for any $\lambda^* > \max_{\lambda\in\sigma(A)} \mathrm{Re}(\lambda)$, there exists $\beta > 0$ such that $\|e^{At}\| \leq \beta e^{\lambda^* t}$ for all $t \geq 0$.

The following lemma extends Hautus' result on detectability to provide a test for α-detectability.

Lemma 3.3: The pair (C, A) is α-detectable if and only if $\mathrm{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \qquad \text{for all } \lambda \in \bar{\mathbb{C}}^+_\alpha.$

IV. Regulator Theory for Affine Systems

This section extends the results of typical regulator theory to accommodate two changes for RCP. First, instead of studying purely linear systems, we extend the theory to affine systems: the state equation, exosystem, and error will all have affine terms. Second, we allow for asymptotically stable exosystems.

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew + a && \text{(3a)} \\ \dot{w} &= Sw + s && \text{(3b)} \\ y &= C_1 x + C_2 w && \text{(3c)} \\ e &= D_1 x + D_2 w + d\,, && \text{(3d)} \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $a \in \mathbb{R}^n$, $E \in \mathbb{R}^{n\times q}$, $S \in \mathbb{R}^{q\times q}$, $s \in \mathbb{R}^q$, $C_1 \in \mathbb{R}^{p\times n}$, $C_2 \in \mathbb{R}^{p\times q}$, $D_1 \in \mathbb{R}^{r\times n}$, $D_2 \in \mathbb{R}^{r\times q}$, $d \in \mathbb{R}^r$. Equation (3a) describes the plant, (3b) describes the exosystem, (3c) is the measurement, and (3d) is the error to be regulated to zero. The exosystem models two behaviours: desired reference behaviour for the plant and, disturbances or external dynamics acting on the plant.

A. Output Regulation with Full Information

We introduce the problem of output regulation with full information.

Problem 4.1 (Regulation with Full Information): Consider the system (3) and let $0 < \alpha < \alpha^*$. Find $u = H_1 x + H_2 w + h$ such that the following conditions hold:

(AS) $\sigma(A + BH_1) \subset \mathbb{C}^-_\alpha$.

(R) There exists $\beta > 0$, such that for all $(x(0), w(0))$ and for all $t \geq 0$, the closed-loop system satisfies $||e(t)|| \leq \beta e^{-\alpha^* t}\|e(0)\|$.

The regulation requirement for this problem statement is different from typical regulator theory. We provide a guaranteed rate of decay on the error bound, as opposed to simply guaranteeing that the error decays to zero. To solve Problem 4.1 we require the following assumptions.

Assumption 4.2: The system (3) satisfies the following:

(A1) (A, B) is α-stabilizable.

(A2) $\sigma(S) \subset \bar{\mathbb{C}}^+_\alpha$.

Before we can prove Theorem 4.4, we introduce a preliminary lemma that will be used in the proof.

Lemma 4.3: Consider the system $\dot{w} = Sw + s$ where $w(t) \in \mathbb{R}^q$ and $\sigma(S) \subset \bar{\mathbb{C}}^+_\alpha$. Let $G \in \mathbb{R}^{p\times q}$ and $g \in \mathbb{R}^p$. If for all initial conditions $w(0) \in \mathbb{R}^q$, $e^{\alpha t}(Gw(t) + g) \to 0$ as $t \to \infty$, then $G = 0$ and $g = 0$.

The next result provides regulator equations for the affine extension of regulator theory. The first and third regulator equations are the usual ones, whereas the second and fourth arise specifically to address the affine nature of the problem.

Theorem 4.4: Consider the system (3) and suppose Assumption 4.2 holds. Then Problem 4.1 is solvable if and only if there exist (Π, Γ, p, γ) such that

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + E && \text{(4a)} \\ \Pi s &= Ap + B\gamma + a && \text{(4b)} \\ 0 &= D_1\Pi + D_2 && \text{(4c)} \\ 0 &= D_1 p + d\,, && \text{(4d)} \end{aligned}$$

where $\Pi \in \mathbb{R}^{n\times q}$, $\Gamma \in \mathbb{R}^{m\times q}$, $p \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}^m$. Moreover a suitable state feedback solving Problem 4.1 is given by

$$u = \Gamma w + K(x - (\Pi w + p)) + \gamma\,, \qquad (5)$$

where K is any matrix such that $\sigma(A + BK) \subset \mathbb{C}^-_\alpha$.

