# A Viability Approach to the Output Reach Control Problem

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***Abstract*— We study the *Output Reach Control Problem* (ORCP) to force the output trajectories of an affine control system to cross a prescribed exit facet of a simplex in output space, without first leaving the simplex. Our approach leverages existing results on the *Reach Control Problem* (RCP) and viability theory.**

## I. INTRODUCTION

The *Reach Control Problem* (RCP) for affine systems on simplices and polytopes has received considerable attention over the past fifteen years. First introduced in [16], the problem is to design a state feedback to force closed-loop trajectories starting anywhere in a polytopic state space to leave the polytope from a prescribed exit facet in finite time. One motivation for the RCP is a shift from classical control results that focus on stabilization to more complex control specifications. Such specifications often include safety constraints, where the system states may need to reach a certain region while avoiding unsafe regions during the transient. In other situations, there may be a temporal order for the control tasks, e.g. do task A and only then do task B twice. The RCP is a tool to deal with such complex specifications. Interesting applications of the RCP include motion of robots in complex environments [4], aircraft and underwater vehicles [5], genetic networks [6], smart buildings, process control [19], among others [14].

The most definitive results on the RCP are focused on reach control on simplices by affine feedback [7], [17], [18], [22]. Some more recent results include [1], [2], [8], [10], [24], which exploit system structure on simplices, particularly the reach control indices [8] and the concept of reach controllability [24]. While there is no such structure for general polytopes, in [20] geometric conditions are imposed to solve the RCP in a monotonic sense. Finally in [12], the authors solve the standard RCP by piecewise-affine output feedback.

This paper presents the Output Reach Control Problem (ORCP), a variant of the standard RCP in which the output is restricted to an *output simplex* $\mathcal{S}$. The control objective is to force output trajectories to cross a prescribed exit facet of the simplex without first leaving the simplex. While a related problem has been studied in [21], transient restrictions were not guaranteed. Since constraints on the output impose constraints on the full state, the state space for the ORCP is a polyhedron. The aim of this paper is to develop a method for solving the ORCP that builds on existing techniques for solving the RCP. Our approach is to further restrict the state space from a polyhedron to a (bounded) polytope $\mathcal{P}$. By properly associating the exit facet of $\mathcal{S}$ in the output space to an exit facet of $\mathcal{P}$ in the state space, we can apply standard RCP techniques. The main result of the paper is to prove that if the standard RCP is solved on $\mathcal{P}$, then the ORCP is solved in $\mathcal{S}$.

The main technical difficulty in mapping the ORCP to the RCP is in ensuring that the so-called *invariance conditions*, necessary conditions for solving the RCP, are solvable on $\mathcal{P}$ [20]. In order to remedy this problem, we turn to viability theory. First introduced in [3], viability theory studies the evolution of dynamical systems under state constraints. Viability algorithms are used to construct sets which are positively invariant under the system dynamics. Since the goal of the RCP is to guarantee that trajectories exit polytopic sets through an exit facet, we modify viability theory to fit with our problem requirements. We employ the algorithm of [13], but we relax the set invariance requirement and allow for an exit facet.

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## II. PRELIMINARIES

We use the following notation. For vectors $x, y \in \mathbb{R}^n$, the notation $x \prec y$ ($x \preceq y$) means $x_i < y_i$ ($x_i \leq y_i$) for all $1 \leq i \leq n$. Let $\mathcal{X}, \mathcal{Y}$ be vector spaces. If $f : \mathcal{X} \to \mathcal{Y}$ is a surjective mapping, and $\mathcal{W} \subset \mathcal{Y}$, then $f(f^{-1}(\mathcal{W})) = \mathcal{W}$, where $f^{-1}(\mathcal{W}) = \{x \in \mathcal{X} \mid f(x) \in \mathcal{W}\}$.

## III. OUTPUT REACH CONTROL PROBLEM

Consider a $p$-dimensional *output simplex* $\mathcal{S} := \text{co}\{v_0, \ldots, v_p\} \subset \mathbb{R}^p$, the convex hull of $p+1$ affinely independent points $v_i \in \mathbb{R}^p$, $i = 0, \ldots, p$. Let its vertex set be $V := \{v_0, \ldots, v_p\}$ and its facets $\mathcal{F}_0, \ldots, \mathcal{F}_p$. The facet is indexed by the vertex it does not contain. Let $h_j \in \mathbb{R}^p$, $j \in \{0, \ldots, p\}$, be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $I := \{1, \ldots, p\}$ and define $I(x)$ to be the minimal index set among $\{0, \ldots, p\}$ such that $x \in \text{co}\{v_i \mid i \in I(x)\}$.

We consider the affine control system

$$\dot{x} = Ax + Bu + a \tag{1}$$
$$y = Cx, \tag{2}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $\text{rank}(B) = m$, and $\text{rank}(C) = p$. Let $\mathcal{B} = \text{Im}\,(B)$, the image of $B$. Note that using an output which is a linear function of the state rather than an affine function $y = Cx + c$, $c \in \mathbb{R}^p$, is no loss of generality because one can always translate the origin of the output space to convert an affine function to a linear function. Let $\phi(t, x_0)$ denote the trajectory of (1) starting at $x_0$ under some control law $u$. Let $y(t, x_0) := C\phi(t, x_0)$ be the *output trajectory* corresponding to $\phi(t, x_0)$. We are interested in studying reachability by the system output of the exit facet $\mathcal{F}_0$ of $\mathcal{S}$.

*Problem* 1 (Output Reach Control Problem (ORCP)). Consider system (1)-(2) and the output simplex $\mathcal{S} \subset \mathbb{R}^p$. find a state feedback $u = f(x)$ such that for each initial condition $x_0 \in \mathbb{R}^n$ with $Cx_0 \in \mathcal{S}$, there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $y(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$;
(ii) $y(T, x_0) \in \mathcal{F}_0$; and
(iii) $y(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

The problem formulation of the ORCP differs from the standard RCP in that conditions (i)-(iii) are normally imposed on the state trajectory $\phi(\cdot, x_0)$, whereas here they are stated in terms of the output trajectory $y(\cdot, x_0)$.

