Resolving Control to Facet Problems for Affine Hypersurface Systems on Simplices

Zhiyun Lin and Mireille E. Broucke

***Abstract*— We study control to facet problems for affine hypersurface systems on simplices. An easily verifiable necessary and sufficient condition is derived that allows one to determine whether there is a linear affine feedback driving all trajectories of the closed-loop system out of the simplex through a desired facet. In cases where the condition is violated, methods are developed to establish the largest subset of the simplex such that there is still a feedback control (not necessarily linear) which can steer all trajectories starting from this set to the desired facet without crossing other facets. The results are demonstrated on a circuit with a piecewise linear resistor.**

I. Introduction

The control to facet problem for systems on polytopes is to find, if possible, a feedback control such that all trajectories of the closed-loop system leave the polytope via some desired facets. The problem was first introduced by Habets and van Schuppen in [4]. It is a subproblem of reachability analysis of hybrid systems. That is, does there exist a controller driving a system from an initial operating region to a desired one while not entering an unsafe region? Partitions of the state space of the system with adjacent polytopic cells are made in terms of either the piecewise linear nature of the system itself, the control specification such as safety or reachability, or the switching control paradigm in which different controllers operate in each region of the state space. What arises is a class of hybrid systems. The reachability problem of hybrid systems has been widely studied over the past decade. More recently, the control to facet problem has received considerable attention in reachability analysis, see for example, [1]–[3], [5]–[7], [12]–[14].

In this paper we restrict our attention to the study of affine hypersurface systems; that is, linear affine systems with n-dimensional state and $n-1$ independent control inputs. Such systems received significant attention in controllability studies (see [8], [9]) as a first step in solving a more general setup. For the same reason, this paper is devoted to the control to facet problem for affine hypersurface systems. The contributions of the paper are threefold. We redevelop an easily verifiable necessary and sufficient condition for existence of a linear affine feedback control solving the problem, which requires only a few algebraic steps. The result improves that of [14] where necessary and sufficient conditions for construction of a linear affine feedback only are given. Here we clarify the role of linear affine feedback in obtaining new necessary and sufficient conditions. A second contribution is that, if the conditions fail, we find the largest subset of the simplex such that there is still a feedback control (not necessarily continuous or linear affine as required in [5], [6], [12]) which can steer all trajectories starting from this set to a selected facet without crossing other facets. This result is important toward refinement of partitions in order to solve the reachability problem in general. Our approach extends results in [11] where a classification of failures called vertex, facet, and region failures is introduced and analyzed. Third, the analysis also produces a necessary and sufficient condition for solving the control to facet problem, in contrast to the results in [5], [6], [12] where it is only sufficient. We show that if there is a control solving the problem then a piecewise linear affine control will do so.

The authors are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: linz@control.utoronto.ca; broucke@control.utoronto.ca). Supported by the Natural Sciences and Engineering Research Council of Canada.

II. Preliminaries and Problem Description

Consider an n-simplex in $\mathbb{R}^n$ described by $\mathcal{S} = \text{co}\{v_1, \ldots, v_{n+1}\}$, where $v_1, \ldots, v_{n+1}$ are its vertices and $\text{co}\{\cdots\}$ represents the convex hull. The convex hull of any $m+1$ of the $n+1$ points is also a simplex, called an *m-face*. The $(n-1)$-faces are called the *facets*. For $i \in \bar{I} := \{1, \ldots, n+1\}$, denote the facet $\text{co}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}\}$ by $\mathcal{F}_i$ (which is indexed by the vertex it does not contain). Let h_i denote the corresponding normal vector and by convention, h_i is of unit length and points out of the simplex. Write $H = [h_1 \cdots h_n] \in \mathbb{R}^{n \times n}$ and let $H_i \in \mathbb{R}^{n \times (n-1)}$ be the matrix obtained from H by removing the i-th column.

Remark 1: With the above conventions, we can see that for every $j \neq i$: $h_i \cdot (x - v_j) = 0$ for every $x \in \mathcal{F}_i$ and $h_i \cdot (x - v_j) < 0$ for every $x \in \mathcal{S} - \mathcal{F}_i$. In addition, $\{h_1, \ldots, h_n\}$ are linearly independent (i.e., H is invertible).

We now consider a linear affine hypersurface system on the simplex $\mathcal{S}$. As long as the state x is in $\mathcal{S}$, the system is governed by the affine differential equation

$$\dot{x} = Ax + a + Bu, \quad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n \times (n-1)}$ with $\text{rank}(B) = n-1$. We consider the following problems.

Problem 1: Consider system (1). Find, if possible, a feedback $u = f(x)$, $f : \mathcal{S} \to \mathbb{R}^{n-1}$ such that for every $x_0 \in \mathcal{S}$ there exists a unique solution $x(t, x_0)$ and there exist $t_1 \geq 0$ and $\epsilon > 0$ satisfying

(i) $x(t, x_0) \in \mathcal{S}$ for all $t \in [0, t_1]$,
(ii) $x(t_1, x_0) \in \mathcal{F}_{n+1}$,
(iii) $x(t, x_0) \notin \mathcal{S}$ for all $t \in (t_1, t_1 + \epsilon)$.

The problem requires that all solutions of the closed-loop system leave the simplex $\mathcal{S}$ in finite time through only one specified facet, called the *exit facet*, that without loss of generality, is $\mathcal{F}_{n+1}$. In addition, no trajectory of the closed-loop system exits through other facets, which are called *restricted facets*. We denote $I = \bar{I} - \{n+1\}$ the index set of restricted facets. The problem was first studied in [6] and [12], but only linear affine control $u = Fx+g$ is considered. In this paper we study the existence of any type control (not necessarily linear affine feedback) to solve the problem. But first we reformulate a necessary and sufficient condition for the existence of linear affine feedback to solve Problem 1 from [14]. The proof is omitted due to page limitations.

