Reachability and Control of Affine Hypersurface Systems on Polytopes

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***Abstract*— This paper studies the reachability problem with state constraints; that is, to reach a target without leaving a polytope. First, we provide a conceptual framework for the constrained reachability problem. Then we obtain necessary and sufficient conditions for the problem for affine hypersurface systems. In addition, we provide a geometric characterization of the maximal set of initial states from which the objective can be met. Finally, algorithms on constructing piecewise affine feedback controls that accomplish the objective are presented.**

I. Introduction

Problems of reachability and invariance for dynamical systems have been extensively studied in the control literature for a long time. These problems have attracted renewed interest due to the emergence of a class of practically important systems — hybrid systems. The problem of reachability with state constraints arises in practice for the reason of meeting safety or performance specifications when synthesizing controllers. A typical example of the problem is motion planning of multiple vehicles with collision avoidance.

More recently, a relevant problem called the *control-to-facet problem* was introduced by Habets and van Schuppen in [4] and later studied in [2], [3], [5], [8], [10]. In [2], [4], [5], [10], necessary and sufficient conditions are derived for facet reachability of simplices by affine state feedback, while in [8], the facet reachability problem of simplices is extended to be solved by any type of feedback control if affine state feedback does not exist. In [3] a similar problem is studied but for rectangular multi-affine systems. The generalization of this problem to affine systems on polytopes is more difficult and remains open.

This paper studies the problem on polytopes and the starting point is reachability with state constraints (using open-loop control). The purpose of the paper is to provide a systematic view and methodology for the problem. Consider an affine system and consider a polytope in the state space and a target set on the boundary of the polytope. The first problem is to determine whether all the states in the polytope can be steered by an open-loop control to reach the target set while the trajectories remain in the polytope before reaching the target. The second problem is to find the maximal set (or a well-approximation of it) of initial states in the polytope from which it is possible to reach the target, assuming the entire polytope cannot. Finally, find a state feedback control (of any type) that accomplishes the reachability specification when the reachability problem with state constraint can be solved by open-loop control. Here we particularly consider affine hypersurface systems; that is, affine systems with n-dimensional state and $n-1$ independent control inputs. The contribution of the paper is that we completely solve these problems for affine hypersurface systems. Necessary and sufficient conditions are derived for the reachability problem (using open loop control) with state constraint in a polytope, and then we show that if there is an open loop control for the reachability problem, there is also a piecewise affine state feedback that accomplishes this. Algorithms on constructing such a piecewise affine state feedback are also provided.

Throughout the paper, we use the following notations: rank(B) and Im(B) denote the rank and the image of a matrix B. Let $\mathcal{A}, \mathcal{B}$ be two sets. $\overset{o}{\mathcal{A}}$, conv$(\mathcal{A})$, vert$(\mathcal{A})$, and aff$(\mathcal{A})$ denote the interior of $\mathcal{A}$, the convex hull of $\mathcal{A}$, the vertices of $\mathcal{A}$, and the smallest affine space containing $\mathcal{A}$, respectively. dist$(x, \mathcal{A})$ expresses the distance from a point x to $\mathcal{A}$ and $\mathcal{A} \setminus \mathcal{B}$ expresses the set difference.

Several proofs are omitted in the paper due to space limitations. More details can be found in [7].

II. Preliminaries and Problem Formulation

Consider a control system whose dynamics are affine on an n-dimensional polytope $\mathcal{P}$,

$$\Sigma : \quad \dot{x} = Ax + a + Bu =: f(x, u), \quad x \in \mathcal{P}, \qquad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $a \in \mathbb{R}^n$, and the control $u \in \mathbb{R}^m$ lives in the space of piecewise continuous functions.

We call the above control system an *affine system*. If in addition rank$(B) = n-1$, then we call Σ an *affine hypersurface system*. Given any piecewise continuous function $u : t \mapsto u(t)$ and any initial state $x_0 \in \mathcal{P}$, let $\phi_t^u(x_0)$ denote the solution of Σ starting from x_0 with the control $u(t)$.

Let $\mathcal{F}$ be an $(n-1)$-dimensional polytope on the boundary of $\mathcal{P}$, which will be the target set in our reachability problem. Notice it is not necessarily a facet of $\mathcal{P}$.

A. Reachability with State Constraints and $\mathcal{P}$-Invariant Sets

In this section we establish two preliminary results which provide a conceptual framework for the problems that will be studied in the paper. First we show that any closed set can be partitioned into states that can reach a target set and states that cannot. Second, we define $\mathcal{P}$-invariant sets, which will be a key conceptual tool in our development.

Definition 2.1: (a) A point $x \in \mathcal{P}$ can *reach* $\mathcal{F}$ *with constraint in* $\mathcal{P}$, denoted by $x \xrightarrow{\mathcal{P}} \mathcal{F}$, if there exist a piecewise continuous control $u : t \mapsto u(t)$ and $T > 0$ such that the solution $\phi_t^u(x)$ satisfies $\phi_t^u(x) \in \mathcal{P}$ for all

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$t \in [0,T]$ and $\phi_T^u(x) \in \mathcal{F}$. Otherwise, we say x cannot reach $\mathcal{F}$ with constraint in $\mathcal{P}$, denoted by $x \overset{\mathcal{P}}{\not\longrightarrow} \mathcal{F}$.

(b) A set $\mathcal{P}' \subseteq \mathcal{P}$ can *reach $\mathcal{F}$ with constraint in $\mathcal{P}$*, denoted by $\mathcal{P}' \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$, if $x \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$ for every $x \in \mathcal{P}'$.

(c) A set $\mathcal{P}' \subseteq \mathcal{P}$ is said to be *a failure set to reach $\mathcal{F}$ with constraint in $\mathcal{P}$*, denoted by $\mathcal{P}' \overset{\mathcal{P}}{\not\longrightarrow} \mathcal{F}$, if $x \overset{\mathcal{P}}{\not\longrightarrow} \mathcal{F}$ for all $x \in \mathcal{P}'$.

Proposition 2.1: Let $\mathcal{P}'$ be the maximal set in $\mathcal{P}$ such that $\mathcal{P}' \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$ and let $\mathcal{P}''$ be the maximal set in $\mathcal{P}$ such that $\mathcal{P}'' \overset{\mathcal{P}}{\not\longrightarrow} \mathcal{F}$. Then (a) $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$ and $\mathcal{P}' \cap \mathcal{P}'' = \emptyset$, (b) $\mathcal{P}' \overset{\mathcal{P}'}{\longrightarrow} \mathcal{F}$ and $\mathcal{P}'' \overset{\mathcal{P}}{\not\longrightarrow} \mathcal{P}'$.

