On a Reachability Problem for Affine Hypersurface Systems on Polytopes

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Abstract

This paper studies the problem for an affine hypersurface system (with $n-1$ inputs) to reach a polytopic target set starting from inside a polytope in the state space. We present a solution which begins with a characterization of solvability by open-loop control and concludes with a procedure to synthesize a feedback control. Our emphasis is on methods of subdivision, triangulation, and covers which explicitly account for the capabilities of the control system. In contrast with previous literature, the partition methods are guaranteed to yield a correct feedback synthesis, assuming the problem is solvable by open-loop control.

Key words: Piecewise affine systems, reachability, hybrid systems, switched systems

1 Introduction

This paper studies the problem for an affine system to reach a polytopic target set in finite time assuming the state space is a polytope. Promising new ideas have appeared in the last five years in this area [7–9,14,16]. The analogous problem for simplices was first formulated in [7] by Luc Habets and Jan van Schuppen. This paper specifically focuses on affine hypersurface systems on polytopes. The goal is to devise partition methods so that dynamic programming methods as in [9] are guaranteed to terminate. The contribution of the paper is the focus on subdivision (especially triangulation) methods guaranteeing a control synthesis. Indeed, there is no previous work linking triangulation procedures and control synthesis in the literature. Affine hypersurface systems on simplices were studied in [15,10]. Our investigation of polytopes and more general feedbacks was initiated in [11]. The reader is referred to an archived paper [12] that provides all missing proofs and examples supporting this paper. Our main result is: if the problem is solvable by open-loop controls, then it is solvable by piecewise affine feedback.

We conclude our introduction by mentioning that the problems studied here fit into a larger context concerning the use of piecewise affine feedback on polytopes for other control specifications such as stabilization, optimal control, and set invariance. The recent text [3] presents an overview of methods for set invariance, which can be viewed as dual to the problem of reachability. More generally, piecewise affine systems have been the subject of a large number of papers. A small sampling of recent papers includes [1,2,4,6,13]. Several interesting applications of piecewise affine modeling have recently been explored, for example [5].

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2 Problem Formulation

Let $\mathcal{P} \subset \mathbb{R}^n$ be an n-dimensional polytope and let $\mathcal{F}$ denote a target set which is an $(n-1)$-dimensional polytope in the boundary of $\mathcal{P}$. Consider an affine control system

$$\dot{x} = Ax + a + Bu =: f(x, u), \qquad x \in \mathcal{P}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $a \in \mathbb{R}^n$. We assume that $\mathrm{rank}(B) = n-1$, in which case (1) is called an *affine hypersurface system.* We also assume that (A, B) is a controllable. Given a piecewise continuous function $u : t \mapsto u(t)$ and an initial state $x_0 \in \mathcal{P}$, let $\phi_u(t, x_0)$ denote the unique solution of (1) starting from x_0.

Problem 1 (Reach Control Problem (RCP)) *Consider system (1) defined on $\mathcal{P}$. Find a feedback control $u(x)$ such that for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and*

$\epsilon > 0$ *satisfying: (i)* $\phi_u(t, x_0) \in \mathcal{P}$ *for all* $t \in [0, T]$*; (ii)* $\phi_u(T, x_0) \in \mathcal{F}$*; (iii)* $\phi_u(t, x_0) \notin \mathcal{P}$ *for all* $t \in (T, T+\epsilon)$.

Let $\mathbb{U}$ denote a control type, including affine feedback, piecewise affine feedback, open-loop controls, and so forth. We say a point $x_0 \in \mathcal{P}$ can reach $\mathcal{F}$ with constraint in $\mathcal{P}$ with control type $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{P}} \mathcal{F}$, if there exists a control u of type $\mathbb{U}$ such that properties (i)-(iii) of Problem 1 hold. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by control type $\mathbb{U}$ if for every $x_0 \in \mathcal{P}$, $x_0 \xrightarrow{\mathcal{P}} \mathcal{F}$ with control of type $\mathbb{U}$. If we do not state a control type, then it is inferred that we mean open-loop controls.

Let $\mathcal{B}$ denote the $(n-1)$-dimensional subspace spanned by the column vectors of B (namely, $\mathcal{B} = \mathrm{Im}(B)$, the image of B). Define $\mathcal{O} := \{x \in \mathbb{R}^n \; : \; Ax + a \in \mathcal{B}\}$. When the pair (A, B) is controllable it can be shown that $\mathcal{O}$ is an $(n-1)$-dimensional affine space. Notice that $f(x, u)$ on $\mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. We make the following standing assumption until Section 6.

Assumption 2 *If* $\mathcal{P} \cap \mathcal{O} \neq \emptyset$*, then* $\mathcal{P} \cap \mathcal{O}$ *is a* κ*-dimensional face of* $\mathcal{P}$*, where* $0 \le \kappa \le n-1$.