Remark 4.5: The affine regulator equations (4) can also be derived from the standard linear regulator equations by defining an extended exoysystem

$$\begin{aligned} \dot{x} &= Ax + Bu + \begin{bmatrix} E & a \end{bmatrix} w_e \\ \dot{w}_e &= \begin{bmatrix} S & s \\ 0 & 0 \end{bmatrix} w_e \\ y &= C_1 x + C_2 w_e \\ e &= D_1 x + \begin{bmatrix} D_2 & d \end{bmatrix} w_e\,, \end{aligned}$$

and setting $w_e(0) = \begin{bmatrix} w_{e_1}(0) & 1 \end{bmatrix}^T$.

B. Output Regulation with Partial Information

In typical engineering systems it is not common to have knowledge of the exosystem's initial conditions or even the initial conditions of the full state. This brings us to the problem of output regulation with partial state information. We consider dynamic feedback of the form

$$\begin{aligned} \dot{\xi} &= F\xi + Gy + f && \text{(7a)} \\ u &= H\xi + h\,, && \text{(7b)} \end{aligned}$$

where $F \in \mathbb{R}^{n_c \times n_c}$, $G \in \mathbb{R}^{n_c \times p}$, $f \in \mathbb{R}^{n_c}$, $H \in \mathbb{R}^{m \times n_c}$, and $h \in \mathbb{R}^m$.

Problem 4.6 (Regulation with Measurement Feedback): Consider the system (3) and let $0 < \alpha < \alpha^*$. Find a dynamic feedback (7) such that the following conditions hold:

(AS) $\sigma\left(\begin{bmatrix} A & BH \\ GC_1 & F \end{bmatrix}\right) \subset \mathbb{C}_\alpha^-$.

(R) For all $(x(0), \xi(0), w(0))$ and for all $t \geq 0$, there exists $\beta > 0$ such that the closed-loop system satisfies $||e(t)|| \leq \beta e^{-\alpha^* t}\|e(0)\|$.

Assumption 4.7: System (3) satisfies the following:

(A3) The pair $\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\right)$ is α-detectable.

Theorem 4.8: Consider the system (3) and suppose Assumptions 4.2 and 4.7 hold. Then Problem 4.6 is solvable if and only if there exist (Π, Γ, p, γ) such that (4) hold. Moreover a suitable dynamic feedback solving Problem 4.6 is given by

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \\ & \quad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}(y - \hat{y}) + \begin{bmatrix} a \\ s \end{bmatrix} \end{aligned} \tag{8a}$$

$$\hat{y} = C_1\xi_1 + C_2\xi_2 \tag{8b}$$

$$u = \Gamma\xi_2 + K(\xi_1 - (\Pi\xi_2 + p)) + \gamma\,, \tag{8c}$$

where K and $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ are any matrices such that

$$\sigma(A+BK) \subset \mathbb{C}_\alpha^-\,, \sigma\left(\begin{bmatrix} A - G_1C_1 & E - G_1C_2 \\ -G_2C_1 & S - G_2C_2 \end{bmatrix}\right) \subset \mathbb{C}_\alpha^-\,.$$

Proof: First we prove that, under Assumptions 4.2 and 4.7, if there exists a state feedback law of the form (7) such that (AS) and (R) hold, then (4) are solvable. With the regulator of the form (7), and $w(t) = 0$, the linearisation of the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & BH \\ GC_1 & F \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix}$$

which by assumption is stable. Now consider the Sylvester equations

$$\begin{bmatrix} \Pi \\ \Sigma \end{bmatrix} S = \begin{bmatrix} A & BH \\ GC_1 & F \end{bmatrix}\begin{bmatrix} \Pi \\ \Sigma \end{bmatrix} + \begin{bmatrix} E \\ GC_2 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} p \\ \sigma \end{bmatrix}[0] = \begin{bmatrix} A & BH \\ GC_1 & F \end{bmatrix}\begin{bmatrix} p \\ \sigma \end{bmatrix} + \begin{bmatrix} Bh + a - \Pi s \\ f - \Sigma s \end{bmatrix} \tag{10}$$