Let β be a unit length vector normal to the range of B, denoted by $\mathcal{B}$. Since $\{h_1, \dots, h_n\}$ are linearly independent and span $\mathbb{R}^n$, β can be written as a linear combination $\beta = \lambda_1 h_1 + \cdots + \lambda_n h_n$, or in matrix form, $\beta = H\lambda$, where $\lambda = [\lambda_1 \ \cdots \ \lambda_n]^T$. Note that β is not unique (it can point in two opposite directions). In the case when β can be spanned by either a non-positive or non-negative combination of $h_1, \dots, h_n$, *we choose* β *to be the non-positive combination*. In what follows, we denote by $\mathcal{B}_x$ the hyperplane parallel to $\mathcal{B}$ passing through the point $x \in \mathbb{R}^n$.

Let $\lambda_{max} := \max\{\lambda_1, \dots, \lambda_n\}$. We give an interpretation for the possible signs of λ_{max} which elucidates the geometrically distinct cases that appear in the main theorem of this section. $\lambda_{max} > 0$ if and only if $\mathcal{B}_{v_{n+1}}$ is not a supporting plane for the simplex $\mathcal{S}$ (see Fig. 1(i)). When $\lambda_{max} = 0$, we let $J = \{i \in I : \lambda_i = 0\}$ and let m be the cardinality of J (clearly, $1 \le m \le n-1$). $\lambda_{max} = 0$ if and only if $\mathcal{B}_{v_{n+1}}$ is a supporting plane of $\mathcal{S}$ containing at least two vertices. Specifically, it contains the vertex v_{n+1} and the vertices $v_i, i \in J$ (that is, it touches an m-face of the simplex) (see Fig. 1(ii)). Condition $\lambda_{max} < 0$ occurs if and only if $\mathcal{B}_{v_{n+1}}$ is a supporting plane of $\mathcal{S}$ containing only one vertex v_{n+1}. (see Fig. 1(iii)).

Theorem 1: There exists a linear affine feedback $u = Fx+g$ solving Problem 1 if and only if one of the following conditions holds:

(i) $\lambda_{max} > 0$;
(ii) $\lambda_{max} = 0$ and $\beta \cdot (Av_i + a) \ge 0$ for all $i \in J \cup \{n+1\}$;
(iii) $\lambda_{max} < 0$, $\beta \cdot (Av_{n+1} + a) > 0$, and $\exists k \in I$ such that $\beta \cdot (Av_k + a) > 0$.

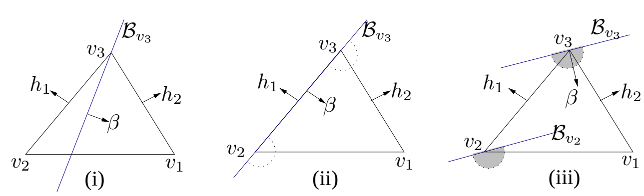

Fig. 1. Illustration for Theorem 1 in 2D.

Theorem 1 provides a very simple condition to check the existence of linear affine control solving Problem 1. Once one of the conditions in Theorem 1 is satisfied, one can choose a vector ξ based on the following rules: if $\lambda_{max} > 0$, set ξ to be any one of $\{h_i : i \in I\}$; if $\lambda_{max} = 0$, set ξ to be any one of $\{h_i : i \in J\}$; if $\lambda_{max} < 0$, set ξ to be any one of $\{h_i : i \in I$ such that $\beta \cdot (Av_i + a) > 0\}$. Next solve the following linear inequalities at each vertex to obtain u_i:

$$\begin{cases} H_i^T (Av_i + a + Bu_i) \preceq 0 \\ \xi^T (Av_i + a + Bu_i) < 0 \end{cases} \quad \text{for } i \in I$$

and

$$\begin{cases} H^T (Av_{n+1} + a + Bu_{n+1}) \preceq 0 \\ \xi^T (Av_{n+1} + a + Bu_{n+1}) < 0. \end{cases}$$

Finally, a unique linear affine feedback $u = Fx + g$ can be constructed based on these correspondences $u_i = Fv_i + g$, $i = 1, \dots, n+1$ (see [6], [12]).

Instead, we would like to consider the following problems. If no linear affine control can solve Problem 1, does there exist some other feedback control (noncontinuous or nonlinear) to solve it? Furthermore, if no control can solve Problem 1, what is the feasible initial set such that there is a control to make all trajectories starting in this set leave the simplex through the exit facet?

Problem 2: Consider system (1) and a subset $\mathcal{X}$ of $\mathcal{S}$. Find, if possible, a feedback $u = f(x)$ such that for every $x_0 \in \mathcal{X}$ there exists a unique solution $x(t, x_0)$ and there exist $t_1 \ge 0$ and $\epsilon > 0$ satisfying the conditions (i–iii) of Problem 1.

If there is a subset $\mathcal{X}$ of $\mathcal{S}$ such that Problem 2 is solvable by any control (not necessarily continuous linear affine), then it is called a *feasible set*; otherwise it is called a *failure set*. Clearly, if there is a controller so that all trajectories starting in some subset of $\mathcal{S}$ enter into a known feasible set without crossing restricted facets, then this subset is also a feasible set. Our goal is to find the largest such set $\mathcal{X}$ in $\mathcal{S}$.

