Stabilization of Infinitesimally Rigid Formations of Multi-Robot Networks

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***Abstract*— This paper proposes a local gradient control law to stabilize a group of robots to a target formation. The control is derived from a potential function based on an undirected infinitesimally rigid graph that specifies the target formation. It is shown that infinitesimal rigidity is a sufficient condition for local asymptotical stability of the equilibrium manifold describing the target formation.**

I. Introduction

This paper considers distributed control of systems of agents that are interconnected dynamically or have a common objective, and where control is local, with the possible exception of high-level intermittent centralized supervision. Undoubtedly these kinds of systems will become more and more prevalent as embedded hardware evolves. An interesting example and area of ongoing research is the control of a group of autonomous mobile robots, ideally without centralized control or a global coordinate system, so that they work cooperatively to accomplish a common goal. The aims of such research are to achieve systems that are scalable, modular, and robust. These goals are similar to those of sensor networks—networks of inexpensive devices with computing, communications, and sensing capabilities. Such devices are currently commercially available and include products like the Intel Mote. A natural extension of sensor networks would be to add simple actuators to the sensors to make them mobile, and then to adapt the network configuration to optimize network coverage.

An interesting approach to formation control is that of [5]. The robots are point masses (double integrators) with limited vision, and he proposes using rigid graph theory to define the formation; he also proposes a gradient control law involving prescribed distances. The limitation is that the network is not homogeneous—special so-called γ-agents are required to achieve flocking. Anderson et al. [1] propose a novel modification of rigidity within the context of directed visibility graphs and provide control laws not derived from potential functions. The starting point for our paper is [6]. Following that paper, we use graphs to define formations, but instead of global rigidity we use infinitesimal rigidity and instead of the double integrator model we use the simpler single integrator (kinematic point). More substantially, we provide a more detailed stability analysis. In particular, [6] has no topological analysis of the equilibrium set and does not note that the equilibrium set is not compact. Moreover, [6] uses a LaSalle argument to prove stability, but since the equilibrium set is not compact, this approach is open to question. Finally, [6] does not address if the trajectories have a limit on the equilibrium set.

The main contribution of the paper is a decentralized gradient control law to stabilize a group of point mass robots to any formation corresponding to an infinitesimally rigid framework. A complete stability analysis is provided in Section V. Regular polygon formations are studied in Section VI where it is shown that the conditions of our theory can be applied to this case.

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II. Background

Notation. We denote the Jacobian of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ evaluated at a point x as $J_f(x)$. In the special case when $f : \mathbb{R}^n \to \mathbb{R}$, the Jacobian of f is the gradient of f and we denote it by $\nabla f(x)$. Occasionally for convenience during calculations of the Jacobian, the notation $\frac{\partial}{\partial x}$ will be used to represent $J_f(x) = \frac{\partial}{\partial x} f(x)$.

A. Graph Theory

A directed graph $G = (V, E)$ is a pair consisting of a finite set of vertices $V := \{1, \ldots, n\}$ and a set of edges $E \subset V \times V$. We assume the edges are ordered; that is $E = \{1, \ldots, m\}$, where $m \in \{1, \ldots, n(n-1)\}$. We exclude the possibility of self loops. An undirected graph is a directed graph such that if there is an edge e_i from vertex j to vertex k, then there is also an edge e_l from vertex k to vertex j. For undirected graphs, we omit the arrows in the pictorial representation of the graph. A special undirected graph is the graph K_n, the *complete graph* with n vertices, which has an edge between every pair of vertices. A useful matrix associated with a graph G is the $m \times n$ *incidence matrix*, H. It is determined by the edges e_i of G: row i of H is determined by e_i and has two non-zero entries: a 1 in column k and a -1 in column j, where e_i is the edge between vertex j and vertex k. Thus, by definition, $H\mathbf{1} = 0$, where $\mathbf{1}$ is the vector with a 1 in each component.

Lemma 1: ([3], p. 23) The incidence matrix H has rank $n - c$ where c is the number of connected components of G.

For the remainder of this work we assume that all graphs are connected and thus Ker(H) is one dimensional. Also, directed graphs are considered connected if the corresponding undirected graph is connected.

B. Graph Rigidity

To introduce the notion of rigidity of graphs we must view a graph as a framework embedded in the plane, $\mathbb{R}^2$. Let $G = (V, E)$ be an undirected graph with n vertices. We embed G

into $\mathbb{R}^2$ by assigning to each vertex i a location $p_i \in \mathbb{R}^2$. Define the composite vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^{2n}$. A *framework* is a pair (G, p).

We define the *rigidity function* associated with the framework (G, p) as the function $g_G : \mathbb{R}^{2n} \to \mathbb{R}^{|E|}$ given by

$$g_G(p) := (\ldots, \|p_k - p_j\|^2, \ldots),$$

The ith component of $g_G(p)$, $\|p_k - p_j\|^2$, corresponds to the edge e_i in E, where vertices j and k are connected by e_i. Note that this function is not unique and depends on the ordering given to the edges.