Definition 2.2: A set $\mathcal{A} \subseteq \mathcal{P}$ is called *$\mathcal{P}$-invariant* if for all $x_0 \in \mathcal{A}$ and for all piecewise continuous functions $u : t \mapsto u(t)$, each continuous trajectory segment $\phi_t^u(x_0)$ including x_0 that is in $\mathcal{P}$ is also in $\mathcal{A}$.

Note that the time interval that a continuous trajectory segment $\phi_t^u(x_0)$ including x_0 is in $\mathcal{P}$ is either a closed interval $[0,T]$ with $T < \infty$, or it is $[0,\infty)$; i.e., $\phi_t^u(x_0) \in \mathcal{P}$ for all $t \in [0,T]$ or $[0,\infty)$. The definition means that the trajectories cannot leave $\mathcal{A}$ before leaving $\mathcal{P}$.

Proposition 2.2: A set $\mathcal{A} \subset \mathcal{P}$ is $\mathcal{P}$-invariant if and only if $\mathcal{A} \overset{\mathcal{P}}{\not\longrightarrow} \mathcal{P} \setminus \mathcal{A}$.

We now present a fundamental result which characterizes the set of states that can reach $\mathcal{F}$ with constraint in $\mathcal{P}$ in terms of $\mathcal{P}$-invariant sets.

Proposition 2.3: $\mathcal{P} \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$ if and only if no $\mathcal{P}$-invariant set is in $\mathcal{P} \setminus \mathcal{F}$.

From above, it is clear that if a set $\mathcal{A} \subseteq \mathcal{P} \setminus \mathcal{F}$ is $\mathcal{P}$-invariant, then $\mathcal{A}$ is a failure set to reach $\mathcal{F}$ with constraint in $\mathcal{P}$. But the converse is not true: trajectories starting in a failure set $\mathcal{A}$ do not reach $\mathcal{F}$ in finite time, but they need not be in $\mathcal{A}$ for the same time that they are in $\mathcal{P}$. However, the following statement is true: The maximal failure set to reach $\mathcal{F}$ with constraint in $\mathcal{P}$ is $\mathcal{P}$-invariant. This follows from Proposition 2.1 and Proposition 2.2.

B. Problem Formulation

This paper addresses the following problems for affine hypersurface systems.

Problem 2.1: Find necessary and sufficient conditions such that $\mathcal{P} \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$ (using open loop control).

Problem 2.2: Find $\mathcal{P}' \subseteq \mathcal{P}$, the maximal set of initial states, such that $\mathcal{P}' \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$.

One difficulty that arises is that the maximal reachable set may not be closed, and this results in two practical problems. One is that the time to reach the target set $\mathcal{F}$, while finite for each initial condition, may not be uniformly upper bounded on $\mathcal{P}'$. Second, the control effort can tend to infinity as the initial condition approaches the boundary of $\mathcal{P}'$. To bypass these problems, we develop in Section III-B a procedure to form an arbitrarily good, closed, full-dimensional approximation of $\mathcal{P}'$.

Problem 2.3: Find a state feedback $u = h(x)$ that accomplishes $\mathcal{P} \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$.

We first introduce a set that plays an important role in our reachability problem (as well as in controllability problems [6]). Define $\mathcal{O} = \{x \in \mathbb{R}^n \ : \ Ax + a \in \text{Im}(B)\}$. It is fairly easy to prove that $\mathcal{O} = \emptyset$ when $\text{Im}(A) \subseteq \text{Im}(B)$ and $a \notin \text{Im}(B)$; $\mathcal{O} = \mathbb{R}^n$ when $\text{Im}(A) \subseteq \text{Im}(B)$ and $a \in \text{Im}(B)$; and $\mathcal{O}$ is a hyperplane, otherwise. Notice that the vector field $f(x,u)$ on $\mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. When $\mathcal{O}$ is a hyperplane (called *the dividing hyperplane* in [9]), it divides the state space into two disjoint regions inside of which the normal component of $\dot{x}$ to $\text{Im}(B)$ points to the opposite direction and does not vanish. When $\mathcal{O}$ is empty, the normal component of $\dot{x}$ to $\text{Im}(B)$ points to a same direction and does not vanish on the entire state space.

In the following, we consider two cases depending on $\mathcal{O}$: (A) $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$; (B) $\overset{o}{\mathcal{P}} \cap \mathcal{O} \neq \emptyset$.

III. Reachability on Polytopes – Case A

For notational convenience, denote by $\mathcal{B}$ the subspace $\text{Im}(B)$ and denote by $\mathcal{B}_x$ the hyperplane parallel to $\mathcal{B}$ and going through a point x. For case A, let β be the unit normal vector to $\mathcal{B}$ satisfying $\beta^T(Ax + a) \leq 0$ for all $x \in \mathcal{P}$. For any $z \in \mathbb{R}^n$, define half spaces $\mathcal{H}_z^+ = \{x \in \mathbb{R}^n : \beta^T x \geq \beta^T z\}$ and $\mathcal{H}_z^- = \{x \in \mathbb{R}^n : \beta^T x \leq \beta^T z\}$, respectively. Also, let v_- (v_+) be a point in $\arg\min\{\beta^T x : x \in \mathcal{F}\}$ ($\arg\max\{\beta^T x : x \in \mathcal{F}\}$) and denote $\partial\mathcal{P}_{max} = \arg\max\{\beta^T x : x \in \mathcal{P}\}$ and $\partial\mathcal{P}_{min} = \arg\min\{\beta^T x : x \in \mathcal{P}\}$.

A. Necessary and Sufficient Conditions

In this subsection we give the solution of Problem 2.1.

Theorem 3.1: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. Then $\mathcal{P} \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}$ if and only if none of the following conditions holds:
(a) $(\mathcal{H}_{v_-}^- \cap \mathcal{P}) \setminus (\mathcal{F} \cup \mathcal{O}) \neq \emptyset$; (b) $\partial\mathcal{P}_{max} \subset \mathcal{O} \cap \overset{o}{\mathcal{H}}{}_{v_+}^{+}$;
(c) $\mathcal{B}$ is parallel to $\mathcal{O}$ when $\mathcal{F} \subset \mathcal{O}$.