III. Maximal Feasible Set

In this section, we focus on Problem 2 to find the maximal feasible initial set so that there is a control to drive the state from this set to the exit facet. From Theorem 1, we know that there is no linear affine feedback when one of the following happens:

1) $\lambda_{max} = 0$ and $\exists i \in J \cup \{n+1\}$ such that $\beta \cdot (Av_i + a) < 0$;
2) $\lambda_{max} < 0$ and additionally either $\beta \cdot (Av_{n+1} + a) \le 0$ or $\beta \cdot (Av_i + a) \le 0$ for all $i \in I$.

We first deal with the failure 1) which we call a *face failure*. Recall that when $\lambda_{max} = 0$, the hyperplane $\mathcal{B}_{v_{n+1}}$ contains only an m-face of $\mathcal{S}$ (where m is the cardinality of J). Specifically, this m-face can be represented by $\mathcal{E} = \text{co}\{v_{n+1}, v_i|_{i \in J}\}$, or simply $\mathcal{E} = \bigcap_{i \notin J \cup \{n+1\}} \mathcal{F}_i$.

Theorem 2: Suppose $\lambda_{max} = 0$ and $\exists i \in J \cup \{n+1\}$ such that $\beta \cdot (Av_i + a) < 0$. Then the feasible set solving Problem 2 is $\mathcal{X} = \mathcal{S} - \mathcal{E}$.

Proof: If necessary, we renumber the vertices $v_1, \dots, v_n$ so that $\lambda_1, \dots, \lambda_r < 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$, where $r = n - m < n$. Then $J = \{r+1, \dots, n\}$ and

$$\beta = \lambda_1 h_1 + \cdots + \lambda_r h_r. \tag{2}$$

Let $v'_{n+1} = (1-\varepsilon)v_{n+1} + \varepsilon v_1$, where $\varepsilon > 0$ is chosen to be sufficiently small. Then we have a new simplex $\mathcal{S}' =$

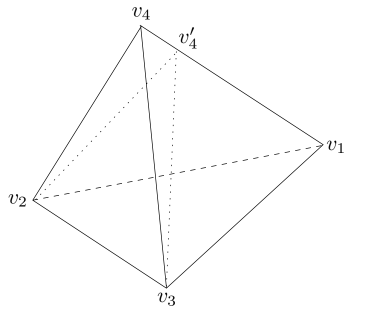

Fig. 2. New simplex $\mathcal{S}' = \text{co}\{v_1, v_2, v_3, v_4'\}$.

$\text{co}\{v_1, \ldots, v_n, v'_{n+1}\}$ inside the original one (see Fig. 2 for an example).

By (2) and Remark 1, it follows that

$$\begin{aligned} \beta \cdot (v_1 - v_{n+1}) &= (\lambda_1 h_1 + \cdots + \lambda_r h_r) \cdot (v_1 - v_{n+1}) \\ &= \lambda_1 h_1 \cdot (v_1 - v_{n+1}) > 0 \\ \beta \cdot (v'_{n+1} - v_n) &= (\lambda_1 h_1 + \cdots + \lambda_r h_r) \cdot (v'_{n+1} - v_n) \\ &= \lambda_1 h_1 \cdot (v'_{n+1} - v_n) > 0. \end{aligned}$$

Thus, from above we have

$$\begin{aligned} &\beta \cdot (v_1 - v'_{n+1}) = \beta \cdot (v_1 - (1-\varepsilon) v_{n+1} - \varepsilon v_1) > 0 \\ &\beta \cdot (v_n - v'_{n+1}) < 0. \end{aligned}$$

This means that $\mathcal{B}_{v'_{n+1}}$ is not a supporting hyperplane for the simplex $\mathcal{S}'$. Accordingly, for the linear affine system (1) on the simplex $\mathcal{S}'$, we have $\lambda_{max} > 0$. By Theorem 1, this shows there is a linear affine feedback control such that all the trajectories of the closed-loop system leave $\mathcal{S}'$ in finite time via the facet $\mathcal{F}'_{n+1}$, which is exactly $\mathcal{F}_{n+1}$. Furthermore, notice that $\mathcal{S}' \to \mathcal{S}$ as $\varepsilon \to 0$, so for any $x_0 \in \mathcal{S} - \mathcal{F}_1$, we can choose a sufficiently small $\varepsilon > 0$ so that $x_0 \in \mathcal{S}'$. This further implies that Problem 2 is solvable for the initial set $\mathcal{S} - \mathcal{F}_1$.

For $i = 1, \ldots, r$, let $v'_{n+1} = (1-\varepsilon) v_{n+1} + \varepsilon v_i$. By the same argument, we show that Problem 2 is solvable for the initial set $\mathcal{S} - \mathcal{F}_i$, $i = 1, \ldots, r$. That is, the feasible set solving Problem 2 is $\mathcal{X} = (\mathcal{S} - \mathcal{F}_1) \cup \cdots \cup (\mathcal{S} - \mathcal{F}_r) = \mathcal{S} - \mathcal{E}$. ■

Remark 2: Although the m-face $\mathcal{E}$ is called the failure m-face, it does not mean that for any initial state $x_0 \in \mathcal{E}$, Problem 2 is unsolvable. Indeed, if $\beta \cdot (Av_i + a) < 0$ for all $i \in J \cup \{n+1\}$, then for no initial state in $\mathcal{E}$ Problem 2 is solvable. Otherwise, there may exist some initial state in $\mathcal{E}$ such that Problem 2 is solvable. However, the failure set has to be in the m-face, $m \geq 1$, rather than in some lower dimensional face, because the failure arises from a condition of the form $\beta \cdot (Av_i + a) < 0$ which holds in an open neighborhood in $\mathbb{R}^n$.