While the ORCP stated is ultimately the problem we would like to solve, in order to use the existing RCP literature, we must bound the state space to a polytope. Since, in general, the states $x_0 \in \mathbb{R}^n$ such that $Cx_0 \in \mathcal{S}$ form a polyhedron, we further restrict these state to form a (bounded) polytope. Therefore, we pose a related, but modified version of the ORCP.

*Problem* 2. Consider system (1)-(2) and the output simplex $\mathcal{S} \subset \mathbb{R}^p$. Find a state feedback $u = f(x)$ and a polytope $\mathcal{P} \subset \mathbb{R}^n$ such that for each initial condition $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $y(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$;
(ii) $y(T, x_0) \in \mathcal{F}_0$; and
(iii) $y(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \gamma)$.

## IV. From ORCP to RCP

In this section we develop our method to solve Problem 2 with attention on how to find the polytope $\mathcal{P}$. The main challenge in solving Problem 2 is that it is formulated in the output space so there are no explicit constraints on the full state vector, unlike the standard RCP. We seek to impose extra constraints on the states in order to guarantee that the evolution of the output meets the requirements of Problem 2. Additionally, we hope to leverage the existing theoretical tools for solving the standard RCP, since there is now a substantial literature available [1], [7], [8], [10], [18], [20], [22], [24]. In essence, Problem 2 will be lifted to the state space, additional constraints will be imposed on the states, a feasible state set will be constructed, and finally, we invoke the standard RCP on that feasible state set. If our procedure works correctly, then the solution of the standard RCP will result in a solution of Problem 2 on $\mathcal{S}$.

The main ideas of our methodology are as follows. We begin by constructing a polytope $\mathcal{P} \subset \mathbb{R}^n$ with the property that if the initial state $x_0$ satisfies $x_0 \in \mathcal{P}$, then $y_0 := Cx_0 \in \mathcal{S}$. The polytope $\mathcal{P}$ is constructed by first "lifting" $\mathcal{S}$ into the state space to create an (unbounded) $n$-dimensional polyhedron, and second imposing additional state constraints to ensure that $\mathcal{P}$ is bounded, i.e. it is an $n$-dimensional polytope. Existing theory of polyhedra tells us that the lift of the exit facet of $\mathcal{S}$ to $\mathcal{P}$ is again a facet of $\mathcal{P}$, so we show this lifted exit facet can serve as the exit facet for $\mathcal{P}$.

Once the initial condition set $\mathcal{P}$ has been formed and an exit facet identified, one would like to solve a standard RCP on $\mathcal{P}$. It is well-known that a necessary condition for solvability of the RCP is that the so-called *invariance conditions* are solvable [20]. Unfortunately, these conditions are not guaranteed to be solvable on $\mathcal{P}$. Thus, we invoke viability theory [3], [11] to construct (a polytopic estimate) of the largest subset of $\mathcal{P}$ on which the invariance conditions are solvable. An algorithm inspired by [13] is proposed with the crucial property that any intermediate solution of the algorithm $\mathcal{P}^k$ includes the exit facet of $\mathcal{P}$, and moreover, the invariance conditions are solvable on $\mathcal{P}^k$. This implies that the algorithm can be terminated after any number of iterations to obtain an estimate $\mathcal{P}^j$ of $\mathcal{P}^*$, the largest subset of $\mathcal{P}$ containing the exit facet and such that the invariance conditions are solvable. The differences between the usual application of viability theory and our approach for reach control will be highlighted in the sequel. Finally, it is worth pointing out that our algorithm can be applied to any instance of the standard RCP on a polytope when the invariance conditions do not hold a priori, not only problems originating from Problem 2.

### A. Computing $\mathcal{P}$

In this section we develop our method to construct the initial polytope $\mathcal{P}$. It consists of two steps: first, lift the output simplex $\mathcal{S} \subset \mathbb{R}^p$ into the state space $\mathbb{R}^n$; second, impose additional constraints on the states so that the resulting set is a bounded polytope. Define the *output map*

$$y(x) := Cx\,.$$

Also, for $\mathcal{V} \subset \mathbb{R}^n$, let $y(\mathcal{V}) := \{Cx \mid x \in \mathcal{V}\}$, and for $\mathcal{W} \subset \mathbb{R}^p$, let $y^{-1}(\mathcal{V}) := \{x \in \mathbb{R}^n \mid Cx \in \mathcal{V}\}$. The *lift* of $\mathcal{S}$ to $\mathbb{R}^n$ is defined to be $y^{-1}(\mathcal{S})$. It is easily shown that $y^{-1}(\mathcal{S})$ is an $n$-dimensional polyhedron. To convert it to a (bounded) $n$-dimensional polytope we must impose additional constraints on the states, particularly the states in $\operatorname{Ker} C$. To that end, let $\mathbb{R}^n = \operatorname{Im} C^T \overset{\perp}{\oplus} \operatorname{Ker} C$. Also let $C' \in \mathbb{R}^{n \times p}$ be a maximal rank solution of $CC' = 0$. Define the coordinate transformation $x = T\tilde{x}$, where $T = \begin{bmatrix} C^T & C' \end{bmatrix}$. Let $\tilde{x} := (\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_1 \in \mathbb{R}^p$ and $\tilde{x}_2 \in \mathbb{R}^{n-p}$. Let $a, b \in \mathbb{R}^{n-p}$ with $a \prec b$. Define

$$\mathcal{P}_{box} := \{x = T\tilde{x} \mid a \preceq \tilde{x}_2 \preceq b\}\,.$$