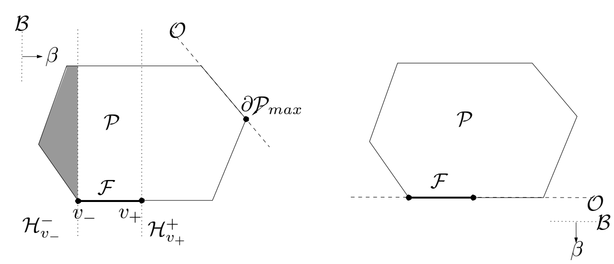

Fig. 1. Illustration for Theorem 3.1.

Fig. 1 illustrates the three failure cases of the theorem. In the left figure, $(\mathcal{H}_{v_-}^- \cap \mathcal{P}) \setminus (\mathcal{F} \cup \mathcal{O}) \neq \emptyset$ (shaded region) and $\partial\mathcal{P}_{max} \subset \mathcal{O} \cap \overset{o}{\mathcal{H}}{}_{v_+}^{+}$ where $\partial\mathcal{P}_{max}$ is a point. In the right figure, $\mathcal{F} \subset \mathcal{O}$ and $\mathcal{B}$ is parallel to $\mathcal{O}$.

We provide two lemmas to prove Theorem 3.1.

Lemma 3.1: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. Let z be a point in $\mathcal{P}$.

(a) The sets $\mathcal{H}_z^- \cap \mathcal{P}$ and $\overset{o}{\mathcal{H}}{}_z^{-} \cap \mathcal{P}$ are $\mathcal{P}$-invariant;

(b) If $\beta^T(Ax + a) = 0$ for all $x \in \mathcal{B}_z \cap \mathcal{P}$, then $\mathcal{B}_z \cap \mathcal{P}$ and $\mathcal{P} \setminus \mathcal{B}_z$ are $\mathcal{P}$-invariant.

Lemma 3.2: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. Let y, z be distinct points and let l be the line segment joining them.

(a) If $z, y \notin \mathcal{O}$ and $y \in \overset{o}{\mathcal{H}}{}_z^-$, then $z \xrightarrow{l} y$.

(b) If $z, y \in \mathcal{O}$ and $y \in \mathcal{B}_z$, then $z \xrightarrow{l} y$.

Sketch Proof of Theorem 3.1 ($\Longrightarrow$) Define

$$\mathcal{A}_- = \begin{cases} \mathcal{D}_1 & \text{if Thm. 3.1(a) holds and } \mathcal{B}_{v_-} \cap \mathcal{F} \cap \mathcal{O} = \emptyset, \\ \mathcal{D}_2 & \text{if Thm. 3.1(a) holds and } \mathcal{B}_{v_-} \cap \mathcal{F} \cap \mathcal{O} \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\mathcal{D}_1 = (\mathcal{H}_{v_-}^- \cap \mathcal{P}) \setminus \mathcal{F}$ and $\mathcal{D}_2 = \mathcal{D}_1 \setminus (\mathcal{B}_{v_-} \cap \mathcal{O})$.

$$\mathcal{A}_+ = \begin{cases} \partial \mathcal{P}_{max} & \text{if Thm. 3.1(b) holds,} \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

$$\mathcal{A}_o = \begin{cases} \mathcal{P} \setminus \mathcal{O} & \text{if Thm. 3.1(c) holds,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Applying Lemma 3.1, one obtains that these sets are $\mathcal{P}$-invariant and also in $\mathcal{P} \setminus \mathcal{F}$. Thus, the conclusion follows from Proposition 2.3.

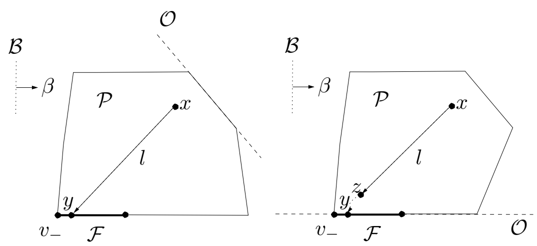

Fig. 2. Illustration for case I (left) and case II (right).

($\Longleftarrow$) Let x be a point in $\mathcal{P} \setminus (\mathcal{F} \cup \mathcal{O})$. (I): Suppose $\mathcal{F}$ is not in $\mathcal{O}$. Since the assumption (a) does not hold, it follows that there is a point y in $\mathcal{F} \setminus \mathcal{O}$ and in the small neighborhood of v_- satisfying the assumption in Lemma 3.2(a) (see Fig. 2), so $x \xrightarrow{l} y$, where l is the line segment joining x and y. Hence, $x \xrightarrow{\mathcal{P}} \mathcal{F}$. (II): Suppose $\mathcal{F}$ is in $\mathcal{O}$. Then by assumption, we know that $\mathcal{B}$ is not parallel to $\mathcal{O}$. Let y be the point selected as in case (I). Then for a proper control with $f(y, u)$ pointing outside of $\mathcal{P}$, there is a point z on the backward trajectory of y satisfying $z \in \overset{o}{\mathcal{H}}{}_x^-$ (see Fig. 2). Thus, by Lemma 3.2(a) we have $x \xrightarrow{l} z$ and therefore $x \xrightarrow{\mathcal{P}} \mathcal{F}$.

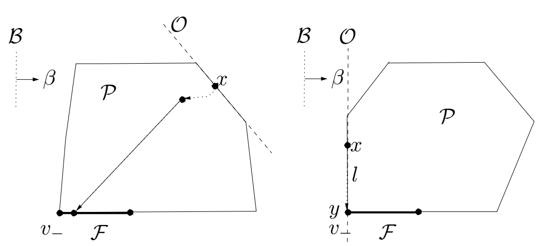

Fig. 3. Illustration for case III (left) and IV (right).