Problem 3 *We are given system* (1) *such that Assumption 2 holds. (a) Find necessary and sufficient conditions such that* $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ *by open-loop controls. (b) If* $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$*, then synthesize a feedback* $u(x)$ *such that* $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ *using* $u(x)$.

In the special case when $\mathcal{P}$ is a simplex $\mathcal{S}$, we have the following solution of the problem.

Theorem 4 *[10] If* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by open-loop controls, then* $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by piecewise affine feedback.*

3 Open-Loop Reachability

In this section we study necessary and sufficient conditions for solvability of RCP by open-loop control. For a set $\mathcal{A} \subset \mathbb{R}^n$, notation $\overset{o}{\mathcal{A}}$, $\mathrm{conv}(\mathcal{A})$, and $\mathrm{vert}(\mathcal{A})$ denote the interior of $\mathcal{A}$, the convex hull of $\mathcal{A}$, and the vertices of $\mathcal{A}$, respectively. Denote by $\mathcal{B}_x$ the hyperplane parallel to $\mathcal{B}$ and going through a point x. Let β be the unit normal vector to $\mathcal{B}$ satisfying $\beta^T(Ax + a) \le 0$ for all $x \in \mathcal{P}$. Such β always exists by our Assumption 2 that $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. Let v^- be a point in $\arg\min\{\beta^T x : x \in \mathcal{F}\}$ and v^+ a point in $\arg\max\{\beta^T x : x \in \mathcal{F}\}$. Define the sets $\mathcal{H}^- := \{x \in \mathcal{P} \mid \beta^T x \le \beta^T v^-\}$, $\mathcal{H}^+ := \{x \in \mathcal{P} \mid \beta^T x \ge \beta^T v^+\}$, and $\mathcal{P}^+ := \arg\max\{\beta^T x \mid x \in \mathcal{P}\}$. Also, for any $z \in \mathbb{R}^n$, define $\mathcal{H}^-(z) := \{x \in \mathcal{P} \mid \beta^T x \le \beta^T z\}$ and $\mathcal{H}^+(z) := \{x \in \mathcal{P} \mid \beta^T x \ge \beta^T z\}$.

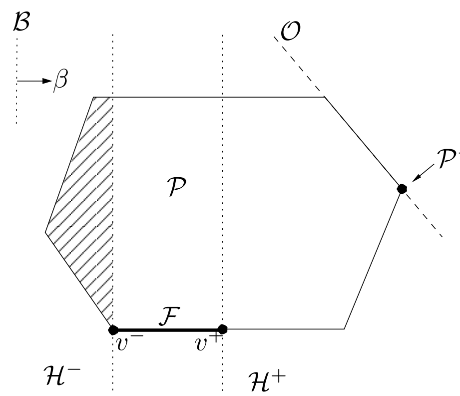

Fig. 1. Illustration for Theorem 6

Because $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$, we know that for each initial condition in $\mathcal{P}$, all trajectories will only flow in one direction relative to hyperplane $\mathcal{B}$. In particular, the β-component of any trajectory, $\beta \cdot \phi_u(t, x_0)$, is always non-increasing by the convention that $\beta \cdot (Ax + a) \le 0, \forall x \in \mathcal{P}$. Now the points v^- and v^+ mark the points in $\mathcal{F}$ with minimum and maximum β components. It is clear that if there is any $x_0 \in \mathcal{P}$ with a β component smaller than v^-, then no $\phi_u(t, x_0)$ can reach $\mathcal{F}$. This suggests that a first necessary condition for $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ is that $\mathcal{H}^- \setminus \mathcal{F}$ is empty. This is not quite right. In particular, if $x, y \in \mathcal{B}_x \cap \mathcal{O}$, then $x \xrightarrow{l} y$ where l is the line segment joining x and y. This follows from the observation that points in $\mathcal{O}$ can steer along $\mathcal{B}$, and by assumption $y - x \in \mathcal{B}$. Therefore the first necessary condition for solvability via open-loop controls is that the first *failure set* $\mathcal{A}^- := \mathcal{H}^- \setminus (\mathcal{F} \cup \mathcal{B}^-)$ is the empty set, where $\mathcal{B}^- := \mathcal{B}_{v^-} \cap \mathcal{O}$ if $\mathcal{B}_{v^-} \cap \mathcal{O} \cap \mathcal{F} \neq \emptyset$, and otherwise $\mathcal{B}^- = \emptyset$. Figure 1 shows a shaded region corresponding to failure of this condition.