Then we define

$$\mathcal{P} := y^{-1}(\mathcal{S}) \cap \mathcal{P}_{box}\,. \tag{3}$$

**Lemma 3.** *$\mathcal{P}$ is an $n$-dimensional polytope.*

*Proof.* First, since $\operatorname{rank}(C) = \operatorname{rank}(C^T) = p$, $\operatorname{rank}(CC^T) = p$ and $(CC^T)^{-1}$ exists. Define the sets

$$\begin{aligned} \widetilde{\mathcal{W}}_1 &:= \{\tilde{x}_1 \in \mathbb{R}^p \mid \tilde{x}_1 = (CC^T)^{-1}y, y \in \mathcal{S}\} \\ \widetilde{\mathcal{W}}_2 &:= \{\tilde{x}_2 \in \mathbb{R}^{n-p} \mid a \preceq \tilde{x}_2 \preceq b\}\,. \end{aligned}$$

The set $\widetilde{\mathcal{W}_1}$ is compact since $\mathcal{S}$ is compact, and $\widetilde{\mathcal{W}_2}$ is compact by construction. Hence, $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ is compact. We claim that

$$\mathcal{P} = \{x = T\tilde{x} \mid \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in \widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}\}. \qquad (4)$$

To prove the claim, let $x \in \mathcal{P}$. By definition of $\mathcal{P}$, $y = Cx \in \mathcal{S}$. Then $y = CT\tilde{x} = CC^T\tilde{x}_1 + CC'\tilde{x}_2$. But $CC' = 0$, so $y = CC^T\tilde{x}_1$ and $\tilde{x}_1 = (CC^T)^{-1}y$. Since $y \in \mathcal{S}$, we conclude $\tilde{x}_1 \in \widetilde{\mathcal{W}_1}$. Also since $x \in \mathcal{P}_{box}$, $a \preceq \tilde{x}_2 \preceq b$, so $\tilde{x}_2 \in \widetilde{\mathcal{W}_2}$. We conclude $\tilde{x} \in \widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$, as desired. Conversely, suppose $x = T\tilde{x}$ with $\tilde{x} \in \widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$. Then $a \preceq \tilde{x}_2 \preceq b$, so $x \in \mathcal{P}_{box}$. Also since $\tilde{x}_1 \in \widetilde{\mathcal{W}_1}$, $y = Cx = CC^T\tilde{x}_1 \in \mathcal{S}$, so $x \in y^{-1}(\mathcal{S})$. We conclude $x \in \mathcal{P} = y^{-1}(\mathcal{S}) \cap \mathcal{P}_{box}$.

Since $\mathcal{P}$ is related to $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ through a (nonsingular) coordinate transformation and $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ is a polytope, so is $\mathcal{P}$. It remains to show that $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ is $n$-dimensional, from which we conclude $\mathcal{P}$ is an $n$-dimensional polytope. Let $\tilde{v}_i = (CC^T)^{-1}v_i$ for $i = 0, \ldots, p$. Since $v_i \in \mathcal{S}$, $\tilde{v}_i \in \widetilde{\mathcal{W}_1}$. Observe that $(\tilde{v}_i, c) \in \widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ for any $c \in \mathbb{R}^{n-P}$ with $a \preceq c \preceq b$. We will show there are $n+1$ points of this form in $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ which are affinely independent, from which we conclude $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$ and $\mathcal{P}$ have dimension $n$. To that end, consider the $n+1$ points in $\widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$:

$$\begin{aligned} \tilde{w}_i &= (\tilde{v}_i, a), \quad i = 0, \ldots, p \\ \tilde{w}_{p+1} &= (\tilde{v}_p, (b_1, a_2, \ldots, a_{n-p})) \\ &\vdots \\ \tilde{w}_n &= (\tilde{v}_p, (b_1, b_2, \ldots, b_{n-p})). \end{aligned}$$

We will show that if

$$\lambda_0\tilde{w}_0 + \cdots + \lambda_n\tilde{w}_n = 0 \qquad (5)$$

with $\lambda_i \in \mathbb{R}^n$ such that $\lambda_0 + \cdots + \lambda_n = 0$, then $\lambda_0 = \cdots = \lambda_n = 0$. The first $p$ components of each vector in (5) give $\lambda_0\tilde{v}_0 + \cdots + \lambda_{p-1}\tilde{v}_{p-1} + (\lambda_p + \cdots + \lambda_n)\tilde{v}_p = 0$. Since $\lambda_0 + \cdots + \lambda_{p-1} + (\lambda_p + \cdots + \lambda_n) = 0$ and $\tilde{v}_0, \ldots, \tilde{v}_p$ are affinely independent in $\mathbb{R}^p$, $\lambda_0 = \cdots = \lambda_{p-1} = (\lambda_p + \cdots + \lambda_n) = 0$. The $(p+1)$th component of each vector in (5) gives $(\lambda_0 + \cdots + \lambda_p)a_1 + (\lambda_{p+1} + \cdots + \lambda_n)b_1 = 0$, which implies that $\lambda_p(a_1 - b_1) = 0$, and therefore $\lambda_p = 0$. Repeating this argument with the remaining components leads to $\lambda_0 = \cdots = \lambda_n = 0$, as desired. ☐

We have now constructed an initial $n$-dimensional polytope $\mathcal{P}$ on which we can formulate a standard RCP. However, we still need to define a suitable exit facet for $\mathcal{P}$ based on the exit facet of $\mathcal{S}$. The exit facet $\mathcal{F}_0$ of $\mathcal{S}$ naturally lifts to an exit facet $\mathcal{F}_0^{\mathcal{P}}$ of $\mathcal{P}$ using standard arguments about projections of polytopes [25].