1) Rigidity and Global Rigidity: There are several equivalent definitions of rigidity. The definitions below are taken from [2].

Definition 2: A framework (G, p) is *rigid* if there exists a neighbourhood $\mathcal{U} \subset \mathbb{R}^{2n}$ of p such that $g_G^{-1}(g_G(p)) \cap \mathcal{U} = g_K^{-1}(g_K(p)) \cap \mathcal{U}$, where K is the complete graph with the same vertices as G.

It is also possible to define a global version of rigidity.

Definition 3: A framework (G, p) is *globally rigid* if $g_G^{-1}(g_G(p)) = g_K^{-1}(g_K(p))$.

The level set $g_G^{-1}(g_G(p))$ consists of all possible points that have the same edge lengths as the framework (G, p). For the complete graph K the set $g_K^{-1}(g_K(p))$ consists of points related by rotations and translations, i.e., rigid body motions, of the framework (K, p). We conclude that a graph G is rigid if the level set $g_G^{-1}(g_G(p))$ in a neighbourhood of p contains only points corresponding to rotations and translations of the formation at p.

2) Infinitesimal Rigidity: We refer to the matrix $J_{g_G}(p)$ as the *rigidity matrix* of (G, p). The rigidity matrix is useful in defining some other concepts related to graph rigidity.

Definition 4: A point p is a *regular point* of the graph G with n vertices if

$$\mathrm{rank} J_{g_G}(p) = \max \left\{ \mathrm{rank} J_{g_G}(q) \mid q \in \mathbb{R}^{2n} \right\}.$$

In Figure 1(a) we see that the graph K_3 is embedded at a regular point. Instead, Figure 1(b) shows the graph K_3 embedded at a point that is not regular.

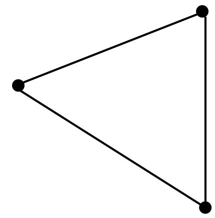
The idea of infinitesimal rigidity is to allow the vertices to move infinitesimally, while keeping the rigidity function constant up to first order. Let δp be an infinitesimal motion of the framework (G, p). Then the Taylor series expansion of g_G about p is

$$g_G(p + \delta p) = g_G(p) + J_{g_G}(p)\delta p + \text{ higher order terms.}$$

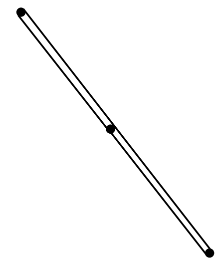
The rigidity function remains constant up to first order when $J_{g_G}(p)\delta p = 0$, that is, when δp belongs to $\mathrm{Ker} J_{g_G}(p)$. The dimension of this kernel is at least 3 because $g_G(p)$ will not change if p is perturbed by a rigid body motion. Infinitesimal rigidity is when the dimension of the kernel is not larger than 3.

Definition 5: ([2]) A framework (G, p) is *infinitesimally rigid* in the plane if $\dim(\mathrm{Ker} J_{g_G}(p)) = 3$, or equivalently if

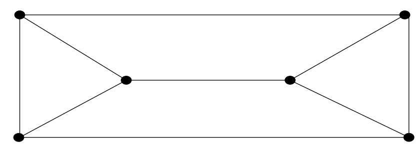
$$\mathrm{rank} J_{g_G}(p) = 2n - 3.$$



(a) A rigid and infinitesimally rigid framework.



(b) A rigid but not infinitesimally rigid framework.



(c) A rigid but not infinitesimally rigid framework.

Fig. 1.

If a framework is infinitesimally rigid, then it is also rigid. The converse is not true. The following theorem outlines when rigidity and infinitesimal rigidity are equivalent.

Theorem 6: ([2]) A framework (G, p) is infinitesimally rigid if and only if (G, p) is rigid and p is a regular point.

Observe that for a graph to be infinitesimally rigid in the plane it must have at least $2n - 3$ edges. If it has exactly $2n - 3$ edges, we say that the graph is *minimally rigid*.

The two different embeddings of K_3 shown in Figure 1(a)-(b) illustrate some of the rigidity properties. Both frameworks shown are embeddings of the complete graph. They are both rigid and globally rigid. The framework shown in Figure 1(a) is also infinitesimally rigid. If we check the rigidity matrix for any point p where the vertices are not collinear we will find it has rank 3. The framework in Figure 1(b) is not infinitesimally rigid. We can check this using the rigidity matrix. Let the embedding of the points in the plane be $z_1 = (0, 0)$, $z_2 = (0, 1)$, $z_3 = (0, 2)$. The rigidity function for this graph is

$$g_G(z) = \begin{bmatrix} ||z_1 - z_2||^2 \\ ||z_2 - z_3||^2 \\ ||z_3 - z_1||^2 \end{bmatrix}.$$

Then

$$J_{g_G}(p) = 2 \begin{bmatrix} z_1^T - z_2^T & z_2^T - z_1^T & 0 \\ 0 & z_2^T - z_3^T & z_3^T - z_2^T \\ z_1^T - z_3^T & 0 & z_3^T - z_1^T \end{bmatrix}.$$

If we check the rank at a collinear point p we obtain rank $J_{g_G}(p) = 2 < 2n - 3$. As the rigidity matrix does not have maximal rank, p is not a regular point; consistent with Theorem 6, a rigid framework is not infinitesimally rigid at a non-regular point.