Another failure leading to a second necessary condition is as follows. If $x_0 \in \mathcal{P} \cap \mathcal{O}$, then the instantaneous motion from this point is only along $\mathcal{B}$. If $\mathcal{P} \cap \mathcal{O} \subset \mathcal{B}$ and no direction in $\mathcal{B}$ points into $\mathcal{P}$, then the set $\mathcal{P} \cap \mathcal{O}$ becomes *controlled invariant* when the invariance conditions for $\mathcal{P}$ are imposed. The only points where $\mathcal{B}$ does not point into $\mathcal{P}$ are at the extreme values of $\beta^T x$ for $x \in \mathcal{P}$. In particular, it should not be allowed that $\mathcal{P}^+ \subset \mathcal{B}$ within the region $\{x \mid \beta^T x > \beta^T v^+\}$. See the right side of Figure 1. Therefore, we can define the second failure set $\mathcal{A}^+ = \mathcal{P}^+$, if $\mathcal{P}^+ \subset \mathcal{O} \cap \{x \mid \beta^T x > \beta^T v^+\}$; and otherwise $\mathcal{A}^+ = \emptyset$.

The argument to show that the necessary conditions are also sufficient relies on two properties: the system is controllable, so it has sufficient maneuverability on $\mathcal{O}$, and the following lemma which provides the required maneuverability off of $\mathcal{O}$.

Lemma 5 *Let* $y \neq z \in \mathcal{P}$ *and let* l *be the line segment joining them.*
(i) If $z, y \in \mathcal{O}$ *and* $y \in \mathcal{B}_z$*, then* $z \xrightarrow{l} y$.

(ii) If $z, y \notin \mathcal{O}$ and $y \in \overset{o}{\mathcal{H}}{}^-(z)$, then $z \xrightarrow{l} y$.

Theorem 6 *Suppose Assumption 2 holds and (A, B) is controllable. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ if and only if (a) $\mathcal{A}^- = \emptyset$ and (b) $\mathcal{A}^+ = \emptyset$.*

PROOF. We present only the sufficiency proof. Suppose conditions (a) and (b) hold. For a point $x \in \mathcal{F}$, one can easily show that under (a) and (b), $x \xrightarrow{\mathcal{P}} \mathcal{F}$. Instead, let $x \in \mathcal{P} \setminus (\mathcal{F} \cup \mathcal{O})$. By assumption (a) $x \notin \mathcal{H}^-$ or equivalently $v^- \in \overset{o}{\mathcal{H}}{}^-(x)$. Consequently, there is a point $y \in \mathcal{N}(v^-) \cap \mathcal{F}$ satisfying $y \in \overset{o}{\mathcal{H}}{}^-(x)$, where $\mathcal{N}(v^-)$ is a sufficiently small neighborhood of v^-. If $\mathcal{F}$ is not in $\mathcal{O}$, such a point y can be chosen not in $\mathcal{O}$. Then these two points x and y satisfy the assumption in Lemma 5(ii), so $x \xrightarrow{l} y$, where l is the line segment joining x and y. Clearly, l is in $\mathcal{P}$ as $\mathcal{P}$ is convex. Hence, $x \xrightarrow{\mathcal{P}} \mathcal{F}$. Otherwise, suppose $\mathcal{F} \subset \mathcal{O}$. Because (A, B) is controllable, $\mathcal{B}$ is not parallel to $\mathcal{O}$. It means we can select a control u so that $f(y, u)$ points outside of $\mathcal{P}$. Thus, there is a sufficiently small $\epsilon > 0$ such that $\phi_u(t, y), t \in (-\epsilon, 0)$ is in $\overset{o}{\mathcal{P}}$. Note that $\phi_u(t, y)$ is continuous and $y \in \overset{o}{\mathcal{H}}{}^-(x)$, so there is a point $z \in \phi_u(t, y), t \in (-\epsilon, 0)$ satisfying $z \in \overset{o}{\mathcal{H}}{}^-(x)$ and therefore $z \in \overset{o}{\mathcal{P}}$. Thus, $\beta^T(Az + a) < 0$ by assumption. Applying Lemma 5(ii) for the two points x and z leads to $x \xrightarrow{l} z$, where l is the line segment in $\mathcal{P}$ joining x and z. Considering $z \xrightarrow{\mathcal{P}} y \in \mathcal{F}$, we then have $x \xrightarrow{\mathcal{P}} \mathcal{F}$.

Finally, let $x \in (\mathcal{P} \cap \mathcal{O}) \setminus \mathcal{F}$. Clearly, x is on the boundary of $\mathcal{P}$. If $\mathcal{B}_x \cap \mathcal{P} \not\subset \mathcal{O}$, then select a point $y \in (\mathcal{B}_x \cap \mathcal{P}) \setminus \mathcal{O}$ and let u be chosen such that $f(x, u) = Ax + a + Bu = y - x$, which is possible because both $(Ax + a)$ and $(y - x)$ are in $\mathrm{Im}(B)$. Note that $f(x, u)$ points inside the polytope $\mathcal{P}$. This implies the trajectory instantaneously enters the interior of $\mathcal{P}$, which is not in $\mathcal{O}$ any more. Then by the previous argument, it can be driven to reach $\mathcal{F}$ through a line. Otherwise, if $\mathcal{B}_x \cap \mathcal{P} \subset \mathcal{O}$, then the whole set $\mathcal{B}_x \cap \mathcal{P}$ is on the boundary of $\mathcal{P}$, and moreover it comprises either $\mathcal{P}^+$ or $\arg\min\{\beta^T x : x \in \mathcal{F}\}$.