**Definition 4.** A *projection of polytopes* $f : \mathcal{P} \to \mathcal{P}'$ is an affine map $f : \mathbb{R}^n \to \mathbb{R}^p$, where $\mathcal{P} \subseteq \mathbb{R}^n$ is an $n$-dimensional polytope, $\mathcal{P}' \subseteq \mathbb{R}^p$ is a $p$-dimensional polytope, and $f(\mathcal{P}) = \mathcal{P}'$.

**Lemma 5.** *[25] Let $f : \mathcal{P} \to \mathcal{P}'$ be a projection of polytopes. Then for every face $\mathcal{F}'$ of $\mathcal{P}'$, the preimage $f^{-1}(\mathcal{F}') = \{x \in \mathcal{P} \mid f(x) \in \mathcal{F}'\}$ is a face of $\mathcal{P}$.*

The next result shows that the linear map $y : \mathbb{R}^n \to \mathbb{R}^p$ is a projection of polytopes.

**Lemma 6.** *Let $\mathcal{S} \subset \mathbb{R}^p$ and $\mathcal{P} = y^{-1}(\mathcal{S}) \cap \mathcal{P}_{box}$. Then $y(\mathcal{P}) = \mathcal{S}$.*

*Proof.* Let $\bar{y} \in y(\mathcal{P})$. By the definition of $\mathcal{P}$, $\bar{y} \in y(y^{-1}(\mathcal{S}))$. Since $y : \mathbb{R}^n \to \mathbb{R}^p$ is surjective, $y(y^{-1}(\mathcal{S})) = \mathcal{S}$, and therefore $\bar{y} \in \mathcal{S}$. Conversely, let $\bar{y} \in \mathcal{S}$. Select any $\tilde{x}_2 \in \mathbb{R}^{n-p}$ such that $a \preceq \tilde{x}_2 \preceq b$ and define $\tilde{x} := ((CC^T)^{-1}\bar{y}, \tilde{x}_2)$. By the proof of Lemma 3, $\tilde{x} \in \widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$, and therefore $x := T\tilde{x} \in \mathcal{P}$. This leads to $Cx = C(C^T\tilde{x}_1 + C'\tilde{x}_2) = (CC^T)(CC^T)^{-1}\overline{y} = \overline{y} \in y(\mathcal{P})$, as desired. ☐

Using the previous two lemmas we can now define a feasible exit facet of $\mathcal{P}$ by lifting $\mathcal{F}_0$, the exit facet of $\mathcal{S}$:

$$\mathcal{F}_0^{\mathcal{P}} := y^{-1}(\mathcal{F}_0) \cap \mathcal{P}. \qquad (6)$$

**Lemma 7.** *$\mathcal{F}_0^{\mathcal{P}}$ is a facet of $\mathcal{P}$ with outward normal vector given by $h_0^{\mathcal{P}} = \frac{C^T h_0}{||C^T h_0||}$.*

*Proof.* The exit facet of $\mathcal{S}$ is $\mathcal{F}_0 = \mathrm{co}\{v_1, \ldots, v_p\}$. Define $\tilde{v}_i := (CC^T)^{-1}v_i$, $i = 1, \ldots, p$. By Lemma 5, $\mathcal{F}_0^{\mathcal{P}}$ is a face of $\mathcal{P}$. We show it is a facet of $\mathcal{P}$ by showing $\mathcal{F}_0^{\mathcal{P}}$ is $(n-1)$-dimensional. To that end, let $w_i := T\tilde{w}_i$, $i = 1, \ldots, n$, with $\tilde{w}_i$ given in the proof of Lemma 3. We will show that $w_i \in \mathcal{F}_0^{\mathcal{P}}$, $i = 1, \ldots, n$. Then since $\{\tilde{w}_1, \ldots, \tilde{w}_n\}$ are affinely independent, so are $\{w_1, \ldots, w_n\}$, so we conclude $\mathcal{F}_0^{\mathcal{P}}$ is $(n-1)$-dimensional. To show $w_i \in \mathcal{F}_0^{\mathcal{P}}$, first recall from the proof of Lemma 3 that $\tilde{w}_i \in \widetilde{\mathcal{W}_1} \times \widetilde{\mathcal{W}_2}$. Then by (4), $w_i = T\tilde{w}_i \in \mathcal{P}$, $i = 1, \ldots, n$. Second, we have $Cw_i = CT\tilde{w}_i = CC^T\tilde{v}_i = v_i \in \mathcal{F}_0$, for $i = 1, \ldots, n$. Thus, $w_i \in y^{-1}(\mathcal{F}_0)$. We conclude $w_i \in \mathcal{P} \cap y^{-1}(\mathcal{F}_0) = \mathcal{F}_0^{\mathcal{P}}$, as desired.

Next, we assume without loss of generality (w.l.o.g.) that $0 \in \mathcal{F}_0$, so $0 \in y^{-1}(\mathcal{F}_0)$. We observe that if $x \in y^{-1}(\mathcal{F}_0)$, then $y = Cx \in \mathcal{F}_0$, so $h_0 \cdot Cx = 0$. Equivalently, $(C^T h_0) \cdot x = 0$. Now $Ker(C^T) \neq \{0\}$ since $\mathrm{rank}(C^T) = p$, so $h_0 \neq 0$ implies $h_0 \notin \mathrm{Ker}\, C^T$. Hence $C^T h_0 \neq 0$ so the unit vector $h_0^{\mathcal{P}} = \frac{C^T h_0}{||C^T h_0||}$ is well-defined. Finally, it is easy to show since $h_0$ is the *outward* normal vector of $\mathcal{F}_0$, then $h_0^{\mathcal{P}}$ is also the outward normal vector of $\mathcal{F}_0^{\mathcal{P}}$. ☐

We have now constructed an $n$-dimensional polytope $\mathcal{P}$ and an appropriate exit facet $\mathcal{F}_0^{\mathcal{P}}$ which consistently lift the requirements of Problem 2 on the output simplex $\mathcal{S}$ up to the full state space. The next step is to solve the standard RCP on this polytope and show that the solution of the RCP on $\mathcal{P}$ results in solving Problem 2 on $\mathcal{S}$. Unfortunately, our work is not complete because to solve the standard RCP, it is necessary that so-called invariance conditions are solvable on $\mathcal{P}$ [18], [22]. In the next section we address this gap by proposing a viability algorithm introduced in [13] but adapted to the RCP to help ensure the invariance conditions are met on a possibly smaller polytope in $\mathcal{P}$.