Next, let x be a point in $(\mathcal{P} \cap \mathcal{O}) \setminus \mathcal{F}$. Clearly, x is on the boundary of $\mathcal{P}$ by assumption $\mathcal{O} \cap \overset{o}{\mathcal{P}} = \emptyset$. (III): Suppose $\mathcal{B}_x \cap \mathcal{P} \not\subset \mathcal{O}$. Then for a proper control with $f(x, u)$ pointing inside of $\mathcal{P}$, the trajectory starting from x instantaneously enters the interior of $\mathcal{P}$ (see Fig. 3), which is not in $\mathcal{O}$ any more. Then by the previous argument, it can be driven to reach $\mathcal{F}$ through a line. (IV): Suppose $\mathcal{B}_x \cap \mathcal{P} \subset \mathcal{O}$ (see Fig. 3). Then it follows from the assumptions that $\mathcal{B}_x \cap \mathcal{F} \neq \emptyset$. Now we select a point $y \in \mathcal{B}_x \cap \mathcal{F}$. Clearly, x and y satisfy the assumption in Lemma 3.2(b). Thus, $x \xrightarrow{l} y$, where l is the line segment joining from x to y, and so $x \xrightarrow{\mathcal{P}} \mathcal{F}$. ■

B. Failure Sets and Partition of the Polytope

Theorem 3.1 gives necessary and sufficient conditions for the reachability problem $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$. This result, in turn, can be tied to failure sets apropos Proposition 2.3. In this subsection we classify all failure sets. First, if Theorem 3.1(c) holds, then no point in $\mathcal{A}_o$ can reach $\mathcal{F}$. We call this a *total failure*. If Theorem 3.1(b) holds, then the failure set is $\mathcal{A}_+ = \partial \mathcal{P}_{max}$, which is always a face of $\mathcal{P}$. We call this a *face failure*. If Theorem 3.1(a) holds, there are several cases. If the failure set $\mathcal{A}_-$ is a full-dimensional subpolytope in $\mathcal{P}$ we call this a *region failure*. There are also two extreme situations. One is a *total failure* when the closure of $\mathcal{A}_-$ is $\mathcal{P}$ which occurs when $\mathcal{B}$ is parallel to $\mathcal{F}$. The other situation is a *face failure* when $\mathcal{A}_-$ is a face when $\overset{o}{\mathcal{H}}{}_{v_-}^- \cap \mathcal{P} = \emptyset$. We arrive at the following corollary of Theorem 3.1.

Corollary 3.1: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. Let $\mathcal{P}' = \mathcal{P} \setminus \mathcal{A}$ where $\mathcal{A} = \mathcal{A}_- \cup \mathcal{A}_+ \cup \mathcal{A}_o$. Then $\mathcal{P}' \xrightarrow{\mathcal{P}'} \mathcal{F}$ and $\mathcal{A} \not\xrightarrow{\mathcal{P}} \mathcal{F}$.

Remark 3.1: It can be easily checked that Theorem 3.1 (b) and (c) are mutually exclusive. This means only one of the sets $\mathcal{A}_+$ and $\mathcal{A}_o$ is not empty. If Theorem 3.1(c) holds, we do not need to check (b) as $\mathcal{A}_+$ must be empty. Also we do not need to check (a) as $\mathcal{A}_-$ equals to $\mathcal{A}_o$ or is empty.

We have identified the maximal set $\mathcal{P}' \subseteq \mathcal{P}$ for Problem 2.2. This set, in general, is not closed. This leads to difficulties with unbounded control effort and unbounded time to reach $\mathcal{F}$. Consequently, once failure sets have been identified, it is desirable to remove them via a procedure that both well-approximates the set of states that can reach $\mathcal{F}$ and also yields a closed full-dimensional polytope $\mathcal{P}''$ such that $\mathcal{P}'' \xrightarrow{\mathcal{P}''} \mathcal{F}$. The approach is to "cut off" failure sets from $\mathcal{P}$ by one of two procedures. One procedure is for removing region and face failures $\mathcal{A}_-$ by cutting along a hyperplane which is parallel to a slightly shifted version of $\mathcal{B}$. The second procedure is for removing face failures $\mathcal{A}_+$ by cutting exactly along a hyperplane parallel to $\mathcal{B}$. These cuts are chosen arbitrarily close to the failure set and so that the remaining polytope has no failure sets. Of course, if there is a total failure, there is no need to partition $\mathcal{P}$.

The following procedure partitions $\mathcal{P}$ into a disjoint union of a closed full-dimensional polytope and at most two other polytopes that over-approximate the failure sets $\mathcal{A}_-$ and $\mathcal{A}_+$.

Algorithm 1: (Let $\epsilon > 0$ be sufficiently small.)

1) If $\mathcal{A}_- \neq \emptyset$, select enough number of points $z_1, \ldots, z_k$ in $\overset{o}{\mathcal{P}} \cap \overset{o}{\mathcal{H}}{}_{v_-}^+$ such that $\mathcal{G} := \text{aff}\{z_1, \ldots, z_k, \mathcal{F} \cap \mathcal{B}_{v_-}\}$ is of dimension $n - 1$ and $\max_{x \in \mathcal{P} \cap \mathcal{G}} \text{dist}(x, \mathcal{B}_{v_-}) = \epsilon$. Then divide $\mathcal{P}$ along $\mathcal{G}$.
2) If $\mathcal{A}_+ \neq \emptyset$, select a point $z \in \mathcal{P}$ such that $\max_{x \in \mathcal{A}_+} \text{dist}(x, \mathcal{B}_z) = \epsilon$. Then divide $\mathcal{P}$ along $\mathcal{B}_z$.

Let $\mathcal{A}_{\epsilon_-}$, $\mathcal{A}_{\epsilon_+}$, and $\mathcal{P}_\epsilon$ be the collection of sets after the application of the division rules in Algorithm 1, where $\mathcal{A}_{\epsilon_-}$ contains $\mathcal{A}_-$, $\mathcal{A}_{\epsilon_+}$ contains $\mathcal{A}_+$, and $\mathcal{P}_\epsilon$ is the remainder. Clearly, these three sets (if not empty) are n-polytopes and $\mathcal{F} \subset \mathcal{P}_\epsilon \subseteq \mathcal{P}'$. Then we have the following corollary which follows directly from Algorithm 1 and Theorem 3.1.

Corollary 3.2: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. Then $\mathcal{P}_\epsilon \xrightarrow{\mathcal{P}_\epsilon} \mathcal{F}$.