From condition (b), $\mathcal{P}^+ \subset \mathcal{H}^-(v^+)$ and this implies $\mathcal{P}^+ \cap \mathcal{F} \neq \emptyset$. From condition (a), $\overset{o}{\mathcal{H}}{}^- = \emptyset$ so $\arg\min\{\beta^T x : x \in \mathcal{F}\} \subset \mathcal{H}^+(v^-)$, which implies $\arg\min\{\beta^T x : x \in \mathcal{F}\} \cap \mathcal{F} \neq \emptyset$. For both cases, we get $\mathcal{B}_x \cap \mathcal{F} \neq \emptyset$. Now we select a point $y \in \mathcal{B}_x \cap \mathcal{F}$. Then these two points satisfy the assumption in Lemma 5(i). Thus, it follows that $x \xrightarrow{l} y$, where l is the line segment joining from x to y, and so $x \xrightarrow{\mathcal{P}} \mathcal{F}$.

In [11] (see also [12]) it was shown that one can "cut off" the failure sets $\mathcal{A}^+$ and $\mathcal{A}^-$ to obtain a smaller polytope $\mathcal{P}'$ with the same exit facet $\mathcal{F}$ and such that $\mathcal{P}' \xrightarrow{\mathcal{P}'} \mathcal{F}$. We assume that this procedure has been applied, that $\mathcal{P}'$ is renamed as $\mathcal{P}$, and $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

4 Control Synthesis on Polytopes

We now begin our investigation of state feedback synthesis on polytopes. We want to show that if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ using open-loop control then there exists a piecewise affine feedback solving the reachability problem. The idea is to triangulate the polytope, transform the reachability problem within a polytope into a set of reachability problems for simplices, and then devise appropriate piecewise affine controllers on each simplex using Theorem 4 of the previous section. The triangulation must be performed properly otherwise the procedure may fail. First we present a lemma that aids in finding a proper triangulation. See Figure 2.

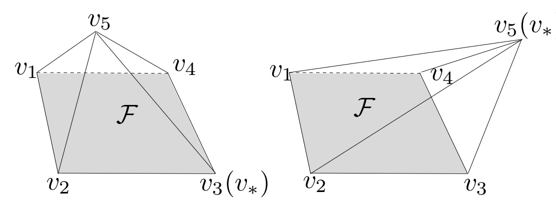

Fig. 2. Illustration for Lemma 7.

Lemma 7 *If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$, then there exists a vertex v_* of $\mathcal{P}$ in $\mathcal{P}^+$ such that either $v_* \notin \mathcal{O}$ or $v_* \in \mathcal{F}$.*

PROOF. Suppose by contradiction that for any vertex $v \in \mathcal{P}^+$ we have $v \in \mathcal{O}$ and $v \notin \mathcal{F}$. Note that $v \notin \mathcal{F}$ for all vertices $v \in \mathcal{P}^+$ implies, by convexity, $\mathcal{P}^+ \subset \overset{o}{\mathcal{H}}{}^+$. Moreover, since $v \in \mathcal{O}$ for all $v \in \mathcal{P}^+$, it follows from the convexity of $\mathcal{O}$ that $\mathcal{P}^+ \subset \mathcal{O}$. Hence, by Theorem 6, this contradicts $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

Basic Triangulation of $\mathcal{P}$:

(1) Select v_* as in Lemma 7.
(2) Triangulate each facet $\mathcal{F}_j$ of $\mathcal{P}$. Denote $\{\mathcal{S}^i_{\mathcal{F}_j} : i = 1, \ldots, k_j\}$ the triangulation for $\mathcal{F}_j$.
(3) Let $\mathbb{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_q\} := \{\mathrm{conv}(v_*, \mathcal{S}^i_{\mathcal{F}_j}) :$ $\mathcal{F}_j$ is any facet of $\mathcal{P}$ not containing $v_*\}$.