In general, frameworks that are rigid but fail to be infinitesimally rigid have collinear or parallel edges. For instance the graph in Figure 1(c) is rigid but not infinitesimally rigid because the framework could undergo an infinitesimal distortion by perturbing the top link horizontally; the two triangles would then rotate infinitesimally, and the middle link rotate infinitesimally.

III. Problem Formulation

Consider n robots in the plane, $\mathbb{R}^2$. The robots are wheeled vehicles with sensors that allow them to measure the relative positions of some of the other vehicles. Such data can be obtained using a camera or a radar system. The simplest model for a wheeled vehicle is the kinematic unicycle. To simplify the analysis, using a standard procedure we assume the unicycle model has been feedback linearized about a point some distance in front of each unicycle. The robots then have a point kinematic model given by the differential equation

$$\dot{z}_i = u_i, \quad i \in \{1, \dots, n\} \tag{1}$$

where $z_i = (x_i, y_i) \in \mathbb{R}^2$ is the location of the ith robot in the plane and $u_i \in \mathbb{R}^2$ is the control input for the ith robot. We define the composite state vector $z = (z_1, \dots, z_n)$, as a vector in $(\mathbb{R}^2)^n$.

The *target formation* is described by a pair $\{G, d\}$ where G is an undirected graph whose vertices represent the robots, and vector $d \in \mathbb{R}^m$ specifies m target lengths for the edges. We refer to G as the *formation graph*. The robots achieve the target formation when the length of edge i is the prescribed distance $d_i > 0$.

Associated with the formation control problem is also a *sensor graph* that describes the sensor data seen by each robot in the closed-loop system. The sensor graph is a directed graph with each robot represented as a vertex in the graph. Given a controller u, if u_i is a function of z_j, then the sensor graph will have an edge from vertex i to vertex j. Also, we require that the control be a function only of relative measurements. For example if robot 1 can see robots 3 and 5, then the measurements available to robot 1 are $z_3 - z_1$ and $z_5 - z_1$, and u_1 can be a function of these two measurements. We refer to this as a *distributed control law*. We have the following problem.

Problem 1: Given the system (1) and a target formation $\{G, d\}$ such that $g_G^{-1}(d) \neq \emptyset$ and such that the framework (G, p) is infinitesimally rigid at each $p \in g_G^{-1}(d)$, design a distributed control law u whose sensor graph is G such that every $p \in g_G^{-1}(d)$ is a stable equilibrium of the closed-loop system.

IV. Gradient Control

In this section we propose a controller to solve Problem 1. We start with the framework (G, p). It has certain edges joining certain vertices. Using exactly the same link structure, define relative positions between robot positions, that is, define $e_i = z_k - z_j$, where p_k, p_j are linked on the framework. Without loss of generality $j < k$. Notice that e_i is an error vector in the direction of edge i and $\|e_i\|^2$ is the ith term in the rigidity function, $g_G(z)$.[1] We also form the composite vector $e = (e_1, \dots, e_m) \in \mathbb{R}^{2m}$. This vector is a linear function of z via the incidence matrix, $H \in \mathbb{R}^{m \times n}$, of the graph G; namely, with the definition

$$\hat{H} := H \otimes I_2 \in \mathbb{R}^{2m \times 2n}, \tag{2}$$

we have

$$e = \hat{H} z. \tag{3}$$

A. Control Law

We now consider a gradient control law to maintain an arbitrary formation of robots. First we define a vector norm function $v : \mathbb{R}^{2m} \to \mathbb{R}^m$:

$$v(e) = (||e_1||^2, \dots, ||e_m||^2).$$

Then using (3) we define $g : \mathbb{R}^{2n} \to \mathbb{R}^m$ by

$$g(z) := v(e) = v(\hat{H} z). \tag{4}$$

Notice that $g(z)$ is precisely the rigidity function $g_G(z)$ (henceforth the subscript is dropped).