## V. Viable Polytope for Reach Control

In this section we present an algorithm that provides a polytopic under-approximation of the original state space polytope $\mathcal{P}$ given in (3) such that the under-approximation satisfies the invariance conditions associated with the RCP. There are two additional considerations to address, beyond satisfaction of the invariance conditions. First, the exit facet for each iterate $\mathcal{P}^k$ must be well-defined to guarantee that solutions of the affine system under a suitable feedback do indeed exit through the given exit facet $\mathcal{F}_0^{\mathcal{P}}$. Second, it is necessary to introduce a bound on the control inputs to make the algorithm computationally tractable. We begin this section by stating the invariance conditions. Then we define the notion of an exit set in $\mathcal{F}_0^{\mathcal{P}}$. This exit set will become the first iterate $\mathcal{P}^0$ of the algorithm. Finally, we present the algorithm and give comparisons to a standard viability algorithm.

We consider an $n$-dimensional polytope

$$\mathcal{P} := \mathrm{co}\{p_0, \dots, p_r\}$$

with vertex set $V^{\mathcal{P}} := \{p_0, \dots, p_r\}$ and facets $\mathcal{F}_0^{\mathcal{P}}, \dots, \mathcal{F}_q^{\mathcal{P}}$, where $\mathcal{F}_0^{\mathcal{P}}$ is the exit facet. Let $h_j^{\mathcal{P}}$ be the unit normal to each facet $\mathcal{F}_j^{\mathcal{P}}$ pointing outside the polytope. Define the index set $J = \{1, \dots, q\}$. For each $x \in \mathcal{P}$ define the closed, convex cone

$$\mathcal{C}(x) := \left\{ y \in \mathbb{R}^n \mid h_j^{\mathcal{P}} \cdot y \le 0, j \in J \text{ s.t. } x \in \mathcal{F}_j^{\mathcal{P}} \right\}.$$

Next we introduce a bound on the control inputs. To that end, we define $U = \mathrm{co}\{u_1, \dots, u_M\}$ to be a polytope that bounds the inputs. The invariance conditions will be stated in terms of this bound.

**Definition 8.** We say the *invariance conditions are solvable* on a polytope $\mathcal{P}$ if for each $v \in V^{\mathcal{P}}$ there exists $u \in U$ such that

$$Av + Bu + a \in \mathcal{C}(v). \tag{7}$$

Next we define the exit set of an exit facet. The exit set can be thought of as the set of points on the exit facet such that there exists a velocity vector of the affine system to force trajectories to immediately leave the polytope.

**Definition 9** (Exit Set)**.** Consider the affine system (1) on a polytope $\mathcal{P}$. Let $\mathcal{F}_0^{\mathcal{P}}$ be a facet of $\mathcal{P}$. The exit set of $\mathcal{F}_0^{\mathcal{P}}$ is the set

$$\mathcal{F}_{exit}^{\mathcal{P}} := \mathrm{cl}\left\{ x \in \mathcal{F}_0^{\mathcal{P}} \mid (\exists u \in U)\ h_0^{\mathcal{P}} \cdot (Ax + Bu + a) > 0 \right\}.$$

There are three pathologies which can arise with $\mathcal{F}_{exit}^{\mathcal{P}}$. First it may be empty. In that case, the RCP is not solvable, and there is no point to proceed with the algorithm. Second, $\mathcal{F}_{exit}^{\mathcal{P}}$ may not be a full dimensional facet of $\mathcal{P}$. But this is impossible as $\mathcal{F}_{exit}^{\mathcal{P}}$ has been constructed as the closure of an open subset in $\mathcal{F}_0^{\mathcal{P}}$. Finally, $\mathcal{F}_{exit}^{\mathcal{P}}$ may not be a polytopic set, a requirement for the algorithm. This is resolved by finding any polytopic under-approximation of $\mathcal{F}_{exit}^{\mathcal{P}}$. We omit this step and we assume that $\mathcal{F}_{exit}^{\mathcal{P}}$ is already presented as an $(n-1)$-dimensional polytope

$$\mathcal{P}^0 := \mathcal{F}_{exit}^{\mathcal{P}} = \mathrm{co}\ \{v \in V^0\},$$

where $V^0$ is the vertex set of $\mathcal{P}^0$. By construction $\mathcal{P}^0$ is an $n-1$ dimensional polytope in the exit set of $\mathcal{F}_0^{\mathcal{P}}$.

Now we introduce the notation of the algorithm. We use $\mathcal{P}^0$ to designate the initial polytope, as defined above. The polytope at iteration $i$ is $\mathcal{P}^i$ and it's vertex set is $V^i$. The polytope $\mathcal{P}$ given by (3) is the largest possible polytope that the algorithm could construct. The vertex set $D$ bookkeeps the vertices in $V^i$ not yet used in the $i$th iteration.