Since the eigenvalues of S and of the closed-loop system matrix are disjoint, and the eigenvalues of the closed-loop system matrix are disjoint from zero, by Sylvester's theorem there exists a unique solution for (Π, Σ, p, σ) satisfying the above two equations. Alternatively, (p, σ) can be solved by inverting the matrix in condition (AS). Setting $\Gamma = H\Sigma$ and $\gamma = Hp + h$, we obtain (4a)-(4b).

Next let $z_1 := x - (\Pi w + p)$ and $z_2 := \xi - (\Sigma w + \sigma)$. Then using (9) - (10) we derive that

$$\dot{z} = \begin{bmatrix} A & BH \\ GC_1 & F \end{bmatrix} z =: \tilde{A}z\,.$$

Considering the error signal, we have that

$$\begin{aligned} e(t) &= D_1x(t) + D_2w(t) + d \\ &= D_1x(t) - D_1(\Pi w(t) + p) + D_1(\Pi w(t) + p) + \\ &\quad D_2w(t) + d \\ &= \begin{bmatrix} D_1 & 0 \end{bmatrix} z(t) + (D_1\Pi + D_2)w(t) + (D_1p + d)\,. \end{aligned}$$

We define the transformations $\tilde{e}(t) := e^{\alpha t}e(t)$, $\tilde{z}(t) := e^{\alpha t}z(t)$, and $\tilde{w}(t) = e^{\alpha t}w(t)$. Considering $\tilde{z}(t)$ we have $\|\tilde{z}(t)\| \leq e^{\alpha t}\|e^{\tilde{A}t}\|\|z(0)\|$. By (AS), $\sigma(\tilde{A}) \subset \mathbb{C}_\alpha^-$ so there exists $\lambda^* > \max_{\lambda\in\sigma(\tilde{A})}\{\mathrm{Re}(\lambda)\}$ such that $-\alpha > \lambda^* > \max_{\lambda\in\sigma(\tilde{A})}\{\mathrm{Re}(\lambda)\}$. By Lemma 3.2 there exists $\beta > 0$ such that $\|\tilde{z}(t)\| \leq \beta e^{(\alpha+\lambda^*)t}\|z(0)\|$. Since $\alpha + \lambda^* < 0$, we have that $\tilde{z}(t) \to 0$ as $t \to \infty$. Similarly it can be shown that $\tilde{e}(t) \to 0$ as $t \to \infty$. Since $\tilde{e}(t) = \begin{bmatrix} D_1 & 0 \end{bmatrix}\tilde{z}(t) + (D_1\Pi + D_2)\tilde{w}(t) + e^{\alpha t}(D_1p + d)$, it must be that $e^{\alpha t}[(D_1\Pi + D_2)w(t) + (D_1p + d)] \to 0$ as $t \to \infty$. By Lemma 4.3, this implies that $D_1\Pi + D_2 = 0$ and $D_1p + d = 0$, which give (4c) and (4d).

We now assume that there exist solutions (Π, Γ, p, γ) of (4a)-(4d), and show that Problem 4.6 is solvable. By Assumption 4.2, we can create u given by (8a) - (8c), where K is chosen such that $\sigma(A + BK) \subset \mathbb{C}_\alpha^-$. The main idea is to construct an observer for the composite state $x_c = (x, w)$. Then the composite system is

$$\dot{x}_c = A_cx_c + B_cu + a_c, \quad y = C_cx_c, \quad e = D_cx_c + d\,,$$

where

$$A_c = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}, \qquad B_c = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad a_c = \begin{bmatrix} a \\ s \end{bmatrix}$$
$$C_c = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad D_c = \begin{bmatrix} D_1 & D_2 \end{bmatrix}.$$

An observer for the composite system is

$$\begin{aligned} \dot{\hat{x}}_c &= A_c\hat{x}_c + B_cu + G(y - \hat{y}) + a_c \\ \hat{y} &= C_c\hat{x}_c \\ \hat{e} &= D_c\hat{x}_c + d\,. \end{aligned}$$