Next we are going to study the failure 2). Before presenting our main result, we prove several useful lemmas first.

Lemma 1 ([11]): If $\beta = h_{n+1}$ and $\beta \cdot (Av_i + a) \leq 0$ for all $i \in I$, then the failure set is the whole simplex $\mathcal{S}$.

The lemma says if this condition holds, then for any initial state in $\mathcal{S}$ there is no control which can drive the state out of $\mathcal{S}$ through the facet $\mathcal{F}_{n+1}$ only. The next few lemmas establish some properties which can be used to find the largest feasible set and the failure set.

Lemma 2: Suppose $\lambda_{max} < 0$. Then for every $i \in I$

$$\beta \cdot (v_i - v_{n+1}) > 0.$$

Proof: Recall that for every $i \in I$, by Remark 1 the following holds:

$$h_i \cdot (v_j - v_{n+1}) = 0 \ \forall j \neq i \quad \text{and} \quad h_i \cdot (v_i - v_{n+1}) < 0.$$

Substituting $\beta = \lambda_1 h_1 + \cdots + \lambda_n h_n$, one obtains, for every $i \in I$,

$$\begin{aligned} \beta \cdot (v_i - v_{n+1}) &= (\lambda_1 h_1 + \cdots + \lambda_n h_n) \cdot (v_i - v_{n+1}) \\ &= \lambda_i h_i \cdot (v_i - v_{n+1}) > 0 \end{aligned}$$

because $\lambda_i \leq \lambda_{max} < 0$. ■

Since β is with unit length, we have $\beta \cdot (v_i - v_{n+1}) = \|v_i - v_{n+1}\| \cos(\theta)$ where θ is the angle between the vectors $v_i - v_{n+1}$ and β. This means $\beta \cdot (v_i - v_{n+1})$ is just the distance from the point v_{n+1} to the hyperplane $\mathcal{B}_{v_i}$. In addition, from Lemma 2, since $\beta \cdot (v_i - v_{n+1}) > 0$, then $\cos(\theta) > 0$ and that the angle $|\theta| < \pi/2$. Meanwhile, with $\lambda_{max} < 0$, it can be easily shown that the hyperplane $\mathcal{B}_{v_i}$ has a unique intersection point with every line containing v_{n+1}, v_j, $j \in I$. In what follows, for any two points p_1, p_2 in $\mathbb{R}^n$, we will use $[p_1, p_2]$, (p_1, p_2), and $(p_1, p_2]$ to represent a closed, open, and semi-open line segment joining these two points, respectively.

Lemma 3: Suppose $\lambda_{max} < 0$. Let $k = \arg\min_{i \in I} \beta \cdot (v_i - v_{n+1})$ and for every $i \in I$, let $v^i_{v_k}$ be the unique intersection point of the hyperplane $\mathcal{B}_{v_k}$ and the line through v_{n+1} and v_i. Then for every $i \in I$, $v^i_{v_k} \in (v_{n+1}, v_i]$.

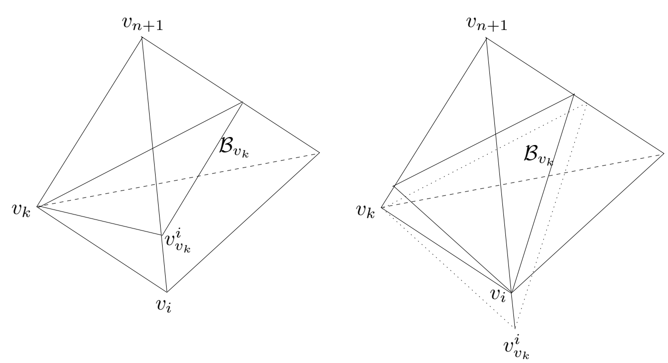

Fig. 3. Illustration for Lemma 3.

Proof: First we show that $v^i_{v_k}$ does not lie in the open semi-line leaving the point v_i. Assume by way of contradiction it does (see the right one in Fig. 3). This means $[v_{n+1}, v_i] \subset [v_{n+1}, v^i_{v_k}]$. So

$$\begin{aligned} &\beta \cdot (v_i - v_{n+1}) = \|v_i - v_{n+1}\| \cos(\theta) \\ &< \|v^i_{v_k} - v_{n+1}\| \cos(\theta) = \beta \cdot (v^i_{v_k} - v_{n+1}). \end{aligned}$$

On the other hand, notice that $\beta \cdot (v^i_{v_k} - v_{n+1}) = \beta \cdot (v_k - v_{n+1})$ because $v^i_{v_k}$ and v_k lie in the same hyperplane $\mathcal{B}_{v_k}$. Thus,

$$\beta \cdot (v_i - v_{n+1}) < \beta \cdot (v_k - v_{n+1}),$$

which contradicts $k = \arg\min_{i \in I} \beta \cdot (v_i - v_{n+1})$.