The algorithm attempts to find the largest polytope $\mathcal{P}^i \subset \mathcal{P}$ such that the invariance conditions hold on $\mathcal{P}^i$. At each iteration, an optimization problem is solved. The objective of the optimization problem is to take a vertex $v$ of $\mathcal{P}^i$, and extend the polytope $\mathcal{P}^i$ by adding new vertices along the rays between $v$ and each of the vertices of $\mathcal{P}$. This creates a new candidate polytope which consists of the convex hull of the new vertices and the old vertices of $\mathcal{P}^i$, except for $v$. The constraint of the optimization problem is that the new polytope must satisfy the invariance conditions. This process continues until the current polytope can no longer be enlarged, while satisfying the invariance conditions.

*Algorithm* 10 (Viable Polytope for Reach Control).

1) Initialization:
$\mathcal{P} = \mathrm{co}\{p_0, \dots, p_r\}; \quad i = 0;$
$V^0 = \{p_0^0, \dots\}; \quad \mathcal{P}^0 = \mathrm{co}\{v \in V^0\}; \quad D := V^0.$
2) If $D = \emptyset$, end with $\mathcal{P}^i$.
3) If $D \neq \emptyset$, select $v \in D$.
4) Solve the optimization problem:
$\mathrm{argmin}_{w_j \in \mathrm{co}\{v, p_j\}} \sum_{j=0}^{r} ||w_j - p_j||_2$
subject to: (7) hold for the polytope $\overline{\mathcal{P}}$ with vertex set $\overline{V} := V^i \cup \{w_0, \dots, w_r\} \setminus \{v\}$.
5) If $\bar{\mathcal{P}} \neq \mathcal{P}^i$, $V^{i+1} = \overline{V}$; $D = V^{i+1}$;
$i = i + 1$. Return to step 2.
6) If $\bar{\mathcal{P}} = \mathcal{P}^i$, $D = D \setminus \{v\}$. Return to step 2.

This algorithm has been modified from the one presented in [13]. The difference is that in [13] each polytope $\mathcal{P}^i$ is positively invariant, while in this algorithm we allow the exit facet to be unrestricted. Note that while the problem has a linear objective function, the constraints are bilinear. Although existing algorithms can be applied to this optimization problem [23], the problem involves converting the polytope $\mathcal{P}^i$ from the $\mathcal{V}$-representation to the $\mathcal{H}$-representation, which can be computationally demanding. What gives this algorithm promise is that not only do we have a viable polytope at each iteration, we also have that the polytopes are non-decreasing in size with each iteration. The following proposition captures the salient properties of the algorithm that we inherit directly from [13].

**Proposition 11** (Prop. 2 of [13])**.** *Each $\mathcal{P}^i$ generated by the algorithm satisfies its invariance conditions. Moreover, $\mathcal{P}^i \subseteq \mathcal{P}^{i+1} \subseteq \mathcal{P}$.*

We also require some further properties which are specific to our problem. These are summarized in the next lemma. The first property guarantees that the ORCP will still be solvable using $\mathcal{P}^i$ rather than the original $\mathcal{P}$. The second property guarantees that each $\mathcal{P}^i$ has a well-defined exit facet.

**Lemma 12.** *Suppose that each $\mathcal{P}^i$ generated by the algorithm is full dimensional. Then for each such $\mathcal{P}^i$,*

(i) $\mathcal{P}^i \subset y^{-1}(\mathcal{S})$.
(ii) $\mathcal{F}_0^{\mathcal{P}_i} := y^{-1}(\mathcal{F}_0) \cap \mathcal{P}^i$ *is the exit facet of* $\mathcal{P}^i$.

*Proof.*

(i) By construction, $\mathcal{P} \subset y^{-1}(\mathcal{S})$ and by Proposition 11, $\mathcal{P}^i \subset \mathcal{P}$. Hence, $\mathcal{P}^i \subset y^{-1}(S)$.
(ii) We need only show that $\mathcal{F}_0^{\mathcal{P}_i}$ is a facet of $\mathcal{P}^i$. The fact that it supplies a consistent exit facet for the ORCP is treated in the next section. Recall from Lemma 7 that $\mathcal{F}_0^{\mathcal{P}} = y^{-1}(\mathcal{F}_0) \cap \mathcal{P}$ is a facet of $\mathcal{P}$. Let $\mathcal{H}$ be the hyperplane in $\mathbb{R}^n$ that contains $\mathcal{F}_0^{\mathcal{P}}$. Since $\mathcal{P}^i \subset \mathcal{P}$, $\mathcal{P}^i$ lies in the closed half-space bounded by $\mathcal{H}$ and containing $\mathcal{P}$. By construction $\mathcal{P}^0 \subset y^{-1}(\mathcal{F}_0)$ and by Proposition 11, $\mathcal{P}^0 \subset \mathcal{P}$ and $\mathcal{P}^0 \subset \mathcal{P}^i$. Thus, $\mathcal{P}^0 \subset y^{-1}(\mathcal{F}_0) \cap \mathcal{P} = \mathcal{F}_0^{\mathcal{P}} \subset \mathcal{H}$. Thus $\mathcal{H} \cap \mathcal{P}^i \neq \emptyset$. We conclude that $\mathcal{H}$ is a supporting hyperplane of $\mathcal{P}^i$ so $\mathcal{F}_0^{\mathcal{P}_i} = y^{-1}(\mathcal{F}_0) \cap \mathcal{P}^i$ is a face of $\mathcal{P}^i$ [15]. Since $\mathcal{P}^0 \subset \mathcal{F}_0^{\mathcal{P}_i}$ and $\mathcal{P}^0$ is, by construction, $(n-1)$-dimensional, we obtain, moreover, that $\mathcal{F}_0^{\mathcal{P}_i}$ is a facet of $\mathcal{P}^i$.