Lemma 8 *The collection $\mathbb{S}$ is a triangulation of $vert(\mathcal{P}) \cup vert(\mathcal{F})$ such that every simplex in $\mathbb{S}$ contains v_* as a vertex.*

PROOF. By construction, it is clear that every simplex $\mathcal{S}_i \in \mathbb{S}$ contains v_* as a vertex, the vertices of $\mathcal{S}_i$ are drawn from $\mathrm{vert}(\mathcal{P}) \cup \mathrm{vert}(\mathcal{F})$, and $\mathcal{S}_i \cap \mathcal{S}_j$ $(i \neq j)$ is a common (possibly empty) face of $\mathcal{S}_i$ and $\mathcal{S}_j$. Next, we show that $\mathcal{P}$ is the union of $\mathcal{S}_1, \ldots, \mathcal{S}_q$. Let x be a point in the union of $\mathcal{S}_1, \ldots, \mathcal{S}_q$. Then it must be in a simplex $\mathcal{S}_i$. Thus, by convexity of $\mathcal{P}$, $x \in \mathcal{P}$. On the other hand, let x be a point in $\mathcal{P}$. Draw a line through v_* and x. It intersects at a point y with a facet (say $\mathcal{F}_j$) of $\mathcal{P}$ that does not contain v_*. It means there exists a simplex $\mathcal{S}^i_{\mathcal{F}_j}$ containing y. So $x \in \mathrm{conv}(v_*, \mathcal{S}^i_{\mathcal{F}_j})$, one of the simplices in $\mathbb{S}$. The conclusion follows.

Now suppose we have a triangulation $\mathbb{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_q\}$ as above, and denote $\mathcal{S}_0 := \mathcal{F}$. We say $\mathcal{S}_i$ and $\mathcal{S}_j$ are *adjacent* (denoted by $\mathcal{S}_i \sim \mathcal{S}_j$) if $\mathcal{F}_{ij} := \mathcal{S}_i \cap \mathcal{S}_j$ is a facet. A sequence $(\mathcal{S}_{i_k}, \ldots, \mathcal{S}_{i_0})$ is called a *path* to reach $\mathcal{S}_{i_0}$ if $\mathcal{S}_{i_j} \sim \mathcal{S}_{i_{j-1}}$ and $\mathcal{S}_{i_j} \xrightarrow{\mathcal{S}_{i_j}} \mathcal{S}_{i_{j-1}}$ for each $1 \leq j \leq k$. The *length* of such a path is k. We propose a greedy algorithm that orders simplices according to minimum β component of exit vertices first. More precisely, at every iteration a pair $(\mathcal{S}_i, \mathcal{S}_j)$ is selected that minimizes the β-component of any vertex on the exit facet $\mathcal{F}_{ij}$. If there is more than one pair achieving the minimum, select a pair which has the maximum number of exit vertices achieving the minimum. In the algorithm below $\mathcal{R}_f$ and $\mathcal{R}_u$ denote the finished and unfinished set of simplices, respectively, and let $w' \in \arg\min\big\{\beta^T x \ : \ x \in \bigcup\{\mathcal{S}_k \cap \mathcal{S}_l \ : \ (\mathcal{S}_k, \mathcal{S}_l) \in \mathcal{R}_u \times \mathcal{R}_f \text{ satisfying } \mathcal{S}_k \sim \mathcal{S}_l\}\big\}$.

Greedy algorithm for path generation in $\mathbb{S}$:

(1) Initialization: $\mathcal{R}_f := \{\mathcal{S}_0\}$, $\mathcal{R}_u := \{\mathcal{S}_1, \ldots, \mathcal{S}_q\}$;
(2) While $(\mathcal{R}_u \neq \emptyset)$, choose $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ such that $\mathcal{S}_i \sim \mathcal{S}_j$, it achieves $\min\limits_{x \in \mathcal{F}_{ij}} \beta^T x = \beta^T w'$, and $\mathcal{F}_{ij}$ contains the maximum number of vertices in $\mathcal{B}_{w'}$. Then move $\mathcal{S}_i$ from $\mathcal{R}_u$ to $\mathcal{R}_f$.

Once the greedy algorithm has generated paths, the synthesis of a piecewise affine control is straightforward. See also [9].

Theorem 9 *Suppose that $\mathcal{F}$ is a facet of $\mathcal{P}$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by open-loop controls, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by piecewise affine feedback.*

The idea of the proof is to show that the path generation algorithm does not terminate until $\mathcal{R}_u = \emptyset$ by showing that for the next selected pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$, the reachability problem $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{S}_j$ can be solved. This is done by applying Theorem 6 and verifying conditions (a) and (b) for the selected pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$. The main effect of our selection of triangulation based on vertex v_* is that condition (b) holds trivially for any such pair. The fact that condition (a) can be made to hold is the main feature of the greedy strategy with respect to β. This strategy guarantees that the vertex $v_0 \in \mathcal{S}_i$ not contained in the exit facet has a strictly larger β-component, and this means that failure set $\mathcal{A}^- = \emptyset$ for $\mathcal{S}_i$. The proof now easily follows from these observations.

PROOF. If the path generation algorithm terminates with $\mathcal{R}_u = \emptyset$ then by straightforward dynamic programming arguments there exists a piecewise affine feedback control that achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$. It is therefore sufficient to show that if $\mathcal{R}_u \neq \emptyset$, there exists a pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ such that $\mathcal{S}_i \cap \mathcal{S}_j =: \mathcal{F}_{ij}$ is a facet and $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{S}_j$.