The estimator error $\tilde{x}_c = x_c - \hat{x}_c$ has dynamics $\dot{\tilde{x}}_c = (A_c - GC_c)\tilde{x}_c$. Since (C_c, A_c) is α-detectable, there exists a G such that $A_c - GC_c \subset \mathbb{C}_\alpha^-$. We'll show that the above is a regulator with $\xi = \hat{x}_c$. First we'll check the asymptotic stability requirement. Suppose $w(t) = 0$. Then the dynamics of the linearised closed-loop system are given by

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}}_c \end{bmatrix} = \begin{bmatrix} (A + BK) & -B\begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix} \\ 0 & (A_c - GC_c) \end{bmatrix}\begin{bmatrix} x \\ \tilde{x}_c \end{bmatrix},$$

which satisfies our asymptotic stability requirement. Next consider the regulation requirement. Define $z = x - (\Pi w + p)$. Using (4a)-(4b), we obtain

$$\dot{z} = (A + BK)z - B\begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix}\tilde{x}_c\,.$$

Combining with the dynamics of $\tilde{x}_c$ we have the composite dynamics

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{x}}_c \end{bmatrix} = \begin{bmatrix} (A + BK) & -B\begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix} \\ 0 & (A_c - GC_c) \end{bmatrix}\begin{bmatrix} z \\ \tilde{x}_c \end{bmatrix} =: \bar{A}\begin{bmatrix} z \\ \tilde{x}_c \end{bmatrix}.$$

We have as above

$$\begin{aligned} e(t) &= D_1 x(t) + D_2 w(t) + d \\ &= \begin{bmatrix} D_1 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \tilde{x}_c(t) \end{bmatrix} + (D_1 \Pi + D_2) w(t) + (D_1 p + d) \\ &= \begin{bmatrix} D_1 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \tilde{x}_c(t) \end{bmatrix} \qquad \text{by (4c),(4d).} \end{aligned}$$

Therefore $\|e(t)\| \leq \| \begin{bmatrix} D_1 & 0 \end{bmatrix} \| \| e^{\bar{A}t} \| \| \begin{bmatrix} z(0) \\ \tilde{x}_c(0) \end{bmatrix} \| \leq \beta e^{\lambda^* t} \|e(0)\|$, where $\beta > 0$ and $-\alpha > \lambda^* > \max_{\sigma(\bar{A})}\{\mathrm{Re}(\lambda)\}$ by Lemma 3.2. This proves our regulation requirement for Problem 4.6. ■

C. Model Reduction

In this section, we discuss necessary conditions for solving Problem 4.6 and how they relate to the assumptions that we have made. It is clear that to achieve the requirement (AS), (A, B) must be α-stablilizable. Also (C_1, A) must be α-detectable since the measurement y is used in the feedback controller. On the other hand, to achieve the regulation requirement, it must be that every eigenvalue that is observable from e which lies in $\bar{\mathbb{C}}^+_\alpha$ is also observable from y. This is a necessary condition, since otherwise it would not be possible to observe if the regulation requirement is satisfied. We state these two new necessary conditions next.

Assumption 4.9: The system (3) satisfies the following:

(A1) (C_1, A) is α-detectable.

(A2) For all $\lambda \in \bar{\mathbb{C}}^+_\alpha$

$$\mathrm{Ker} \begin{bmatrix} A - \lambda I & E \\ 0 & S - \lambda I \\ C_1 & C_2 \end{bmatrix} = \mathrm{Ker} \begin{bmatrix} A - \lambda I & E \\ 0 & S - \lambda I \\ C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}$$