Second we show that $v^i_{v_k}$ does not lie in the closed semi-line leaving the point v_{n+1}. Suppose it does. Let θ be the angle between $v_i - v_{n+1}$ and β. Thus, the angle between $v^i_{v_k} - v_{n+1}$ and β is $\pi - \theta$ and therefore,

$$\beta \cdot (v^i_{v_k} - v_{n+1}) = \|v^i_{v_k} - v_{n+1}\| \cos(\pi - \theta).$$

The right-hand side of the above equation equals to 0 when $v^i_{v_k}$ equals v_{n+1} and it is less than 0 when $v^i_{v_k}$ is in the open semi-line leaving v_{n+1} because $|\theta| < \pi/2$. Considering that $v^i_{v_k}$ and v_k lie in the same hyperplane $\mathcal{B}_{v_k}$, so

$$\beta \cdot (v_k - v_{n+1}) = \beta \cdot (v^i_{v_k} - v_{n+1}) \leq 0,$$

which contradicts Lemma 2. ■

Now we introduce a new set $\mathcal{O}$ defined as

$$\mathcal{O} := \{x | \beta \cdot (Ax + a) = 0\}.$$

It can be easily shown that the set can be either a hyperplane in $\mathbb{R}^n$, the empty set, or the whole space $\mathbb{R}^n$. In case $\mathcal{O}$ is a hyperplane in $\mathbb{R}^n$, we let $\{o_1, \ldots, o_l\}$ be the set of intersection points of $\mathcal{O}$ and the line segments $[v_{n+1}, v^i_{v_k}], i \in I$. Clearly, the set has at most n elements. In addition, it may be empty. Furthermore, we introduce the notation $v^i_{o_j}$, $j \in \{1, \ldots l\}$ and $i \in I$, representing the intersection point of the line segment $[v_{n+1}, v^i_{v_k})$ and the hyperplane $\mathcal{B}_{o_j}$. An example is given in Fig. 4. Clearly, the

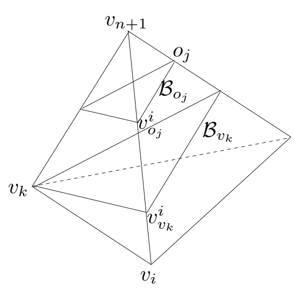

Fig. 4. An example.

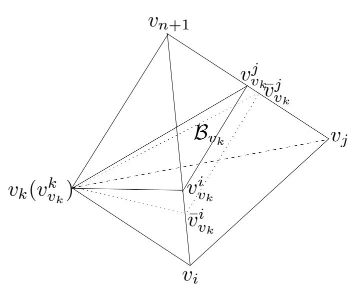

Fig. 5. Illustration.

hyperplane $\mathcal{B}_{o_j}$ is closer to the point v_{n+1} than $\mathcal{B}_{v_k}$. Hence, the top part $\text{co}\{v^1_{o_j}, \ldots, v^n_{o_j}, v_{n+1}\}$ again forms a simplex.

Theorem 3: Suppose $\lambda_{max} < 0$ and let

$$k = \arg\min_{i \in I} \beta \cdot (v_i - v_{n+1}).$$

(i) If $\beta \cdot (Av^i_{v_k} + a) \leq 0$ for all $i \in I$, then the feasible set solving Problem 2 is $\mathcal{X} = \mathcal{S} - \mathcal{S}'$, where $\mathcal{S}'$ is the simplex $\text{co}\{v^1_{v_k} \cdots v^n_{v_k}, v_{n+1}\}$;

(ii) Otherwise,

- (ii-a) if $\beta \cdot (Av_{n+1} + a) > 0$, then the maximal feasible set solving Problem 2 is $\mathcal{X} = \mathcal{S}$;
- (ii-b) if $\beta \cdot (Av_{n+1} + a) = 0$, then the maximal feasible set solving Problem 2 is $\mathcal{X} = \mathcal{S} - \{v_{n+1}\}$;
- (ii-c) if $\beta \cdot (Av_{n+1} + a) < 0$, then the maximal feasible set solving Problem 2 is $\mathcal{X} = \mathcal{S} - \mathcal{S}''$, where $\mathcal{S}''$ is the simplex $\text{co}\{v^1_{o_s} \cdots v^n_{o_s}, v_{n+1}\}$ and $s = \arg\min_{j \in \{1,\ldots,l\}} \beta \cdot (o_j - v_{n+1})$.

Proof: **(i)** If $\beta \cdot (Av^i_{v_k} + a) \leq 0$ for all $i \in I$, it is straightforward from Lemma 1 that the new simplex $\mathcal{S}' = \text{co}\{v^1_{v_k} \cdots v^n_{v_k}, v_{n+1}\}$ is a failure set to solve Problem 2 (see Fig. 5) since $\beta = h'_{n+1}$, the normal vector to the facet $\mathcal{F}'_{n+1}$ of $\mathcal{S}'$, by construction.

For the set $\mathcal{S} - \mathcal{S}'$, it is a polytope of at most $n - 1 + n$ points $v^1_{v_k}, \ldots, v^n_{v_k}$ and $v_1, \ldots, v_n$ since we know for sure at least two points are identical (that is, $v^k_{v_k} = v_k$). For every $i = 1, \ldots, n$, if $v^i_{v_k}$ is not identical to v_i, let $\bar{v}^i_{v_k} = (1-\varepsilon)v^i_{v_k} + \varepsilon v_i$, where $\varepsilon > 0$ is sufficiently small. Thus, we have a new polytope $\mathcal{P}$ composed of the points $v_1, \ldots, v_n$ and the points $\bar{v}^i_{v_k}$. As $\varepsilon \to 0$, the polytope $\mathcal{P}$ tends to $\mathcal{S} - \mathcal{S}'$. Applying the same argument as for (ii-c), conclusion follows.

(ii-a) If $\beta \cdot (Av_{n+1} + a) > 0$ and there is an $i \in I$ such that $\beta \cdot (Av^i_{v_k} + a) > 0$, then for the linear affine system (1) on the simplex $\mathcal{S}' = \text{co}\{v^1_{v_k} \cdots v^n_{v_k}, v_{n+1}\}$, condition (iii) of Theorem 1 holds. So there is a linear affine controller such that for all initial states in $\mathcal{S}'$, the trajectories leave $\mathcal{S}'$ through the facet $\text{co}\{v^1_{v_k} \cdots v^n_{v_k}\}$. In other words, they enter into $\mathcal{S} - \mathcal{S}'$. Then applying the same argument as for (i) shows that $\mathcal{S} - \mathcal{S}'$ is a feasible set to solve Problem 2 and so the largest feasible set is $\mathcal{X} = \mathcal{S}$. The control steering all the states out of $\mathcal{S}$ through the desired exit facet is indeed a piecewise linear affine controller.