Consider any pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ such that $\mathcal{S}_i \cap \mathcal{S}_j =: \mathcal{F}_{ij}$ is a facet. We must verify conditions (a) and (b) of Theorem 6 to show $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{F}_{ij}$. Consider condition (b). We have two observations about v_*. First, from Lemma 8, $v_* \in \mathcal{S}_i$, $\forall i$, and therefore $v_* \in \mathcal{F}_{ij}$. Second, $v_* \in \mathcal{P}^+$ implies $v_* \in \mathcal{S}_i^+$. Applying these two facts, condition (b) for $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{S}_j$ says that $\mathcal{S}_i^+ \not\subset \mathcal{O} \cap \{x \in \mathcal{S}_i : \beta^T x > \beta^T v_*\}$, and this is obviously true.

So far we have shown that for any pair $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{R}_u \times \mathcal{R}_f$ as above, condition (b) of Theorem 6 holds for the problem $\mathcal{S}_i \xrightarrow{\mathcal{S}_i} \mathcal{F}_{ij}$. Now we will show that for the selected pair $(\mathcal{S}_i, \mathcal{S}_j)$, condition (a) holds. Let v_0 be the vertex of $\mathcal{S}_i$ not in $\mathcal{F}_{ij}$. Let $w \in \mathcal{F}_{ij} \cap \mathcal{B}_{w'}$. There are three cases. First, suppose $\beta^T w < \beta^T v_0$. Then condition (a) holds. Second, suppose $\beta^T w > \beta^T v_0$. Also, we know $\beta^T v^- \leq \beta^T v_0 < \beta^T w$ from the assumption $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$. By convexity, for every point y on the line segment joining v^- and v_0, $\beta^T y < \beta^T w$. However, $v^- \in \mathcal{S}_0 \in \mathcal{R}_f$ and $v_0 \in \mathcal{S}_i \in \mathcal{R}_u$, which means the line segment contains a point y on the boundary of $\mathcal{S}_{i'} \in \mathcal{R}_u$ and $\mathcal{S}_{j'} \in \mathcal{R}_f$. This contradicts the choice of the pair $(\mathcal{S}_i, \mathcal{S}_j)$ that achieves $\min\limits_{x \in \mathcal{F}_{ij}} \beta^T x = \beta^T w'$.

Finally, suppose $\beta^T w = \beta^T v_0$. Let $\{v_1, \ldots, v_k\}$ be the set of vertices of $\mathcal{F}_{ij}$ that lie in $\mathcal{B}_{w'}$. If $\mathcal{B}_{w'} \cap \mathcal{P} \subset \mathcal{O}$ then condition (a) holds and we are done. If not, it follows from the assumption $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ that either $\mathcal{B}_{w'} \cap \mathcal{P} \subset \mathcal{F}$ or $\beta^T w' > \beta^T v^-$. For both cases we claim that $\mathcal{G} := \mathrm{conv}\{v_0, v_1, \ldots, v_k\}$ belongs to some $\mathcal{S}_k \in \mathcal{R}_f$. For the former case, it is obvious since $\mathcal{G} \subset \mathcal{B}_{w'} \cap \mathcal{P} \subset \mathcal{S}_0 \in \mathcal{R}_f$. For the latter case, suppose not. Say a point $x \in \mathcal{G}$ does not belong to some $\mathcal{S}_k \in \mathcal{R}_f$. Then since the union of sets in $\mathcal{R}_f$ is a closed set, there exists a point $y \in \mathcal{P}$ near x satisfying $\beta^T y < \beta^T w'$, and y also does not belong to some $\mathcal{S}_k \in \mathcal{R}_f$. This contradicts the choice of the pair $(\mathcal{S}_i, \mathcal{S}_j)$ that achieves $\min\limits_{x \in \mathcal{F}_{ij}} \beta^T x = \beta^T w'$. Therefore $\mathcal{P} \cap \mathcal{O}$ belongs to some $\mathcal{S}_k \in \mathcal{R}_f$ which implies it belongs to some facet $\mathcal{F}_{i'j'} \neq \mathcal{F}_{ij}$ with $\mathcal{F}_{i'j'} = \mathcal{S}_{i'} \cap \mathcal{S}_{j'}$, where

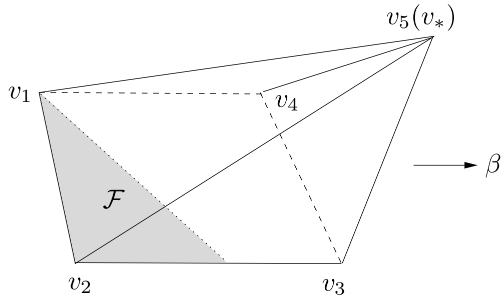

Fig. 3. $\mathcal{F}$ is not a facet of $\mathcal{P}$ but $v_* \notin \bar{\mathcal{F}}$.