We first show that these two assumptions are weaker than Assumption 4.7, and then show through a proposition that there is no loss in using Assumption 4.7. First, to see that Assumption 4.7 implies Assumption 4.9 (A1), we use Hautus' test for detectability. To that end, suppose Assumption 4.9 (A1) does not hold. Then by Lemma 3.3 there exists $\lambda \in \bar{\mathbb{C}}^+_\alpha$ such that $\begin{bmatrix} A - \lambda I \\ C_1 \end{bmatrix}$ is not full column rank. But then with the same λ the matrix $\begin{bmatrix} A - \lambda I & E \\ 0 & S - \lambda I \\ C_1 & C_2 \end{bmatrix}$ fails to be full column rank as well, so Assumption 4.7 does not hold. Second, we show that Assumption 4.7 implies Assumption 4.9 (A2). Observe that for all $\lambda \in \bar{\mathbb{C}}^+_\alpha$ we have

$$\mathrm{Ker} \begin{bmatrix} A - \lambda I & E \\ 0 & S - \lambda I \\ C_1 & C_2 \end{bmatrix} \supseteq \mathrm{Ker} \begin{bmatrix} A - \lambda I & E \\ 0 & S - \lambda I \\ C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}.$$

Since Assumption 4.7 and Hautus' test implies that for all $\lambda \in \bar{\mathbb{C}}^+_\alpha$ we have $\mathrm{Ker} \begin{bmatrix} A - \lambda I & E \\ 0 & S - \lambda I \\ C_1 & C_2 \end{bmatrix} = 0$, we arrive at Assumption 4.9 (A2).

Assumption 4.7, while not a necessary condition for Problem 4.1, implies that an observer can be built for both x and w. The reasoning why this assumption does not involve a loss of generality is based on the fact that we can decompose the exosystem into a part that is observable from e and a part not observable from e. The exosystem can be reduced by eliminating states that are not observable from e, and then apply Theorem 4.8 with the reduced exosystem [5], [15].

Proposition 4.10: Suppose that Assumptions 4.9 hold, but not Assumption 4.7. Consider the composite system

$$\dot{x}_c = A_c x_c + B_c u + a_c, \qquad y = C_c x_c\,,$$

where

$$A_c = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix},\ B_c = \begin{bmatrix} B \\ 0 \end{bmatrix},\ a_c = \begin{bmatrix} a \\ s \end{bmatrix},\ C_c = \begin{bmatrix} C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}.$$

Then there exists a coordinate transformation $\tilde{x}_c = T x_c$ such that, in the new coordinates

$$\begin{aligned} \tilde{A}_c &= T A_c T^{-1} = \begin{bmatrix} A & \tilde{E} \\ 0 & \tilde{S} \end{bmatrix},\ \tilde{B}_c = T B_c = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\ \tilde{a}_c &= T a_c = \begin{bmatrix} a \\ \tilde{s} \end{bmatrix},\ \tilde{C}_c = C T^{-1} = \begin{bmatrix} C_1 & \tilde{C}_2 \\ D_1 & \tilde{D}_2 \end{bmatrix}, \end{aligned}$$

with a partitioned structure $\tilde{S} = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix}$, $\tilde{E} = \begin{bmatrix} \tilde{E}_1 & 0 \end{bmatrix}$, $\tilde{C}_2 = \begin{bmatrix} \tilde{C}_{2_1} & 0 \end{bmatrix}$, $\tilde{D}_2 = \begin{bmatrix} \tilde{D}_{2_1} & 0 \end{bmatrix}$, and $\tilde{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$. Moreover, we have that $\left(\begin{bmatrix} C_1 & \tilde{C}_{2_1} \\ D_1 & \tilde{D}_{2_1} \end{bmatrix}, \begin{bmatrix} A & \tilde{E}_1 \\ 0 & S_{11} \end{bmatrix} \right)$ is α-detectable.

Proof: If the pair (C_c, A_c) is not α-detectable, by an appropriate change of coordinates, one can transform this pair into $(\tilde{C}_c, \tilde{A}_c)$ with the structure $\tilde{C}_c = \begin{bmatrix} \tilde{C}_{c_1} & 0 \end{bmatrix}$ and $\tilde{A}_c = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$, where the pair $(\tilde{C}_{c_1}, A_{11})$ is α-detectable. Since we have that (C_1, A) is α-detectable by assumption, we can pick the transformation to obtain $A_{11} = \begin{bmatrix} A & \tilde{E}_1 \\ 0 & S_{11} \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & S_{21} \end{bmatrix}$, $A_{22} = S_{22}$, and $\tilde{C}_{c_1} = \begin{bmatrix} C_1 & \tilde{C}_{2_1} \\ D_1 & \tilde{D}_{2_1} \end{bmatrix}$. ■