(ii-b) By assumption there is $i \in I$ such that $\beta \cdot (Av^i_{v_k} + a) > 0$. Without loss of generality, say $i = 1$. Furthermore, $\beta \cdot (Av_{n+1} + a) = 0$, so $\mathcal{O}$ is a hyperplane in $\mathbb{R}^n$. Thus it can be easily shown by convexity argument that $\beta \cdot (Av_1 + a) > 0$.

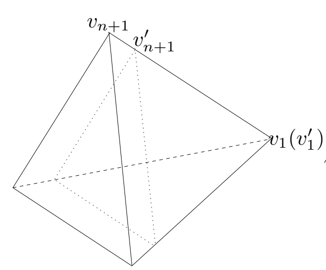

Fig. 6. New simplex $\mathcal{S}'$.

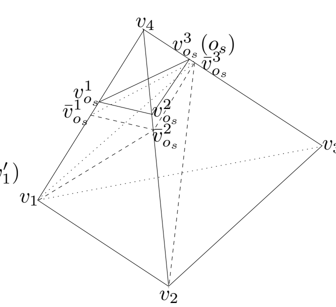

Fig. 7. Illustration.

Let $\varepsilon > 0$ be sufficiently small. We construct a new simplex $\mathcal{S}'$ (see Fig. 6) by defining

$$v'_1 = v_1, \quad v'_i = (1 - \varepsilon)v_i + \varepsilon v_1 \text{ for all } i \neq 1.$$

Clearly, the new simplex $\mathcal{S}'$ has the same normal vectors as the original one. By a convexity argument, it follows that $\beta \cdot (Av'_{n+1} + a) > 0$. Thus, for the linear affine system (1) on the new simplex $\mathcal{S}'$, the condition (iii) of Theorem 1 holds and therefore Problem 2 is solvable for all initial states in $\mathcal{S}'$ since $\mathcal{S}' \subset \mathcal{S}$ and $\mathcal{F}'_{n+1} \subset \mathcal{F}_{n+1}$.

Notice that $\mathcal{S}' \to \mathcal{S}$ as $\varepsilon \to 0$. So for any initial state $x_0 \in \mathcal{S} - \mathcal{F}_1$, we can always construct a simplex $\mathcal{S}'$ so that $x_0 \in \mathcal{S}'$, and therefore Problem 2 is solvable for all initial states in $\mathcal{S} - \mathcal{F}_1$.

Now consider an initial state $x_0 \in \mathcal{F}_1 - \{v_{n+1}\}$, which can be written as a convex combination

$$x_0 = \mu_2 v_2 + \cdots + \mu_{n+1} v_{n+1},$$

where $\mu_i \geq 0,\ i = 2, \ldots n+1$ but not all $\mu_2, \ldots, \mu_n$ are 0. Notice that λ is a negative solution to the equation $H\lambda = \beta$, so $H^T y \preceq 0$ and $\beta^T y = 0$ has only one solution $y = 0$. In other words, in order to make $H^T(Av_{n+1} + a + Bu_{n+1}) = H^T y = 0$, u_{n+1} is chosen so that $Av_{n+1} + a + Bu_{n+1} = y = 0$. Such a u_{n+1} exists because $\beta \cdot (Av_{n+1} + a) = 0$. Also, recall that for each $i \in I - \{1\}$, there is $u_i \in \mathbb{R}^{n+1}$ such that $H_i^T(Av_i + a + Bu_i) \prec 0$. Set $u = \mu_2 u_2 + \cdots + \mu_{n+1} u_{n+1}$. Combining the previous arguments, we obtain $h_1^T(Ax_0 + a + Bu) < 0$. Thus, it is easily shown that the trajectory $x(t, x_0)$ will leave $\mathcal{F}_1$ and enter into $\mathcal{S}$ in a very small time. Then by the previous argument, Problem 2 is again solvable.

Finally, from the above argument we know for whatever $u_{n+1} \in \mathbb{R}^{n-1}$, $H^T(Av_{n+1} + a + Bu_{n+1}) \succeq 0$, implying that the trajectory $x(t, x_0)$ with the initial state $x_0 = v_{n+1}$ either exits the simplex $\mathcal{S}$ via v_{n+1} or remains stationary for ever. So it fails to solve Problem 2 when $x_0 = v_{n+1}$.

(ii-c) If $\beta \cdot (Av_{n+1} + a) < 0$ and there is an $i \in I$ such that $\beta \cdot (Av_{v_k}^i + a) > 0$, then by convexity there is an o_j in the open line segment $(v_{n+1}, v_{v_k}^i)$. Next we are going to show $\beta \cdot (Av_{o_s}^i + a) \leq 0$ for all $i \in I$, where $s = \underset{j \in \{1, \ldots, l\}}{\arg\min}\, \beta \cdot (o_j - v_{n+1})$. Suppose by way of contradiction, there is an $i \in I$ such that $\beta \cdot (Av_{o_s}^i + a) > 0$. Again, by convexity, there is another o_j in the open line segment $(v_{n+1}, v_{o_s}^i)$. Clearly, $\beta \cdot (o_j - v_{n+1}) < \beta \cdot (o_s - v_{n+1})$, a contradiction.