As a candidate potential function, we consider the positive definite function of $g(z) - d$

$$\phi(z) = \frac{1}{2} \|g(z) - d\|^2. \tag{5}$$

Note that $\phi(z)$ is a positive semidefinite function of z and $\phi(z) = 0$ if and only if $g(z) = d$. We propose the gradient control

$$u = -(\nabla \phi(z))^T.$$

It follows from (1) and applying the chain rule to (5) that

$$\begin{aligned} \dot{z} &= -\left(J_g(z)\right)^T \left(g(z) - d\right) \\ &= -\hat{H}^T J_v(\hat{H} z)^T (v(\hat{H} z) - d) \\ &= -\hat{H}^T J_v(e)^T (v(e) - d)\,, \end{aligned} \tag{6}$$

where the Jacobian of v is

$$J_v(e) = 2 \begin{bmatrix} e_1^T & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e_m^T \end{bmatrix}. \tag{7}$$

It is evident that the control is a function only of the relative measurements, as required by the problem specification. More specifically, the control law for each robot is

$$\dot{z}_i = u_i = - \sum_{j \in \{\text{edges leaving } i\}} \frac{1}{2} (\|e_j\|^2 - d_j) e_j, \tag{8}$$

consistent with the problem specification that the sensor graph be identically the same as the formation graph. In the following lemma we list further interesting properties of the controlled system (6). Proofs are omitted since the results are easily verified.

Lemma 7:

1) The centroid $z^\circ := \frac{1}{n} \sum_{i=1}^n z_i$ is stationary: $\dot{z}^\circ = 0$.

[1] The notation e_i is used to refer both to the edge i and as an error vector pointing in the direction of edge i in the framework.

2) The control in (6) is independent of the system of global coordinates.
3) The *collinear set* $\mathcal{C} := \{z \in \mathbb{R}^{2n} \mid (\exists w \in \mathbb{R}^2)(\forall i)\ (z_i - z^\circ) \in \text{span}(w)\}$ is invariant under (6).
4) Solutions of (6) exist and are unique.

B. Coordinate Transformation

In this section we perform a coordinate transformation that separates the centroid dynamics from the remaining dynamics of the system. Let P be an orthonormal matrix whose first two rows are $\frac{1}{n}\mathbf{1}^T \otimes I_2$. Then consider the transformation $\tilde{z} = \begin{bmatrix} z^\circ \\ \overline{z} \end{bmatrix} = Pz$, where z° is the centroid of z, as discussed in Lemma 7. Define $\tilde{H} = \hat{H}P^{-1}$. From the definition of $\tilde{H}$ it is clear that $\tilde{H}\tilde{z} = \hat{H}z$. We now solve for the $\tilde{z}$ dynamics, obtaining

$$\dot{\tilde{z}} = P\dot{z} = -\tilde{H}^T \left(J_v(\tilde{H}\tilde{z})\right)^T (v(\tilde{H}\tilde{z}) - d). \tag{9}$$

So, $\dot{\tilde{z}} = -[\nabla\tilde{\phi}(\tilde{z})]^T$, where $\tilde{\phi}(\tilde{z}) = \frac{1}{2}\|v(\tilde{H}\tilde{z}) - d\|^2$.

Next we consider an interesting property of $\tilde{H}$. Note that since the first two columns of P^{-1} are in Ker($\hat{H}$), $\tilde{H}$ has the form $\begin{bmatrix} 0 & \overline{H} \end{bmatrix}$. From Lemma 1, dim(Ker($H$) = 1, so dim(Ker($\hat{H}$) = 2. Then by using the dimension of Ker($\hat{H}$), the invertibility of P, and the block form of $\tilde{H}$, we know that Ker($\overline{H}$) = $\{0\}$.

Now expand $\tilde{H}\tilde{z} = \begin{bmatrix} 0 & \overline{H} \end{bmatrix}\begin{bmatrix} z^\circ \\ \overline{z} \end{bmatrix} = \overline{H}\overline{z}$. So the $\tilde{z}$ dynamics from (9) can be rewritten as

$$\begin{bmatrix} \dot{z}^\circ \\ \dot{\overline{z}} \end{bmatrix} = -\begin{bmatrix} 0 \\ \overline{H}^T \end{bmatrix} \left(J_v(\overline{H}\overline{z})\right)^T (v(\overline{H}\overline{z}) - d). \tag{10}$$

If we define $\overline{\phi}(\overline{z}) := \frac{1}{2}\|v(\overline{H}\overline{z}) - d\|^2$ then $\dot{\overline{z}} = -(\nabla\overline{\phi}(\overline{z}))^T$, and so $\overline{z}$ is again a gradient system.

C. Existence and Uniqueness of Solutions

Using the coordinate transformation of the previous section it is possible to confirm existence and uniqueness of solutions in the $(z^\circ, \overline{z})$ coordinates. The z° dynamics and the $\overline{z}$ dynamics are decoupled, so we can analyze solutions independently. From Lemma 7 we know that $\dot{z}^\circ = 0$ so solutions trivially exist for all time. The dynamics of $\overline{z}$ evolve according to a gradient system with potential function $\overline{\phi}(\overline{z})$, a radially unbounded function. Consider the sublevel set

$$\mathcal{U}_a := \{\ \overline{z} \in \mathbb{R}^{2n-2} \mid \overline{\phi}(\overline{z}) \leq a\}$$

and define a Lyapunov function to be $V(\overline{z}) := \overline{\phi}(\overline{z})$. Denote by $-L_{\nabla\overline{\phi}}V(\overline{z})$ the Lie derivative of $-\nabla\overline{\phi}(\overline{z})^T$. For the $\overline{z}$ system $-L_{\nabla\overline{\phi}}V(\overline{z}) = -\|\nabla\overline{\phi}(\overline{z})\|^2$, a negative semidefinite function. So the set $\mathcal{U}_a$ is invariant for any $a > 0$. Furthermore, on the set $\mathcal{U}_a$, the function $\nabla\overline{\phi}(\overline{z})$ is Lipschitz continuous. Therefore, solutions $\overline{z}(t)$ exist for all time and are unique, for all initial conditions starting in $\mathcal{U}_a$.