We shall see that, with the help of Proposition 4.10, Assumption 4.7 is without any loss of generality. Let $\begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} = T \begin{bmatrix} x \\ w \end{bmatrix}$, where T is defined as in Proposition 4.10. Then $\tilde{w}$ may be partitioned as $\tilde{w} = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix}$. In these coordinates we have

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + Bu + \tilde{E}_1 w_1 + a \\ \dot{\tilde{w}}_1 &= S_{11}\tilde{w}_1 + s_1 \\ \dot{\tilde{w}}_2 &= S_{21}\tilde{w}_1 + S_{22}\tilde{w}_2 + s_2 \\ y &= C_1\tilde{x} + \tilde{C}_{2_1}\tilde{w}_1 \\ e &= D_1\tilde{x} + \tilde{D}_{2_1}\tilde{w}_1 + d\,. \end{aligned}$$

We observe that the only terms affecting e are $\tilde{x}$ and $\tilde{w}_1$. This means solving the regulation problem of the original system is equivalent to solving the regulation problem with the reduced exosystem

$$\dot{\tilde{w}}_1 = S_{11}\tilde{w}_1 + s_1 \,.$$

For the new plant and exosystem, we have that $\tilde{x}$ and $\tilde{w}_1$ are α-detectable from y and e. Applying Assumption 4.9 *(A2)*, we have that $\tilde{x}$ and $\tilde{w}_1$ are α-detectable from just y. Therefore we have that Assumption 4.7 holds.

V. Reach Control Problem with Disturbance Rejection

Now that a regulator theory for affine systems has been put in place, we apply this theory to the problem of disturbance rejection combined with reach control on a simplex. Our approach is to encode the desired behaviour on each simplex in an exosystem. We present two approaches to solving the disturbance rejection problem with partial state information on simplices. The first method relies directly on the affine formulation we have developed above, while the second method exploits the fact that the desired reference behaviour of the reach control problem is given by phase portraits, not signals.

A. Method 1

Let $\mathcal{P} \in \mathbb{R}^n$ be a full dimensional polyhedron and let $\mathbb{T} = \{\mathcal{S}_1, \ldots, \mathcal{S}_l\}$ be a triangulation of $\mathbb{T}$. Consider the system defined on $\mathcal{P}$

$$\begin{aligned}
\dot{x} &= Ax + Bu + Ew_2 + a && (11a)\\
\dot{w}_1 &= (A + BK^{\kappa(w_1)})w_1 + (a + Bg^{\kappa(w_1)}) && (11b)\\
\dot{w}_2 &= Sw_2 + s && (11c)\\
y &= C_1 x + C_2 w_2 && (11d)\\
e &= x - w_1 \,. && (11e)
\end{aligned}$$

Observe that the exosystem has been split into an exosystem (11b) that describes the desired behavior on each simplex and (11c) that generates the disturbance. In (11b) it is assumed that reach controllers $u^i_{rcp} = K^i x + g^i$, $i = 1, \ldots, l$ are available to generate the desired behaviour on each simplex. The index $\kappa(w_1) = i$ when $w_1 \in \mathcal{S}_i$. We require the following assumptions.

Assumption 5.1: The system (11) satisfies the following:

(A1) (A, B) is α-stabilizable

(A2) $\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\right)$ is α-detectable

(A3) $w_1(0)$ is known.

We have included the third assumption since the desired reference behaviour is specified by the designer, and is therefore known. The assumption can be removed with a slight modification to the proof.