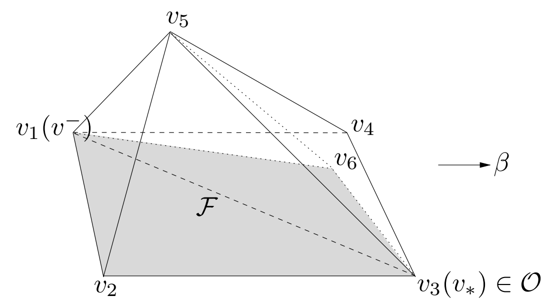

Fig. 4. $\mathcal{F} = \text{conv}\{v_1, v_2, v_3, v_6\} \subset \bar{\mathcal{F}}$ and $v_* = v_3 \in \mathcal{F}$.

$\mathcal{S}_{i'} \in \mathcal{R}_u$, $\mathcal{S}_{j'} \in \mathcal{R}_f$, and $\mathcal{F}_{i'j'}$ has one more vertex, namely v_0, in $\mathcal{B}_{w'}$. This contradicts the choice of $\mathcal{F}_{ij}$.

5 Triangulation with respect to $\mathcal{F}$

In this section we study how the previous results can be extended to solve the control synthesis problem if $\mathcal{F}$ is not given as a facet of $\mathcal{P}$. If the designer has flexibility in modifying the given state constraints, then one can perform a deformation of $\mathcal{P}$ by "pulling out" $\mathcal{F}$ so that $\mathcal{F}$ is a facet of a larger polytope $\mathcal{P}'$. However, this approach has two caveats: (1) The problem $\mathcal{P}' \xrightarrow{\mathcal{P}'} \mathcal{F}'$ may not be solvable even if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ is; (2) If $\mathcal{P}$ is part of a larger subdivision of the state space, then possibly other polytopes in the subdivision must be modified. A more desirable procedure is to use a triangulation method that refines the given subdivision of the state space by splitting $\mathcal{P}$ so that $\mathcal{F}$ becomes a facet of one of the polytopes in the refined subdivision. This approach also has pitfalls, because if one does not refine the subdivision properly, failure sets may emerge even if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by open-loop control. In this section we show one method (among several) to obtain a proper triangulation.

Let $\bar{\mathcal{F}}$ denote the facet of $\mathcal{P}$ containing $\mathcal{F}$. First we consider a simple case when v_* of Lemma 8 can be selected so that $v_* \notin \bar{\mathcal{F}}$. See Figure 3.

Triangulation of $\mathcal{P}$ with respect to $\mathcal{F}$:

(a) Select v_* as in Lemma 7 and so that $v_* \notin \bar{\mathcal{F}}$.
(b) Make a triangulation of $\text{vert}(\bar{\mathcal{F}}) \cup \text{vert}(\mathcal{F})$ such that the interior of each resulting simplex is either entirely in $\mathcal{F}$ or not in $\mathcal{F}$. For the remaining facets $\mathcal{F}_j$ of $\mathcal{P}$, make a triangulation of $\text{vert}(\mathcal{F}_j)$. Denote $\{\mathcal{S}^i_{\mathcal{F}_j} : i = 1, \ldots, k_j\}$ the triangulation for $\mathcal{F}_j$.
(c) Let $\mathbb{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_q\} := \{\text{conv}(v_*, \mathcal{S}^i_{\mathcal{F}_j}) :$ $\mathcal{F}_j$ is any facet of $\mathcal{P}$ not containing $v_*\}$.

The first thing we notice is that nothing about the proof of Lemma 8 is specific to $\mathcal{F}$ being a facet, so the lemma still holds for the new triangulation. Also the proof of Theorem 9 is unchanged since the essential property of v_* (namely Lemma 8) is still true. Therefore, we have the following direct extension of Theorem 9.

Corollary 10 *Suppose that $\mathcal{F}$ is not a facet of $\mathcal{P}$ and there exists v_* as in Lemma 7 such that $v_* \notin \bar{\mathcal{F}}$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by open-loop controls, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by piecewise affine feedback.*