We call $\mathcal{P}_\epsilon$ *the ϵ-approximation of maximal reachable set* of $\mathcal{F}$. To generalize this terminology, we say that the ϵ-approximation of maximal reachable set $\mathcal{P}_\epsilon = \emptyset$ when $\mathcal{P}$ is a total failure, and $\mathcal{P}_\epsilon = \mathcal{P}$ when $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

A simple example is presented to illustrate the possible failure sets and how Algorithm 1 cuts them off. Consider the system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = u.$$

It can be easily verified that the system has a dividing hyperplane; i.e., $\mathcal{O} = \{(x_1, x_2) : x_2 = 0\}$, the x_1 axis, and that $\mathcal{B}$ is just the x_2 axis. Suppose that the polytope $\mathcal{P}$

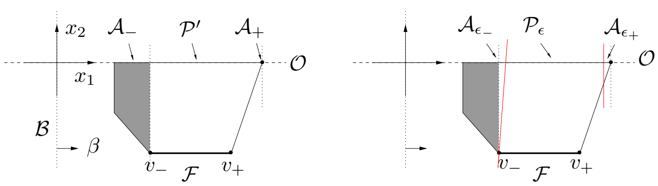

Fig. 4. An example.

and the target set $\mathcal{F}$ are as shown in Fig. 4. The dividing hyperplane touches the polytope but has empty intersection with its interior. From (2), we get, $\mathcal{A}_o = \emptyset$ since $\mathcal{B}$ is not parallel to $\mathcal{O}$; $\mathcal{A}_- = (\mathcal{H}^-_{v_-} \cap \mathcal{P}) \setminus \mathcal{F}$ is the patterned region in Fig. 4; and $\mathcal{A}_+$ is just a point. The set of initial states for which it is possible to reach $\mathcal{F}$ with constraint in $\mathcal{P}$ is the set $\mathcal{P}'$ not including the boundary of $\mathcal{A}_-$ and $\mathcal{A}_+$. This set is not closed. Moreover, if an initial state $x_0 \in \mathcal{P}'$ approaches the boundary of $\mathcal{A}_-$, the control input $u(x_0)$ tends to infinity in order to reach $\mathcal{F}$. Also, if $x_0 \in \mathcal{P}'$ approaches the boundary of $\mathcal{A}_+$, the time to reach $\mathcal{F}$ tends to infinity. Applying Algorithm 1, a good closed ϵ-approximation $\mathcal{P}_\epsilon$ of $\mathcal{P}'$ is given on the right of Fig. 4.

Finally, we present a lemma for the reachability problem with two target sets which will be used in the next section. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $(n-1)$-dimensional polytopes on the boundary of $\mathcal{P}$.

Lemma 3.3: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$ and suppose $\mathcal{B}$ is not parallel to $\mathcal{O}$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_1 \cup \mathcal{F}_2$ but it does not satisfy $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_1$, then $\mathcal{P} \setminus \mathcal{P}^1_\epsilon \subseteq \mathcal{P}^2_\epsilon$, where $\mathcal{P}^1_\epsilon$ and $\mathcal{P}^2_\epsilon$ are the ϵ-approximation ($\epsilon > 0$ sufficiently small) of maximal reachable set of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

IV. Control Synthesis on Polytopes – Case A

This section investigates feedback synthesis on polytopes for case A, namely, $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. We want to show that if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ (using open-loop control) then there exists a piecewise affine feedback solving the reachability problem. The idea is to triangulate the polytope, transform the reachability problem within a polytope into a set of reachability problems for simplices, and then devise appropriate piecewise affine controllers on each simplex. The triangulation must be performed properly otherwise the procedure may fail.

First we present a result on the existence of a piecewise affine feedback that solves the reachability problem on simplices. The result can be obtained from [8] and more details on how to construct such controllers can also be found in [8]. Consider a simplex $\mathcal{S}$ and a facet $\mathcal{E}$ of $\mathcal{S}$. We have the following lemma.

Lemma 4.1: Suppose $\overset{o}{\mathcal{S}} \cap \mathcal{O} = \emptyset$. If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{E}$ (using open loop control) then there exists a piecewise affine feedback $u = F_{\sigma(x)}x + g_{\sigma(x)}$ that accomplishes $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{E}$, where $\sigma : \mathcal{S} \to \{1, 2\}$.

Next we present a lemma that is useful to find a proper triangulation.

Lemma 4.2: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$, then there exists a vertex v_* of $\mathcal{P}$ in $\partial\mathcal{P}_{max}$ such that either $v_* \notin \mathcal{O}$ or $v_* \in \mathcal{F}$.

Proof: Suppose by contradiction that for any vertex $v \in \partial\mathcal{P}_{max}$ we have $v \in \mathcal{O}$ and $v \notin \mathcal{F}$. Note that $v \notin \mathcal{F}$ for all vertices $v \in \partial\mathcal{P}_{max}$ implies, by convexity, $\partial\mathcal{P}_{max} \subset \overset{o}{\mathcal{H}}{}^+_{v_+}$. Moreover, since $v \in \mathcal{O}$ for all $v \in \partial\mathcal{P}_{max}$, it follows from the convexity of $\mathcal{O}$ that $\partial\mathcal{P}_{max} \subset \mathcal{O}$. Hence, by Theorem 3.1, this contradicts $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$. ■

Now we present an algorithm for control synthesis. But first, we introduce some concepts on triangulation; for more details, see [1].

Suppose $\mathcal{V}$ is a finite set of points such that conv($\mathcal{V}$) is n-dimensional. A *subdivision* of $\mathcal{V}$ is a finite collection $\mathbb{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$ of n-polytopes such that the vertices of each $\mathcal{P}_i$ are drawn from $\mathcal{V}$; conv($\mathcal{V}$) is the union of $\mathcal{P}_1, \ldots, \mathcal{P}_m$; and $\mathcal{P}_i \cap \mathcal{P}_j$ ($i \neq j$) is a common (possibly empty) face of $\mathcal{P}_i$ and $\mathcal{P}_j$. A *triangulation* of $\mathcal{V}$ is a subdivision in which each $\mathcal{P}_i$ is a simplex. Consider a triangulation $\mathbb{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_q\}$. We say $\mathcal{S}_i$ and $\mathcal{S}_j$ are *adjacent* (denoted by $\mathcal{S}_i \sim \mathcal{S}_j$) if $\mathcal{S}_i \cap \mathcal{S}_j$ is a facet. A sequence $(\mathcal{S}_{i_k}, \ldots, \mathcal{S}_{i_0})$ is called a *path* to reach $\mathcal{S}_{i_0}$ if $\mathcal{S}_{i_j} \sim \mathcal{S}_{i_{j-1}}$ and $\mathcal{S}_{i_j} \xrightarrow{\mathcal{S}_{i_j}} \mathcal{S}_{i_{j-1}}$ for each $1 \leq j \leq k$. The *length* of such a path is k.