V. Stability Results

In this section we present our main stability result. To begin, the following assumption is crucial to our approach.

Assumption 8: Given a target formation $\{G, d\}$, we assume that $g_G^{-1}(d) \neq \emptyset$ and the framework (G, p) is infinitesimally rigid at each $p \in g_G^{-1}(d)$.

A. Equilibria

We are interested in studying the equilibria of (6). First we have the equilibrium set $\mathcal{E}_1 = g^{-1}(d)$, which represents the desired formations as specified by the formation graph:

$$\mathcal{E}_1 := \{z \mid g(z) - d = 0\ \} \equiv \{z \mid \phi(z) = 0\ \}.$$

Unfortunately, these are not the only equilibria of (6). There is also a larger set of equilibria $\mathcal{E}_2 := \{z \mid J_v(\hat{H}z)^T(g(z) - d) = 0\ \}$. The matrix

$$J_v(\hat{H}z)^T = 2\begin{bmatrix} e_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ & \dots & e_m \end{bmatrix}$$

has a nontrivial kernel if and only if some $e_i = 0$, that is, two robots are collocated. So for a point z to be an equilibrium in $\mathcal{E}_2$, each $||e_i||^2 = d_i$ or $||e_i||^2 = 0$. Finally the complete set of equilibria of (6) is $\mathcal{E} = \{z \mid \nabla\phi(z) = 0\ \}$. Notice that $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}$. Simulation has shown that, in general, $\mathcal{E}_2 \neq \mathcal{E}$. These extra equilibria are not unexpected: The matrix $\hat{H}^T$ is $2n \times 2m$, so if $m > n$, then $\hat{H}^T$ has a nontrivial kernel. In particular, the set $\mathcal{E}$ includes equilibria where the robots are collinear.

It is also possible to define equilibrium sets for the reduced state $\overline{z}$. In particular, the desired target formations are

$$\overline{\mathcal{E}}_1 = \{\ \overline{z} \in \mathbb{R}^{2N-2} \mid v(\overline{H}\overline{z}) = d\ \}.$$

The advantage of using $\overline{\mathcal{E}}_1$ rather than $\mathcal{E}_1$ in the ensuing stability analysis is that (it is easily shown that) $\overline{\mathcal{E}}_1$ is compact, whereas $\mathcal{E}_1$ is not.

To conclude this section, we examine some of the algebraic and geometric properties of $\mathcal{E}_1 = g^{-1}(d)$. First, observe that $\mathcal{E}_1$ is a real algebraic variety, since it is the intersection of the zero level sets of polynomial functions. This implies it has a finite number of connected components [7]. Under Assumption 8, $\mathcal{E}_1$ inherits further properties summarized in the following lemma.

Lemma 9: If Assumption 8 holds, a set $\mathcal{S} \subset g^{-1}(d)$ is a topologically connected component of $g^{-1}(d)$ if and only if for each $p, p' \in g^{-1}(d)$, p and p' are related by a combination of rotations and translations of $\mathbb{R}^2$, and $\mathcal{S}$ is maximal with respect to rotations and translations. Moreover, $\mathcal{E}_1$ is a three dimensional embedded submanifold of $\mathbb{R}^{2n}$.

B. Linearized Dynamics

In order to study the stability of the equilibrium manifold $\mathcal{E}_1$, we will consider the linearized z-dynamics on $\mathcal{E}_1$.

Theorem 10: The matrix $J_f(z)$ evaluated at a point on $\mathcal{E}_1$ has three zero eigenvalues; the rest are real and negative.

Proof: Let $z_0 \in \mathcal{E}_1$ and define $e_0 = \hat{H} z_0$. Also, let $f(z) = -J_g(z)^T(g(z) - d)$, the vector field for the z dynamics. Applying the product rule to f and using the fact that $g(z_0) - d = 0$ it follows that

$$J_f(z_0) = -J_g(z_0)^T J_g(z_0)\,. \tag{11}$$

The matrix $J_f(z_0)$ is symmetric and thus has real eigenvalues, and also $\mathrm{Ker}(J_f(z_0)) = \mathrm{Ker}(J_g(z_0))$. The function $g(z)$ is the rigidity function for graph G and $J_g(z)$ is the rigidity matrix, so by Assumption 8, the rank of $J_g(z)$ is $2n - 3$ at all points on $\mathcal{E}_1$. Therefore, $\dim(\mathrm{Ker} J_g(z_0)) = 3$, so $J_f(z_0)$ has three zero eigenvalues. Moreover, the structure of $J_f(z_0)$ implies that it is a negative semidefinite matrix, so the non-zero eigenvalues are negative. ■