□

## VI. Main Results

We consider again an $n$-dimensional polytope

$$\mathcal{P} := \mathrm{co}\{p_0, \dots, p_r\}$$

with vertex set $V^{\mathcal{P}} := \{p_0, \dots, p_r\}$ and facets $\mathcal{F}_0^{\mathcal{P}}, \dots, \mathcal{F}_q^{\mathcal{P}}$, where $\mathcal{F}_0^{\mathcal{P}}$ is the exit facet. Let $h_j^{\mathcal{P}}$ be the unit normal to each facet $\mathcal{F}_j^{\mathcal{P}}$ pointing outside the polytope. We assume that the standard RCP is solvable on this polytope $\mathcal{P}$. The polytope may have been obtained as the output of Algorithm 10 to guarantee solvability of the invariance conditions. We abuse notation and rename the output of the algorithm as $\mathcal{P}$. It is assumed that the exit facet of $\mathcal{P}$ is $\mathcal{F}_0^{\mathcal{P}} = y^{-1}(\mathcal{F}_0) \cap \mathcal{P}$, with outward normal $h_0^{\mathcal{P}}$. Also $\mathcal{P} \subset y^{-1}(\mathcal{S})$.

Solvability of the RCP on $\mathcal{P}$ means that there exists a state feedback $u(x)$ such that for all $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\gamma > 0$ such that

(i) $\phi(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$,
(ii) $\phi(T, x_0) \in \mathcal{F}_0^{\mathcal{P}}$, and
(iii) $\phi(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T+\gamma)$.

The following lemma will be used in the main theorem. It examines the case when $\phi(T, x_0)$ is on the intersection of several facets. The lemma shows that trajectories cannot cross a restricted facet without crossing the exit facet. The proof will be published elsewhere. A similar result can be found in [17], their proof does not address the issue of chattering; namely, trajectories may cross two facets with infinitely high frequency.

**Lemma 13.** *Let $\mathcal{P}$ be an $n$-dimensional polytope with facets $\mathcal{F}_i^{\mathcal{P}}$ and corresponding outward normal vectors $h_i^{\mathcal{P}}$. Suppose (after possibly an affine coordinate transformation) that $0 \in \mathcal{F}_0^{\mathcal{P}} \cap \cdots \cap \mathcal{F}_k^{\mathcal{P}}$, where $k < q$. Consider the system $\dot{x} = f(x)$ defined on $\mathcal{P}$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz continuous. Further, suppose that*

$$h_i^{\mathcal{P}} \cdot f(x) \leq 0\,, \qquad x \in \mathcal{F}_i^{\mathcal{P}}, i \in \{1, \dots, k\}\,. \tag{8}$$

*Let $\phi(t, 0)$ be the solution starting at $\phi(0, 0)$. There do not exist $\gamma > 0$ and $l \in \{1, \dots, k\}$ such that if $\phi(t, 0) \in \{x \in \mathbb{R}^n \mid \bigcap_{i=1}^{l} h_i^{\mathcal{P}} \cdot x > 0\}$ for all $0 < t < \gamma$, then $\phi(t, 0) \in \{x \mid h_0^{\mathcal{P}} \cdot x \leq 0\}$ for all $0 < t < \gamma$.*

We now present our main theorem.

**Theorem 14.** *Suppose the RCP is solved on $\mathcal{P}$. Suppose w.l.o.g. $\phi(T, x_0) = 0 \in \mathcal{F}_0^{\mathcal{P}} \cap \cdots \cap \mathcal{F}_k^{\mathcal{P}}$, where $0 \leq k < q$ is the largest such integer. Further suppose the following holds.*

(iv) *For each $i \in \{0, \dots, k\}$ there exists $\gamma_i > 0$ such that either $h_i^{\mathcal{P}} \cdot \phi(t, x_0) \leq 0$ for all $t \in (T, T+\gamma_i)$ or $h_i^{\mathcal{P}} \cdot \phi(t, x_0) > 0$ for all $t \in (T, T+\gamma_i)$.*

*Then the ORCP given in Problem 2 is solved on $\mathcal{S}$.*

*Proof.* We must show (i)-(iii) of Problem 2 are satisfied.

(i) Since $\mathcal{P} \subset y^{-1}(\mathcal{S})$, $\phi(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$. This implies that $\phi(t, x_0) \in y^{-1}(\mathcal{S})$ for all $t \in [0, T]$, and thus $y(t, x_0) \in \mathcal{S}\ \forall t \in [0, T]$.

(ii) Since $\phi(T, x_0) \in \mathcal{F}_0^{\mathcal{P}} = y^{-1}(\mathcal{F}_0) \cap \mathcal{P}$, we have that $y(T, x_0) \in y(y^{-1}(\mathcal{F}_0)) = \mathcal{F}_0$.

(iii) We know that $\phi(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T+\gamma)$. First, suppose that $h_i^{\mathcal{P}} \cdot \phi(t, x_0) \leq 0$ for all $t \in (T, T+\gamma)$ and for all $i \in \{1, \dots, q\}$. Then it must be that $h_0^{\mathcal{P}} \cdot \phi(t, x_0) > 0$ for all $t \in (T, T+\gamma)$. By Lemma 7, the outward normal vector of $\mathcal{F}_0^{\mathcal{P}}$ is $h_0^{\mathcal{P}} = \frac{C^T h_0}{||C^T h_0||}$. Thus $h_0^{\mathcal{P}} \cdot \phi(t, x_0) = \frac{1}{||C^T h_0||} h_0 \cdot C\phi(t, x_0) > 0$ for all $t \in (T, T+\gamma)$. This implies $h_0 \cdot y(t, x_0) > 0$ for all $t \in (T, T+\gamma)$, which proves (iii).