In what follows we denote $\mathcal{F}$ by $\mathcal{S}_0$ for notation simplicity.

Algorithm 2:

1) *Triangulation:*

(a) If $\mathcal{F}$ is a facet of $\mathcal{P}$ then select v_* as in Lemma 4.2. If $\mathcal{F}$ is not a facet of $\mathcal{P}$ then select v_* as in Lemma 4.2 and in addition satisfying that v_* is not in the facet containing $\mathcal{F}$, if it exists.

(b) For the facet $\mathcal{F}_j$ of $\mathcal{P}$ that contains $\mathcal{F}$, make a triangulation of vert($\mathcal{F}_j$) $\cup$ vert($\mathcal{F}$) such that the interior of each resulting simplex is entirely either in $\mathcal{F}$ or not in $\mathcal{F}$; for the remaining facet $\mathcal{F}_j$ of $\mathcal{P}$, make a triangulation of vert($\mathcal{F}_j$). Denote $\{\mathcal{S}^i_{\mathcal{F}_j} : i = 1, \ldots, k_j\}$ the triangulation for $\mathcal{F}_j$.

(c) Let $\mathbb{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_q\} := \{\text{conv}(v_*, \mathcal{S}^i_{\mathcal{F}_j}) : \mathcal{F}_j$ is any facet of $\mathcal{P}$ not containing $v_*\}$.

2) *Path Generation:*

(a) Initialization: $\mathcal{R}_f := \{\mathcal{S}_0\}$, $\mathcal{R}_u := \{\mathcal{S}_1, \ldots, \mathcal{S}_q\}$;

(b) While $(\mathcal{R}_u \neq \emptyset)$, do if $\exists(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ such that $\mathcal{S}_i \sim \mathcal{S}_j$ and $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{S}_j$, then move $\mathcal{S}_i$ from $\mathcal{R}_u$ to $\mathcal{R}_f$.

3) *Controller Synthesis:*

(a) Let $\{\ldots, (\ldots, \mathcal{S}_i, \mathcal{S}_j, \ldots, \mathcal{S}_0), \ldots\}$ be the collection of paths to reach $\mathcal{S}_0$ generated from step 2);

(b) Find $u^i(x) := F_{\sigma_i(x)}x + g_{\sigma_i(x)}, \ i = 1, \ldots, q$, that solves $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{F}_{ij}$, where $\mathcal{F}_{ij}$ is the common facet of $\mathcal{S}_i$ and the followed simplex in the path;

(c) For any simplex $\mathcal{S}_i \in \mathbb{S}$, let $u(x) = u^i(x)$ for all $x \in \mathcal{S}_i$. If $x \in \mathcal{P}$ belongs to more than one simplex, set $u(x) = u^j(x)$ where j is the index of a simplex that has the minimum length path to reach $\mathcal{S}_0$.

Lemma 4.3: The collection $\mathbb{S}$ obtained in Algorithm 2 is a triangulation of vert$(\mathcal{P}) \cup$ vert$(\mathcal{F})$ such that every simplex in $\mathbb{S}$ contains v_* as a vertex.

Theorem 4.1: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. There exists a piecewise affine feedback that accomplishes $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ if and only if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ (using open-loop control).

Sketch of proof: $(\Longrightarrow)$ Obvious. $(\Longleftarrow)$ Denote by $\bar{\mathcal{F}}$ the facet of $\mathcal{P}$ containing $\mathcal{F}$. There are three cases.

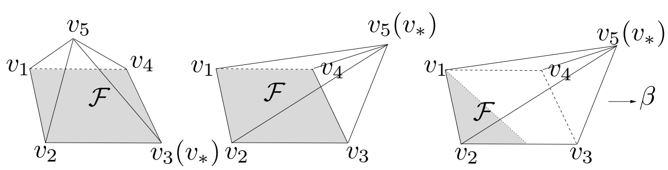

Fig. 5. Illustration for case (I).

(I): Suppose $\mathcal{F}$ is a facet of $\mathcal{P}$ or when $\mathcal{F}$ is not a facet of $\mathcal{P}$, there exists a v_* satisfying the property in Lemma 4.2 and in addition, $v_* \notin \bar{\mathcal{F}}$ (see Fig. 5). For this case, it suffices to show that at every step of the algorithm such that $\mathcal{R}_u \neq \emptyset$, there exists a pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ such that $\mathcal{S}_i \cap \mathcal{S}_j =: \mathcal{F}_{ij}$ is a facet and $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{S}_j$. This can be verified by checking conditions (a)-(c) of Theorem 3.1 when the pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ is chosen such that $\mathcal{F}_{ij}$ has a vertex v with minimum β-coordinate (namely, $\beta^T v \leq \beta^T w$ for all $w \in \mathcal{S}_i \in \mathcal{R}_u$).

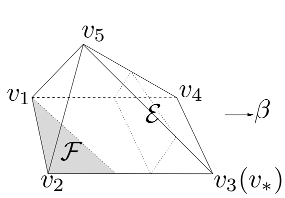

Fig. 6. Illustration for case (II).

(II): Consider the case when $\mathcal{F}$ is not a facet of $\mathcal{P}$ and all v_* satisfying the property of Lemma 4.2 are in $\bar{\mathcal{F}}$ but none of them is in $\mathcal{F}$. So by Lemma 4.2 $v_* \notin \mathcal{O}$. Then, there is a hyperplane $\mathcal{G}$ partitioning $\mathcal{P}$ into two subpolytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that (1) $v_* \in \mathcal{P}_2 \setminus \mathcal{G}$ and $\mathcal{F} \subset \mathcal{P}_1 \setminus \mathcal{G}$, (2) $\mathcal{E} := \mathcal{G} \cap \mathcal{P}$ satisfies $\mathcal{E} \cap \mathcal{O} = \emptyset$, (3) $\arg\max\{\beta^T x : x \in \mathcal{E}\} \not\subset \bar{\mathcal{F}}$ and $\arg\min\{\beta^T x : x \in \mathcal{E}\} \subset \bar{\mathcal{F}}$. Thus, from Theorem 3.1 and the assumption $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$, we know $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{E} \subset \mathcal{P}_1$ and $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$. Furthermore, a point $v'_* \in \arg\max\{\beta^T x : x \in \mathcal{E}\}$ satisfies the property in Lemma 4.2 for the polytope $\mathcal{P}_1$ and also it is not in $\bar{\mathcal{F}}$. Thus, applying Algorithm 2, it gives a piecewise affine controller that achieves $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$ as we just showed. For the polytope $\mathcal{P}_2$, clearly the target set $\mathcal{E}$ is a facet of $\mathcal{P}_2$. So again by Algorithm 2, we have a piecewise affine controller achieving $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{E} \subset \mathcal{P}_1$.