The previous results can also be extended to the reduced system $\dot{\overline{z}} = -(\nabla\overline{\phi}(\overline{z}))^T$. Let $\overline{f}(\overline{z}) := -\overline{H}^T \left(J_v(\overline{H}\overline{z})\right)^T (v(\overline{H}\overline{z}) - d)$. Also define the function $\overline{g} : \mathbb{R}^{2n-2} \to \mathbb{R}^m$ by $\overline{g}(\overline{z}) := v(\overline{H}\overline{z})$.

Corollary 11: The matrix $J_{\overline{f}}(\overline{z})$ evaluated at a point on $\overline{\mathcal{E}}_1$ has one zero eigenvalue; the rest are real and negative.

C. Main Result

Let $\mathcal{S} \subset \mathbb{R}^\nu$ be a set and $x \in \mathbb{R}^\nu$ a point. Then the point to set distance is $\mathrm{dist}(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} ||x - y||$. With respect to a dynamical system with state x we say a set $\mathcal{S}$ is *stable* if $(\forall \epsilon > 0)(\exists \delta > 0)$ $\mathrm{dist}(x(0), \mathcal{S}) < \delta \Rightarrow (\forall t \geq 0)$ $\mathrm{dist}(x(t), \mathcal{S}) < \epsilon$. We say a set $\mathcal{S}$ is *locally asymptotically stable* if it is stable and if $(\exists \delta > 0)$ $\mathrm{dist}(x(0), \mathcal{S}) < \delta \Rightarrow \lim_{t \to \infty} \mathrm{dist}(x(t), \mathcal{S}) = 0$.

Next we review center manifold theory. Consider a system in normal form

$$\dot{\theta} = A\theta + f_1(\theta, \rho) \tag{12}$$

$$\dot{\rho} = B\rho + f_2(\theta, \rho), \tag{13}$$

where $\theta \in \mathbb{R}^{\nu-\kappa}$, $\rho \in \mathbb{R}^\kappa$, A has eigenvalues only on the imaginary axis, B is Hurwitz, $f_1(0,0) = 0$ and $f_2(0,0) = 0$. The C^∞ functions f_1 and f_2 are restricted in order such that $J_{f_1}(0,0) = 0$ and $J_{f_2}(0,0) = 0$. An invariant manifold $\mathcal{M}$ is a *center manifold* of (12)-(13) if it can be locally represented as

$$\mathcal{M} := \{\ (\theta, \rho) \in \mathcal{U} \mid \rho = h(\theta)\ \}$$

where $\mathcal{U}$ is a sufficiently small neighbourhood of the origin, $h(0) = 0$, and $J_h(0) = 0$. It can be shown that a center manifold always exists [4] and the dynamics of (12)-(13) restricted to the center manifold are

$$\dot{\xi} = A\xi + f_1(\xi, h(\xi)) \tag{14}$$

for a sufficiently small $\xi \in \mathbb{R}^{\nu-\kappa}$. The stability of the system (12)-(13) can then be analyzed from the dynamics on the center manifold using the next theorem.

Theorem 12: ([8], p. 195) If the origin is stable under (14), then the origin of (12)-(13) is also stable. Moreover there exists a neighbourhood $\mathcal{W}$ of the origin such that for every $(\theta(0), \rho(0)) \in \mathcal{W}$ there is a solution $\xi(t)$ of (14) and constants $c_i > 0$, $\gamma > 0$ such that

$$\theta(t) = \xi(t) + r_1(t)$$
$$\rho(t) = h(\xi(t)) + r_2(t),$$

where $\|r_i(t)\| < c_i e^{-\gamma t}$.

The following is our main result.

Theorem 13: (Main Result) Suppose Assumption 8 holds. Then $\mathcal{E}_1$ is locally asymptotically stable. Moreover, there exists a neighborhood $\mathcal{U}$ of $\mathcal{E}_1$ such that for each $z(0) \in \mathcal{U}$ there exists a point $p \in \mathcal{E}_1$ where $\lim_{t\to\infty} z(t) = p$.