In the first case we assumed $\phi(t, x_0)$ did not cross a restricted facet $\mathcal{F}_i^{\mathcal{P}}$, $i = 1, \dots, q$, on the interval $(T, T+\gamma)$. Second, suppose w.l.o.g. that the first $l$ restricted facets are crossed at certain times in the interval $(T, T+\gamma)$. By assumption, $\phi(T, x_0) \notin \mathcal{F}_{k+1} \cup \cdots \cup \mathcal{F}_q$. Therefore, there exists $\bar{\gamma} > 0$ such that $\phi(t, x_0$ does not cross the restricted facets $\mathcal{F}_i^{\mathcal{P}}$, $i = k+1, \dots, q$, on the interval $(T, T+\bar{\gamma})$. Let $\gamma' := \min\{\bar{\gamma}, \gamma, \gamma_0, \dots, \gamma_l\}$. Then by (iv), $\phi(t, x_0) \in \{x \in \mathbb{R}^n \mid h_j^{\mathcal{P}} \cdot x > 0\,, j = 1, \dots, l\}$ for all $t \in (T, T+\gamma')$. By Lemma 13, there exists $T' \in (T, T+\gamma')$ such that $\phi(T', x_0) \in \{x \in \mathbb{R}^n \mid h_0^{\mathcal{P}} \cdot x > 0\}$. Then again by (iv), $\phi(t, x_0) \in \{x \in \mathbb{R}^n \mid h_0^{\mathcal{P}} \cdot x > 0\}$ for all $t \in (T, T+\gamma')$. This implies $h_0 \cdot y(t, x_0) > 0$ for all $t \in (T, T+\gamma)$, which proves (iii).

□

## VII. Example

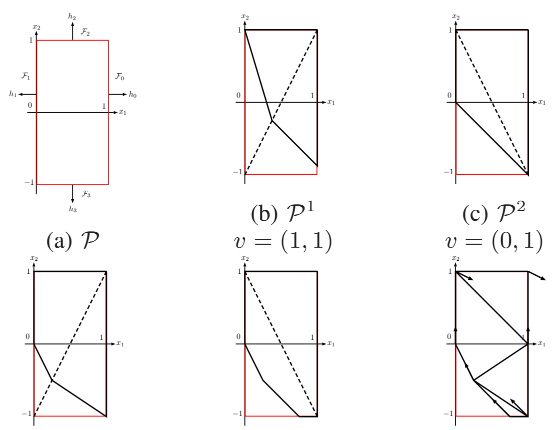
Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u,\ u \in [-1, 1] \\ y &= x_1 \in [0, 1] \end{aligned}$$

where $\mathcal{S} = [0, 1]$ is the output simplex, and $\mathcal{F}_0 = 1$ is the exit facet. The control objective is for the output trajectories to leave through the facet $\mathcal{F}_0$ in finite time, without crossing $y = 0$ first. We be begin by constructing the polytope $\mathcal{P}$. Since $T = [C^T \ C'] = I$ for this problem, we have that $x = \tilde{x}$. Let $\mathcal{P}_{box} = \{x \mid -1 \leq x_2 \leq 1\}$. We have that $\mathcal{P} = y^{-1}(\mathcal{S}) \cap \mathcal{P}_{box} = \{x \mid x \in [0, 1] \times [-1, 1]\}$ shown in Figure 1a below with exit facet $\mathcal{F}_0^{\mathcal{P}} = y^{-1}(\mathcal{F}_0) \cap \mathcal{P}$.

Let $\mathcal{P} := \text{co}\{p_0, p_1, p_2, p_3\}$ as seen in Figure 1a. The invariance conditions fail for vertex $p_2$, and therefore the standard RCP is not solvable for $\mathcal{P}$. We use Algorithm 7 to create a polytope which satisfies the invariance conditions. We have that $\mathcal{P}^0 = \text{cl}\{x \in \mathcal{F}_0^{\mathcal{P}} \mid h_0^{\mathcal{P}}(Ax + a) > 0\} = \{x \in \mathcal{F}_0^{\mathcal{P}} \mid x_2 \geq 0\} = \text{co}\{(1, 1), (1, 0)\} := \text{co}\{p_0^0, p_1^0\}$.

In step 3) we choose $v = p_0^0$. Proceeding to step 4), since $D \neq \emptyset$, we solve the optimization problem which yields $\{w_0, w_1, w_2, w_3\} = \{(0, 1), (1, 1), (0.36, -0.28), (1, -0.95)\}$. Since $\bar{\mathcal{P}} \neq \mathcal{P}^0$, we have a viable polytope $\mathcal{P}^1 = \bar{\mathcal{P}} = \text{co}\{w_0, w_1, w_2, w_3\}$ as shown in Figure 1b below.



(a) $\mathcal{P}$ (b) $\mathcal{P}^1$ $v = (1, 1)$ (c) $\mathcal{P}^2$ $v = (0, 1)$

(d) $\mathcal{P}^3$ $v = (1, 1)$ (e) $\mathcal{P}^3$ $v = (1, -1)$ (f) $\mathcal{P}^3$ $v = (1, -1)$

Fig. 1: Progression of the viability algorithm

The dashed lines are to show that each $w_j \in \text{co}\{v, p_j\}$. Continuing the algorithm selecting $v$ for step 3) as shown in each of the figures, we arrive at $\mathcal{P}^3 = \text{co}\{(0, 1), (1, 1), (0, 0), (0.25, -0.5), (0.75, -1), (1, -1)\}$ where the algorithm terminates.

With a viable polytope, we can try and use standard RCP techniques to ensure trajectories exit the polytope. In this example we choose to triangulate the polytope $\mathcal{P}^3$ and solve the RCP on each simplex. The proposed triangulation is shown in Figure 1f. Also shown in the figure are the chosen closed-loop velocity vectors. Using Algorithm 5.2 of [18] we solve the desired RCP on the polytope $\mathcal{P}^3$.

It is clear from the closed-loop velocity vectors that any initial condition in $\mathcal{P}^3$ has a solution which ensures that $x_1 \geq 0$, and that $x_1$ leaves the output simplex through $\mathcal{F}_0$ as desired.

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