Since we have already showed $\beta \cdot (Av_{o_s}^i + a) \leq 0$ for all $i \in I$, by Lemma 1, the simplex $\mathcal{S}''$ is a failure set to solve Problem 2. Next we show $\mathcal{S} - \mathcal{S}''$ is a feasible set.

Indeed, $\mathcal{S} - \mathcal{S}''$ is a polytope of $n + n$ points $v_{o_s}^1, \ldots, v_{o_s}^n$ (since o_j lies in the open line segment of v_{n+1} and $v_{v_k}^i$) and $v_1, \ldots, v_n$. Note that at least one point of $v_{o_s}^1, \ldots, v_{o_s}^n$ lies in $\mathcal{O}$ by construction. Without loss of generality, say $v_{o_s}^n$ (i.e., $v_{o_s}^n = o_s$). Let $\bar{v}_{o_s}^n = (1 - \varepsilon_1) v_{o_s}^n + \varepsilon_1 v_n$ and let $\bar{v}_{o_s}^i = (1 - \varepsilon_2) v_{o_s}^i + \varepsilon_2 v_i$ for $i = 1, \ldots, n-1$, where $\varepsilon_2 > \varepsilon_1 > 0$ are sufficiently small. Thus, we have a new polytope $\mathcal{P} = \text{co}\{\bar{v}_{o_s}^1, \ldots, \bar{v}_{o_s}^n, v_1, \ldots, v_n\}$, which tends to $\mathcal{S} - \mathcal{S}'$ as $\varepsilon_1, \varepsilon_2 \to 0$ (See Fig. 7).

Now partition the polytope $\mathcal{P}$ into a sequence of simplices:

$$\begin{aligned}
&\mathcal{S}_1 = \text{co}\{\bar{v}_{o_s}^1, \ldots, \bar{v}_{o_s}^n, v_1\}, \\
&\mathcal{S}_2 = \text{co}\{\bar{v}_{o_s}^2, \ldots, \bar{v}_{o_s}^n, v_1, v_2\}, \\
&\quad \vdots \\
&\mathcal{S}_{n-1} = \text{co}\{\bar{v}_{o_s}^{n-1}, \bar{v}_{o_s}^n, v_1, \ldots, v_{n-1}\}, \\
&\mathcal{S}_n = \text{co}\{\bar{v}_{o_s}^n, v_1, \ldots, v_n\}. \quad \text{(see Fig. 7)}
\end{aligned}$$

For the simplex $\mathcal{S}_n$, it is clear that the hyperplane $\mathcal{B}_{\bar{v}_{o_s}^n}$ is a supporting plane of this simplex containing only one vertex $\bar{v}_{o_s}^n$. In this case, we know $\lambda_{max} < 0$. Furthermore, since $v_{o_s}^n = o_s$ is in $\mathcal{O}$ (that is, $\beta \cdot (Av_{o_s}^n + a) = 0$) and $\beta \cdot (Av_{n+1} + a) < 0$, it follows from a convexity argument that $\beta \cdot (A\bar{v}_{o_s}^n + a) > 0$ and $\beta \cdot (Av_n + a) > 0$. Thus, by Theorem 1 there is an affine control steering all trajectories originating from this simplex to the desired exit facet $\mathcal{F}_{n+1}$.

For the simplex $\mathcal{S}_{n-1}$, applying the same idea as the one in proving Theorem 2, it follows that $\mathcal{B}_{\bar{v}_{o_s}^{n-1}}$ is not a supporting hyperplane and so $\lambda_{max} > 0$. Hence by Theorem 1 there is an affine control driving all the states in $\mathcal{S}_{n-1}$ to the facet that is adjacent to $\mathcal{S}_n$.

For the simplex $\mathcal{S}_i, i = 1, n-2$, by the same argument as for $\mathcal{S}_{n-1}$, it follows that there is an affine control driving all the states in $\mathcal{S}_i$ to the facet that is adjacent to $\mathcal{S}_{i+1}$.

In total, this shows that there is a piecewise (state-dependent) affine control such that for every initial state in the polytope $\mathcal{P}$ the trajectory of the closed-loop system is driven out of the simplex $\mathcal{S}$ through $\mathcal{F}_{n+1}$. ■

Theorem 3 can also be interpreted as follows: If we can find a most distant hyperplane parallel to $\mathcal{B}$ through somewhere between v_k and v_{n+1} such that the feasible flow directions at every intersection points are all pointing to the vertex v_{n+1}, then the part consisting of v_{n+1} and this hyperplane is the failure set and the remaining part is the feasible set; Otherwise, there is no failure set, which means there still exists a controller to solve Problem 1 though there is no continuous linear affine controller solving it. Indeed, as we can see in our analysis, this controller is a piecewise linear affine controller. This analysis leads to a necessary and sufficient condition for the existence of any control solving Problem 1 which improves the results of [14].

Corollary 4: Problem 1 is solvable if and only if condition (i) of Theorem 1, or condition (ii) of Theorem 1, or condition (ii-a) of Theorem 3 holds.