Proof: To prove $\mathcal{E}_1$ is stable we study the $(z^\circ, \overline{z})$ dynamics. First apply the linear transformation $P \in \mathbb{R}^{2n \times 2n}$ of Section IV-B to separate the system into $(z^\circ, \overline{z})$ components. The z° dynamics are stationary, so we study only the reduced $\overline{z}$ system. Without loss of generality assume $\overline{z}_0 = 0$. From Corollary 11 and the symmetry of $J_{\overline{f}}(0)$ we know there exists an orthonormal transformation $Q \in \mathbb{R}^{(2n-2)\times(2n-2)}$ such that $QJ_{\overline{f}}(0)Q^T$ is in block diagonal form with a zero for the first term and a block $B \in \mathbb{R}^{(2n-3)\times(2n-3)}$ that is Hurwitz. Then rewrite the $\overline{z}$ dynamics near $0 \in \overline{\mathcal{E}}_1$ as $\dot{\overline{z}} = J_{\overline{f}}(0)\overline{z} + (\overline{f}(\overline{z}) - J_{\overline{f}}(0)\overline{z})$ and define $(\theta, \rho) = Q\overline{z}$. Then it is easily verified that the (θ, ρ) dynamics have the form

$$\dot{\theta} \quad = \quad f_1(\theta, \rho) \tag{15}$$

$$\dot{\rho} \quad = \quad B\rho + f_2(\theta, \rho)\,, \tag{16}$$

where $f_1(0,0) = 0$ and $f_2(0,0) = 0$, and $J_{f_1}(0,0) = 0$ and $J_{f_2}(0,0) = 0$.

Now we claim that $\mathcal{M} := \{(\theta, \rho) \mid (\exists \overline{z} \in \overline{\mathcal{E}}_1)\ (\theta, \rho) = Q\overline{z}\}$ is a center manifold for the system (15)-(16). First, $\mathcal{M}$ is invariant because it consists of equilibria of (15)-(16). Second it is tangent to the θ-axis at 0. This can be seen as follows. Let

$$\tilde{g}(\theta, \rho) := \overline{g}\left(Q^T \begin{bmatrix} \theta \\ \rho \end{bmatrix}\right).$$

Then $\mathcal{M} = \{(\theta, \rho) \mid \tilde{g}(\theta, \rho) - d = 0\}$. We must show that the row vectors $\{d\tilde{g}_1(0), \ldots, d\tilde{g}_m(0)\}$ that span the normal space of $\mathcal{M}$ at 0, have their first entry equal to zero. Now observe that

$$\begin{bmatrix} d\tilde{g}_1(0) \\ \vdots \\ d\tilde{g}_m(0) \end{bmatrix} = J_{\overline{g}}(0)Q^T\,,$$

so we must show that the first column of $J_{\overline{g}}(0)Q^T$ is zero. But this follows from the fact that the first entry of $QJ_{\overline{f}}(0)Q^T = -(J_{\overline{g}}(0)Q^T)^T(J_{\overline{g}}(0)Q^T)$ is zero. Thus, there exists a function $h(\theta)$ such that in a neighborhood $\mathcal{W}_0$ of 0

$$\mathcal{M} \cap \mathcal{W}_0 = \{(\theta, \rho) | \rho = h(\theta)\}.$$

Since $\mathcal{M}$ is an equilibrium manifold, we know that $f_1(\theta, h(\theta)) = 0$ on $\mathcal{W}_0$. It follows that the dynamics restricted to $\mathcal{M}$ are $\dot{\xi} = 0$, and thus $\xi(t) = \xi(0)$.

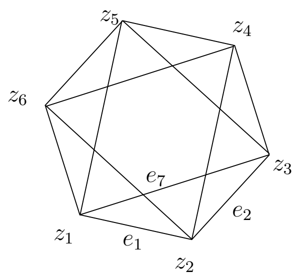


Fig. 2. The graph G_6^*.

Now applying Theorem 12, we obtain the solutions for (θ, ρ) starting in $\mathcal{W}_0$ are

$$\theta(t) = \xi(0) + r_1(t)$$
$$\rho(t) = h(\xi(0)) + r_2(t),$$

where $\|r_i(t)\| < c_i e^{-\gamma t}$ for some $c_1, c_2, \gamma > 0$. This implies $\lim_{t\to\infty}(\theta(t), \rho(t)) = (\xi(0), h(\xi(0)) \in \mathcal{M}$, so $\lim_{t\to\infty} \overline{z}(t) = Q^T(\xi(0), h(\xi(0))) \in \overline{\mathcal{E}}_1$, and $\lim_{t\to\infty} z(t) = P^{-1}(z^\circ(0), Q^T(\xi(0), h(\xi(0)))) \in \mathcal{E}_1$, as desired.

This argument can be repeated for each point on $\overline{\mathcal{E}}_1$ to obtain a cover $\{\mathcal{W}_k\}$ of $\overline{\mathcal{E}}_1$. Since $\overline{\mathcal{E}}_1$ is compact, we pass to a finite subcover to form a neighborhood of $\overline{\mathcal{E}}_1$. Local asymptotic stability of $\overline{\mathcal{E}}_1$ then follows. Finally, this argument can be trivially lifted to $\mathcal{E}_1$ since the center of mass dynamics are stationary. ∎

In summary, the infinitesimal rigidity of the formation graph was the key assumption in proving that the target set is an embedded submanifold and that the linearized dynamics have the required structure to apply center manifold theory.

VI. Regular Polygon Formations

An application of the formation stabilization control developed in the previous sections is to stabilize the robots to a regular polygon. A regular polygon is a useful formation for forming a large aperture antenna array.