Problem 5.2 (Partial State Information): Consider the system (11), RCP controllers $u^1_{rcp}, \ldots, u^l_{rcp}$, and let $0 < \alpha < \alpha^*$. Find a dynamic feedback of the form

$$\begin{aligned}
\dot{\xi} &= F\xi + Gy + f && (12)\\
u^i &= H_1\xi + H_2^{\iota(w_1)} w_1 + h^{\iota(w_1)} \,, && (13)
\end{aligned}$$

where $\iota(w_1)$ is a state-dependent switching signal, such that the following conditions hold:

(AS) $\sigma\left(\begin{bmatrix} A & BH_1 \\ GC_1 & F \end{bmatrix}\right) \subset \mathbb{C}^-_\alpha$,

(R) For all $(x(0), \xi(0), w_1(0), w_2(0))$ and for all $t \geq 0$, there exists $\beta > 0$ such that the closed loop system satisfies $||e(t)|| \leq \beta \mathrm{e}^{-\alpha^* t}||e(0)||$.

Theorem 5.3: Problem 5.2 is solvable if and only if there exists Γ such that $B\Gamma + E = 0$. Moreover a suitable dynamic feedback solving Problem 5.2 is given by

$$\begin{aligned}
\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + && (14a)\\
&\quad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}(y - \hat{y}) + \begin{bmatrix} a \\ s \end{bmatrix}\\
\hat{y} &= C_1\xi_1 + C_2\xi_2 && (14b)\\
u &= K^{\kappa(w_1)}w_1 + g^{\kappa(w_1)} + K(\xi_1 - w_1) + \Gamma\xi_2 && (14c)
\end{aligned}$$

where K and $G = \begin{bmatrix} G_1^T & G_2^T \end{bmatrix}^T$ are any matrices such that $\sigma(A + BK) \subset \mathbb{C}^-_\alpha$ and $\sigma\left(\begin{bmatrix} A - G_1C_1 & E - G_1C_2 \\ -G_2C_1 & S - G_2C_2 \end{bmatrix}\right) \subset \mathbb{C}^-_\alpha$.

Proof: We construct an observer for x and w_2 of the form (14a). Define the estimator error states $\tilde{\xi}_1 = x - \xi_1$ and $\tilde{\xi}_2 = w_2 - \xi_2$. Then we verify

$$\dot{\tilde{\xi}} = \begin{bmatrix} A - G_1C_1 & E - G_1C_2 \\ -G_2C_1 & S - G_2C_2 \end{bmatrix}\tilde{\xi}.$$

By (A2) we can choose G_1 and G_2 such that the estimator error dynamics have poles in $\mathbb{C}^-_\alpha$. Let u be given by (14c). Then the closed-loop system with $w_1 = 0$ is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{\xi}} \end{bmatrix} = \begin{bmatrix} A + BK & -BK & -B\Gamma \\ 0 & A - G_1C_1 & E - G_1C_2 \\ 0 & -G_2C_1 & S - G_2C_2 \end{bmatrix}\begin{bmatrix} x \\ \tilde{\xi} \end{bmatrix} + \begin{bmatrix} Bg^{\kappa(w_1)} + a \\ 0 \\ 0 \end{bmatrix}.$$

The system matrix has its spectrum in $\mathbb{C}^-_\alpha$ by the choice of K and G. Then we can verify that with $H_1 := [B \;\; \Gamma]$,

$$F := \begin{bmatrix} A + BK - G_1C_1 & E + B\Gamma - G_1C_2 \\ -G_2C_1 & S - G_2C_2 \end{bmatrix},$$

and $T := \begin{bmatrix} I & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & -I \end{bmatrix}$, then $T^{-1}\begin{bmatrix} A & BH_1 \\ GC_1 & F \end{bmatrix}T$ is equal to the system matrix above. This proves (AS).

For the regulation requirement (R) it can be verified that using $B\Gamma + E = 0$ and u given in (14c) we have $\dot{e} = (A + BK)e - BK\tilde{\xi}_1 - B\Gamma\tilde{\xi}_2$. Since $\sigma(A + BK) \subset \mathbb{C}^-_\alpha$ and $\tilde{\xi}_1$ and $\tilde{\xi}_2$ decay according to poles in $\mathbb{C}^-_\alpha$, we obtain (R). ■

B. Method 2

The second method exploits the fact that the reach control problem regards achieving a desired phase portrait. We modify the regulation requirement to allow the tracking of a phase portrait instead of tracking an individual signal.