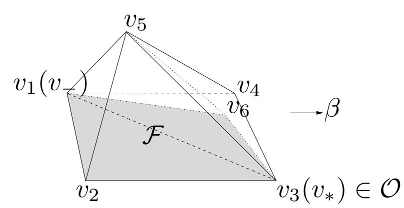

Fig. 7. Illustration for case (III).

(III): Consider the case when $\mathcal{F}$ is not a facet of $\mathcal{P}$ and all v_* satisfying the property of Lemma 4.2 are in $\mathcal{F}$. Then we have $\beta^T v_* \geq \beta^T x \geq \beta^T v_-$ for all $x \in \mathcal{P}$. Construct any hyperplane such that it goes through the points v_- and v_*, and partitions $\mathcal{P}$ into two full-dimensional subpolytopes $\mathcal{P}_1$ and $\mathcal{P}_2$. Let $\mathcal{F}_{12} := \mathcal{P}_1 \cap \mathcal{P}_2$ and let $\mathcal{P}_3$ be the convex hull of $\mathcal{F}$ and $\mathcal{F}_{12}$. Thus, $\mathcal{P}_3$ is a full-dimensional polytope in $\mathcal{P}$ and $\mathcal{F}_{12}$ is in $\mathcal{P}_3$. By Theorem 3.1 and the assumption $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$, it can be easily verified that $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}$. Furthermore, $\mathcal{F}$ is a facet of $\mathcal{P}_3$, so by what we just showed, Algorithm 2 gives a piecewise affine control, say $u(x) = F_{\sigma_3(x)}x + g_{\sigma_3(x)}$, $x \in \mathcal{P}_3$, that achieves $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}$. For the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, again by Theorem 3.1 we know $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}_{12}$ and $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{F}_{12}$. Moreover, $\mathcal{F}_{12}$ is a facet of both $\mathcal{P}_1$ and $\mathcal{P}_2$. So there are piecewise affine controllers, $u(x) = F_{\sigma_1(x)}x + g_{\sigma_1(x)}, x \in \mathcal{P}_1$ and $u(x) = F_{\sigma_2(x)}x + g_{\sigma_2(x)}, x \in \mathcal{P}_2$, that achieve $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}_{12}$ and $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{F}_{12}$, respectively. Since $\mathcal{F}_{12}$ is in $\mathcal{P}_3$, it means that the controllers can drive all the states not in $\mathcal{P}_3$ to $\mathcal{P}_3$. Thus, the following controller

$$u(x) = \begin{cases} F_{\sigma_3(x)}x + g_{\sigma_3(x)} & x \in \mathcal{P}_3 \\ F_{\sigma_1(x)}x + g_{\sigma_1(x)} & x \in \mathcal{P}_1 \setminus \mathcal{P}_3 \\ F_{\sigma_2(x)}x + g_{\sigma_2(x)} & x \in \mathcal{P}_2 \setminus \mathcal{P}_3 \end{cases}$$

achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$. ∎

V. Reachability and Control Synthesis on Polytope – Case B

Now we come to the case B, (namely, $\overset{o}{\mathcal{P}} \cap \mathcal{O} \neq \emptyset$). Recall that this case means either $\mathcal{O}$ is a hyperplane that partitions $\mathcal{P}$ into two full-dimensional polytopes, or $\mathcal{O}$ is the entire state space. We only deal with the first one due to space limitations. For this case, we divide $\mathcal{P}$ along $\mathcal{O}$ that results in two polytopes, denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$. The following algorithm and theorem present solutions to the reachability problem and control synthesis problem.

Algorithm 3:

1) Initialization: $k := 1$, $\mathcal{R}_f := \{\mathcal{Q}_1 = \emptyset, \mathcal{Q}_2 = \emptyset\}$, $\mathcal{R}_u := \{\mathcal{P}_1, \mathcal{P}_2\}$, $\mathbb{F}_{bd} := \{\mathcal{F}_{i0} = \mathcal{P}_i \cap \mathcal{F} \ : \ i = 1, 2;$ $\mathcal{F}_{i0}$ is of $(n-1)$-dimension and $\mathcal{P}_i$ is not a total failure to reach $\mathcal{F}_{i0}\}$.
2) while $(\mathbb{F}_{bd} \neq \emptyset)$, do
   a) for every $\mathcal{F}_{ij} \in \mathbb{F}_{bd}$, find $\mathcal{Q}_i^k \subseteq \mathcal{P}_i$, the ϵ-approximation of maximal reachable set of $\mathcal{F}_{ij}$ and update $\mathcal{Q}_i := \mathcal{Q}_i \cup \mathcal{Q}_i^k$;
   b) for every $\mathcal{Q}_i \in \mathcal{R}_f$, if $\mathcal{Q}_i = \mathcal{P}_i$, then remove $\mathcal{P}_i$ from $\mathcal{R}_u$;
   c) update $\mathbb{F}_{bd} := \{\mathcal{F}_{ij} = \mathcal{P}_i \cap \mathcal{Q}_j : j \neq i; \mathcal{P}_i \in \mathcal{R}_u;$ $\mathcal{Q}_j \in \mathcal{R}_f;$ $\mathcal{F}_{ij} \cap (\mathcal{P}_i \setminus \mathcal{Q}_i)$ is of $(n-1)$-dimension and $\mathcal{P}_i$ is not a total failure to reach $\mathcal{F}_{ij}\}$;
   d) $k := k + 1$.

Theorem 5.1: Suppose $\overset{o}{\mathcal{P}} \cap \mathcal{O} \neq \emptyset$ and $\mathcal{O}$ is a hyperplane. The following are equivalent:

(a) $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ (using open-loop control);
(b) There exits a $\epsilon > 0$ such that Algorithm 3 halts with $\mathcal{R}_u = \emptyset$;
(c) There exists a piecewise affine control that accomplishes $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

Proof: (a) $\Longrightarrow$ (b) If $\mathcal{B}$ is parallel to $\mathcal{O}$, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ implies $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}_{10}$ and $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{F}_{20}$ where both $\mathcal{F}_{10}$ and $\mathcal{F}_{20}$ are of $(n-1)$-dimension. Thus, Algorithm 3 succeeds with $\mathcal{R}_u = \emptyset$.