Now consider a graph denoted G^* with n vertices and $2n$ edges, such that vertex i is connected to vertices $i+1$, $i+2$, $i-1$ and $i-2$. The graph G_6^* is shown in Figure 2. We order the edges in the graph so that the expanded incidence matrix $\hat{H} = H \otimes I_2 \in \mathbb{R}^{4n\times 2n}$ is $\hat{H} := \begin{bmatrix} I_{2n} - P^* \\ I_{2n} - (P^*)^2 \end{bmatrix}$. Note that $\hat{H} = \begin{bmatrix} I_{2n} \\ I_{2n} + P^* \end{bmatrix} (I_{2n} - P^*)$ thus if $e = \hat{H}z$ then $\begin{bmatrix} I_{2n} + P^* & -I_{2n} \end{bmatrix} e = 0$. Thus the components of e have a special form with $e_{i+n} = e_i + e_{i+1}$ for $i = 1, \ldots, n$. Let

$$d^* := \begin{bmatrix} c\mathbf{1} \\ c^*\mathbf{1} \end{bmatrix},$$

where $\sqrt{c} \in \mathbb{R}$ is the side length of the regular polygon and $c^* := 4c\cos^2\frac{\pi}{n}$. We assume that $c \neq 0$. If p is a point where the robots form a regular polygon, then $g_{G^*}(p) = d^*$. By construction, $g_{G^*}^{-1}(d^*) \neq \emptyset$. Techniques from graph theory can be used to show that the framework (G^*, p) is globally rigid and therefore, the robots located at $p \in \mathbb{R}^{2n}$ form a regular polygon if and only if $p \in g_{G^*}^{-1}(d^*)$. Thus, the regular polygon formation is the only formation in the set $\mathcal{E}_1$, with two distinct embeddings (up to translation and rotation), corresponding to reflections of each other. All that remains to be done to apply our theory is to check the rank of the rigidity matrix on $\mathcal{E}_1$.

Lemma 14: The framework (G^*, p) is infinitesimally rigid for all $p \in g_{G^*}^{-1}(d^*)$.

Proof: The rigidity matrix is $J_{g_{G^*}}(p) = J_v(e)\hat{H}$, with $e = \hat{H}p$. The graph G^* is connected, so from Lemma 1 we know that $\dim(\mathrm{Ker}(\hat{H})) = 2$. The strategy of the proof is to show that $\mathrm{Im}(\hat{H}) \cap \mathrm{Ker}(J_v(e)) = 1$, from which it follows that $\mathrm{rank}(J_{g_G^*}(p)) = 2n - \dim(\mathrm{Ker}(\hat{H})) - \dim(\mathrm{Im}(\hat{H}) \cap \mathrm{Ker}(J_v(e))) = 2n - 3$. Without loss of generality, we consider the counterclockwise embedding of G^*. Let $\xi := (\xi_1, \ldots, \xi_{2n}) \in \mathbb{R}^{4n}$, with $\xi_i \in \mathbb{R}^2$, be a vector of the form $\xi = (w, Rw, R^2w, \ldots, R^{n-1}w, (I+R)w, R(I+R)w, \ldots R^{n-1}(I+R)w)$. where $w \in \mathrm{Ker}(e_1^T)$, and $R \in \mathbb{R}^{2\times 2}$ is the rotation matrix by $2\pi/n$ radians. We claim that $\mathrm{Ker}(J_v(e)) = \mathrm{span}\{\xi\}$. Since $J_{g_G^*}(p)$ cannot have rank greater than $2n-3$ the result immediately follows.

From the geometry of the regular polygon we have that for $i = 1, \ldots, n$

$$e_i = R^{i-1}e_1\,, \tag{17}$$
$$e_{n+i} = R^{i-1}(I+R)e_1\,. \tag{18}$$

To show $\xi \in \mathrm{Ker}(J_v(e))$, we must show $e_i^T\xi_i = 0$, $i = 1, \ldots, 2n$. From (17) we have that $e_i^T\xi_i = (R^{i-1}e_1)^T(R^{i-1}w) = e_1^Tw = 0$, for $i = 1, \ldots, n$. From (18) we have that $e_{n+i}^T\xi_{n+i} = (R^{i-1}(I+R)e_1)^T(R^{i-1}(I+R)w) = 0$, for $i = 1, \ldots, n$, as desired. Conversely, suppose $\eta \in \mathrm{Ker}(J_v(e))$; that is, $e_i^T\eta_i = 0$, and $e_{n+i}^T\eta_{n+i} = 0$, for $i = 1, \ldots, n$. But this immediately implies, from the geometry of the plane, that $\eta_i = R^{i-1}\eta_1$ and $\eta_{n+i} = R^{i-1}(I+R)\eta_1$, for $i = 1, \ldots, n$, with $e_1^T\eta_1 = 0$, as desired. ∎

Since graph G^* forms an infinitesimally rigid framework at regular points, our gradient control can be applied to stabilize a regular polygon.

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