Let $\mathcal{P} \in \mathbb{R}^n$ be a full dimensional polyhedron and let $\mathbb{T} = \{\mathcal{S}_1, \ldots, \mathcal{S}_l\}$ be a triangulation of $\mathbb{T}$. Consider the system

$$\begin{aligned}
\dot{x} &= Ax + Bu + Ew + a && (15a)\\
\dot{w} &= Sw + s && (15b)\\
y &= C_1x + C_2w \,, && (15c)
\end{aligned}$$

and reach controllers $u^i_{rcp} = K^i x + g^i$. The main difference between this model and that for Method 1 is that the exosystem only generates the disturbance. We require the following assumption.

Assumption 5.4: The system (15) satisfies the following:

(A1) $\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\right)$ is detectable.

Problem 5.5 (Partial State Information): Given a polytope $\mathcal{P}$ and a triangulation of $\mathcal{P}$, $\mathbb{T} = \{\mathcal{S}_1, \ldots, \mathcal{S}_l\}$, the system (15), and RCP controllers $u^1_{rcp}, \ldots, u^l_{rcp}$. Find dynamic feedbacks on each simplex $\mathcal{S}_i$, $i = 1, \ldots, l$, of the form

$$\begin{aligned} \dot{\xi} &= F\xi + Gy + f && (16) \\ u^i &= H^{\iota(\xi)}\xi + h^{\iota(\xi)}, && (17) \end{aligned}$$

where $\iota(\xi)$ is a state-dependent switching signal, and such that

(R) For all $(x(0), \xi(0), w(0))$ the closed-loop system satisfies $\dot{x}(t) \to (A + BK^{\iota(\xi)})x + (a + Bg^{\iota(\xi)})$ as $t \to \infty$.

Notice that the (AS) requirement has been removed. This is to show that the problem is no longer a tracking problem; it is simply a disturbance rejection problem. Then we can achieve the desired phase portrait without the need to track any individual signal. We also do not require the use of the α-shifted complex plane since we are not building a desired reference behaviour in an exosystem that may have stable poles.

Theorem 5.6: Problem 5.5 is solvable if and only if $\exists\ \Gamma$ such that $B\Gamma + E = 0$. Moreover a suitable dynamic feedback solving Problem 5.5 is given by

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + && (18a) \\ & \quad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} (y - \hat{y}) + \begin{bmatrix} a \\ s \end{bmatrix} && \\ \hat{y} &= C_1\xi_1 + C_2\xi_2 && (18b) \\ u &= K^{\kappa(\xi_1)}\xi_1 + g^{\kappa(\xi_1)} + \Gamma\xi_2 && (18c) \end{aligned}$$

where $G = \begin{bmatrix} G_1^T & G_2^T \end{bmatrix}^T$ is chosen such that $\sigma\left(\begin{bmatrix} A - G_1C_1 & E - G_1C_2 \\ -G_2C_1 & S - G_2C_2 \end{bmatrix}\right) \subset \mathbb{C}^-$.

Proof: We construct an observer for x and w as follows.

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} (y - \hat{y}) + \begin{bmatrix} a \\ s \end{bmatrix} \\ \hat{y} &= C_1\xi_1 + C_2\xi_2 \,. \end{aligned}$$

Define the estimator error states $\tilde{\xi}_1 = x - \xi_1$ and $\tilde{\xi}_2 = w - \xi_2$. Then

$$\begin{bmatrix} \dot{\tilde{\xi}}_1 \\ \dot{\tilde{\xi}}_2 \end{bmatrix} = \left(\begin{bmatrix} A & E \\ 0 & S \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix}\right) \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix}.$$

By (A1) we can choose G_1 and G_2 such that the estimator error dynamics are asymptotically stable. Therefore $\xi_1(t) \to x(t)$, and $\xi_2(t) \to w(t)$ as $t \to \infty$. Let u be given by (18c). Then we have

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew + a \\ &= Ax + B(K^{\kappa(\xi_1)}\xi_1 + g^{\kappa(\xi_1)} + \Gamma\xi_2) + Ew + a \\ &\to (A + BK^{\kappa(\xi_1)})x + (a + Bg^{\kappa(\xi_1)}) \,. \end{aligned}$$

Therefore our regulation requirement has been achieved. ■

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