If $\mathcal{B}$ is not parallel to $\mathcal{O}$, we consider two cases.

First, suppose there exists a $\mathcal{P}_i \in \{\mathcal{P}_1, \mathcal{P}_2\}$ satisfying $\mathcal{P}_i \xrightarrow{\mathcal{P}_i} \mathcal{F}_{i0}$ where $\mathcal{F}_{i0}$ is of $(n-1)$-dimension. Then $\mathcal{Q}_i^1 = \mathcal{P}_i$. Let $\mathcal{P}_j$ be the other polytope in $\{\mathcal{P}_1, \mathcal{P}_2\}$. If furthermore, $\mathcal{P}_j \xrightarrow{\mathcal{P}_j} \mathcal{F}_{j0}$ where $\mathcal{F}_{j0}$ is also of $(n-1)$-dimension. Then $\mathcal{Q}_j^1 = \mathcal{P}_j$. Thus, Algorithm 3 succeeds with $\mathcal{R}_u = \emptyset$. Otherwise, find $\mathcal{Q}_j^1 \subset \mathcal{P}_j$, the ϵ-approximation (ϵ sufficiently small) of maximal reachable set of $\mathcal{F}_{j0}$, that may be empty. Note that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$, so by Lemma 3.3 $\mathcal{P}_j \setminus \mathcal{Q}_j^1$ is contained in $\mathcal{Q}_j^2 \subseteq \mathcal{P}_j$, the ϵ-approximation of maximal reachable set of $\mathcal{F}_{ji} := \mathcal{P}_j \cap \mathcal{Q}_i^1 = \mathcal{P}_j \cap \mathcal{P}_i$. Furthermore, $\mathcal{F}_{ji} \cap (\mathcal{P}_j \setminus \mathcal{Q}_j^1)$ is of $(n-1)$-dimension since otherwise $\mathcal{F}_{ji}$ is in $\mathcal{Q}_j^1$ meaning that $\mathcal{F}_{ji} \xrightarrow{\mathcal{P}_j} \mathcal{F}_{j0}$ and therefore $\mathcal{P}_j \xrightarrow{\mathcal{P}_j} \mathcal{F}_{j0}$, a contradiction. Hence, Algorithm 3 succeeds with $\mathcal{R}_u = \emptyset$.

Second, suppose no $\mathcal{P}_i \in \{\mathcal{P}_1, \mathcal{P}_2\}$ satisfies $\mathcal{P}_i \xrightarrow{\mathcal{P}_i} \mathcal{F}_{i0}$. Without loss of generality, suppose $\mathcal{F}_{10}$ is of $(n-1)$-dimension and $\mathcal{P}$ is not a total failure to reach $\mathcal{F}_{10}$. Find $\mathcal{Q}_1^1 \subset \mathcal{P}_1$, the ϵ-approximation (ϵ sufficiently small) of maximal reachable set of $\mathcal{F}_{10}$, which is not empty by assumption, and find $\mathcal{Q}_2^1 \subset \mathcal{P}_2$, the ϵ-approximation (ϵ sufficiently small) of maximal reachable set of $\mathcal{F}_{20}$, which may be empty. Note that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ and ϵ can be arbitrarily small, so $\mathcal{F}_{21} := \mathcal{P}_2 \cap \mathcal{Q}_1^1$ must not be empty and the states in $\mathcal{P}_2$ that cannot reach $\mathcal{F}_{20}$ must be able to reach $\mathcal{F}_{21}$. Then by Lemma 3.3 $\mathcal{P}_2 \setminus \mathcal{Q}_2^1$ is contained in $\mathcal{Q}_2^2 \subseteq \mathcal{P}_2$, the ϵ-approximation of maximal reachable set of $\mathcal{F}_{21}$. Furthermore, $\mathcal{F}_{21} \cap (\mathcal{P}_2 \setminus \mathcal{Q}_2^1)$ is of $(n-1)$-dimension since otherwise $\mathcal{F}_{21}$ is in $\mathcal{Q}_2^1$ meaning that $\mathcal{F}_{21} \xrightarrow{\mathcal{P}_2} \mathcal{F}_{20}$ and therefore $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{F}_{20}$, a contradiction. Now $\mathcal{Q}_2^1 \cup \mathcal{Q}_2^2 = \mathcal{P}_2$, implying $\mathcal{P}_2 \notin \mathcal{R}_u$. Repeat the argument for $\mathcal{P}_1$. Algorithm 3 will succeed with $\mathcal{R}_u = \emptyset$.

(b) $\Longrightarrow$ (c) Suppose there is a $\epsilon > 0$ such that Algorithm 3 halts with $\mathcal{R}_u = \emptyset$. Then the algorithm may produce one (or two) of the following pathes:

$$\mathcal{Q}_1^k \longrightarrow \mathcal{Q}_2^{k-1} \longrightarrow \cdots \longrightarrow \mathcal{F},$$
$$\mathcal{Q}_2^k \longrightarrow \mathcal{Q}_1^{k-1} \longrightarrow \cdots \longrightarrow \mathcal{F},$$

where $\cup_{m=1,\ldots,k} \mathcal{Q}_i^m = \mathcal{P}_i$, $i = 1, 2$. For any $m = 1, \ldots, k$, since every $\mathcal{Q}_i^m$ is a closed full-dimensional polytope and $\overset{o}{\mathcal{Q}}{}_i^m \cap \mathcal{O} = \emptyset$ from the algorithm, applying Theorem 4.1, we have a piecewise affine control that achieves $\mathcal{Q}_i^m \xrightarrow{\mathcal{Q}_i^m} \mathcal{Q}_j^{m-1}$ or $\mathcal{Q}_i^m \xrightarrow{\mathcal{Q}_i^m} \mathcal{F}$. Note that $\mathcal{Q}_i^k, \ldots, \mathcal{Q}_i^1$ may overlap each other. So for any $x \in \mathcal{P}_i$ belonging to more than one polytopes, set the controller at x to be the one defined for $\mathcal{Q}_i^{m_*}$ where m_* is the minimum of the indices of $\mathcal{Q}_i^m$ containing x. Thus, the piecewise affine control accomplishes $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

(c) $\Longrightarrow$ (a) Obvious. ∎

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