IV. Example

We illustrate our results for a circuit with a piecewise linear resistor, shown in Fig. 8, which appeared in [10]. With time expressed in 10^{-8} seconds, the inductor current in

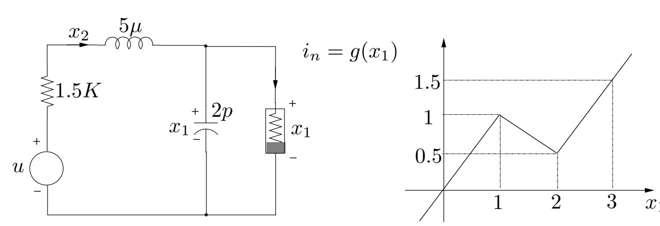

Fig. 8. Circuit with nonlinear (piecewise linear) resistor.

miliAmps and the capacitor voltage in Volts, the dynamics are written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 5g(x_1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u.$$

The control specification is to make the nonlinear resistor work in the linear range $1 \leq x_1 \leq 2$ by applying a voltage input u. In addition, there is a safety requirement on the inductor current that $0 \leq x_2 \leq 2$. Suppose the initial state of the inductor and the capacitor is in the region of $0 \leq x_{10} \leq 3$ and $0 \leq x_{20} \leq 2$ as shown in Fig. 9. Now the control problem becomes to find a control u such that the trajectories of the closed-loop system enter into the region of $1 \leq x_1 \leq 2$ without crossing the constraints $x_2 = 0$ and $x_2 = 2$. Consider a triangulation of the state space (see Fig. 9). Then we would

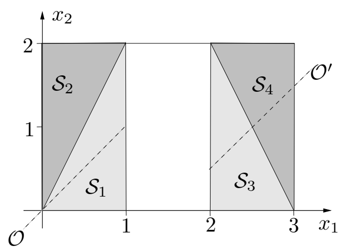

Fig. 9. Triangulation of state space.

like to control the state from the simplex $\mathcal{S}_2$ to the facet, which is adjacent to $\mathcal{S}_1$, with the other two facets restricted. Next control the state from the simplex $\mathcal{S}_1$ to the facet, which is adjacent to the target region, with the other two facets restricted. The same idea applies to the simplices $\mathcal{S}_3$ and $\mathcal{S}_4$. First we will apply our results to check whether it is possible to do so, and if not, what is the failure set.

For the simplex $\mathcal{S}_1$, let $v_1 = (1,0)$, $v_2 = (1,2)$, and $v_3 = (0,0)$. Thus, $h_1 = (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, $h_2 = (0,-1)$, and $h_3 = (1,0)$. On the simplex $\mathcal{S}_1$, the dynamics are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u. \qquad (3)$$

The range of B is the x_2-axis and $\beta = (1,0)$. From the geometry, we can see that $\mathcal{B}_{v_3}$ is a supporting plane of $\mathcal{S}_1$ containing only one vertex and so $\lambda_{max} < 0$. Furthermore, it can be easily checked that $\beta \cdot (Av_3 + a) = 0$, where A is the system matrix in (3) and a is the zero vector. It violates the condition (iii) of Theorem 1, so there is no continuous linear affine feedback driving all the states to the desired facet. Now we apply Theorem 3. Compute $k = \underset{i=1,2}{\arg\min} \beta \cdot (v_i - v_3)$ and obtain that k can be 1 or 2. Arbitrarily choose $k = 1$. Thus, $v^1_{v_1} = v_1$ and $v^2_{v_1} = v_2$. We can check that $\beta \cdot (Av^1_{v_1} + a) = -5 < 0$ and $\beta \cdot (Av^2_{v_1} + a) = 5 > 0$. Hence, it satisfies condition (ii-b) in Theorem 3 and so the largest feasible set is $\mathcal{S}_1 - \{v_3\}$.

Consider the simplex $\mathcal{S}_2$ with vertices $v_1 = (0,0)$, $v_2 = (1,2)$, and $v_3 = (0,3)$. The dynamics on $\mathcal{S}_2$ is the same one in (3). Again, $\beta = (1,0)$. Notice that the hyperplane $\mathcal{B}_{v_3}$ is a supporting plane of $\mathcal{S}_2$ and contains two vertices v_3 and v_1. Thus, $J = \{1\}$. We can check that condition (ii) of Theorem 1 holds since $\beta \cdot (Av_1 + a) = 0 \geq 0$ and $\beta \cdot (Av_3 + a) = 10 \geq 0$. So there is a linear affine control driving all the states to the facet $\mathcal{F}_3$ and into the simplex $\mathcal{S}_1$.

Now consider the simplex $\mathcal{S}_3$ with vertices $v_1 = (2,0)$, $v_2 = (2,2)$, $v_3 = (3,0)$, and normal vectors $h_1 = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, $h_2 = (0,-1)$, $h_3 = (-1,0)$. The dynamics on the simplex $\mathcal{S}_3$ are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 7.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u. \qquad (4)$$

Then we can solve $\beta = (-1,0)$. By the same argument, we know $\lambda_{max} < 0$. Since $\beta \cdot (Av_3 + a) = -7.5 < 0$, it violates the condition of Theorem 1. Compute k in Theorem

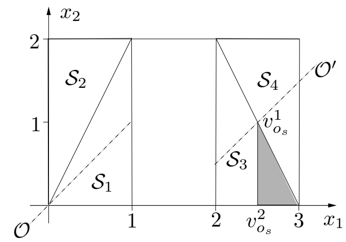

Fig. 10. The failure set in $\mathcal{S}_3$.

3 and get k equalling 1 and 2. We arbitrarily choose $k = 1$. It can be verified that $\beta \cdot (Av^2_{v_1} + a) = 7.5 > 0$. Hence, condition (ii-c) in Theorem 3 holds and so the largest feasible set $\mathcal{X} = \mathcal{S}_3 - \mathcal{S}''$, where $\mathcal{S}'' = \text{co}\{v^1_{o_s}, v^2_{o_s}, v_3\}$ is the shaded triangle in Fig. 10 with $v^1_{o_s} = (2.5, 0)$ and $v^2_{o_s} = (2.5, 1)$.

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