When there does not exist $v_* \notin \bar{\mathcal{F}}$, the problem is more complex because Lemma 8 breaks down. Nevertheless, we would like to build upon our previous triangulation and control methods by appropriately subdividing $\mathcal{P}$. A natural idea would be to form $\mathcal{P}_1 := \text{conv}(\mathcal{F}, v \mid v \in \text{vert}(\mathcal{P}) \setminus \bar{\mathcal{F}})$, a polytope for which $\mathcal{F}$ is a facet. There are two problems to be addressed. First, can $\mathcal{P}_1$ have failure sets for the problem $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$ even if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$? Theorem 6 tell us that $\mathcal{H}^- \setminus (\mathcal{F} \cap \mathcal{B}^-) = \emptyset$ and we observe that this condition is identical for any polytope with the same exit facet $\mathcal{F}$. Therefore, condition (a) holds for $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$. Instead, it is condition (b) which is problematic because generally $\mathcal{P}_1^+ \neq \mathcal{P}^+$ and equilibria can appear on $\mathcal{P}_1^+$ when we try to solve $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$. A more careful approach is needed, and inspiration is provided by the proof of Theorem 9: for any n-dimensional polytope $\mathcal{P}_1 \subset \mathcal{P}$ with exit facet $\mathcal{F}$, if $\mathcal{P}_1^+ \cap \mathcal{F} \neq \emptyset$, then condition (b) automatically holds. For example, in Figure 4 a polytope with this property is $\mathcal{P}_1 = \text{conv}\{v_1, v_2, v_3, v_6, v_5\}$. Thus, we have the following.

Proposition 11 *Suppose there exists v_* a vertex of $\mathcal{F}$ such that $v_* \in \mathcal{F} \cap \mathcal{P}^+$. Let $\mathcal{P}_1 \subset \mathcal{P}$ be an n-dimensional polytope such that $\mathcal{F}$ is a facet of $\mathcal{P}_1$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ implies $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$.*

PROOF. Consider condition (b) of Theorem 6 for $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$. We have to show that $\mathcal{P}_1^+ \not\subset \mathcal{O} \cap \{x \in \mathcal{P}_1 \mid \beta^T x > \beta^T v^+\}$. But $v_* \in \mathcal{F} \cap \mathcal{P}^+$ implies $\mathcal{P}_1^+ = \{x \in \mathcal{P}_1 \mid \beta^T x = \beta^T v^+\}$, so condition (b) is obviously true.

For condition (a), Theorem 6 tells us that $\mathcal{H}^- \setminus (\mathcal{B}^- \cup \mathcal{F}) = \emptyset$ and since $\mathcal{H}_1^- = \mathcal{H}^-$ and $\mathcal{B}_1^- = \mathcal{B}^-$, condition

(a) obviously holds for $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$.

Proposition 11 gives some indication of how the polytope $\mathcal{P}_1$ which has $\mathcal{F}$ as a facet could be constructed. Now we face the second problem. The set $\mathcal{P} \setminus \mathcal{P}_1$ is of course not a polytope. For example, in Figure 4 with $\mathcal{P}_1 = \mathrm{conv}\{v_1, v_2, v_3, v_6, v_5\}$, the remainder $\mathcal{P} \setminus \mathcal{P}_1$ is not convex. How shall this remainder be subdivided and what reachability problems need to be assigned to avoid new failure sets from appearing? The problem is difficult due to the generality of the description of $\mathcal{F}$. Instead, we will search for other polytopes which do not use $\mathcal{F}$ as their exit facet but which effectively channel trajectories toward $\mathcal{P}_1$. Considering again Figure 4, let $\mathcal{P}_1 = \mathrm{conv}\{v_1, v_2, v_3, v_6, v_5\}$, $\mathcal{P}_3 := \mathrm{conv}\{v_1, v_3, v_4, v_5\}$, and suppose $\mathcal{P}_3$'s exit facet is $\mathcal{F}_3 := \mathrm{conv}\{v_1, v_3, v_5\}$. If trajectories can be made to exit $\mathcal{P}_3$ through $\mathcal{F}_3$, then they arrive in $\mathcal{P}_1$, for which a control strategy is already known. Why can this be achieved for $\mathcal{P}_3$? It is explained in the next result which gives a procedure for identifying certain polytopes which do not use $\mathcal{F}$ as the exit facet, but which guarantee that trajectories arrive in $\mathcal{P}_1$, nonetheless.

Proposition 12 *Suppose there exists v_*, a vertex of $\mathcal{F}$, such that $v_* \in \mathcal{F} \cap \mathcal{P}^+$. Let $\mathcal{P}_3 \subset \mathcal{P}$ be an n-dimensional polytope and let $\mathcal{F}_3$ be an $(n-1)$-dimensional polytope which is a facet of $\mathcal{P}_3$. Suppose that $v^-, v_* \in \mathcal{F}_3$ and $\mathcal{B}_{v^-} \cap \mathcal{F} \subset \mathcal{B}_{v^-} \cap \mathcal{F}_3$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ implies $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}_3$.*

PROOF. By the same argument as in Proposition 11, condition (b) for $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}_3$ obviously holds. Consider condition (a) for $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$. It says that $\{x \in \mathcal{P} \mid \beta^T x \leq \beta^T v^-\} \setminus (\mathcal{F} \cup \mathcal{B}^-) = \emptyset$. Equivalently, $\{x \in \mathcal{P} \mid \beta^T x < \beta^T v^-\} = \emptyset$ and if $x \in \mathcal{B}_{v^-}$ then either $x \in \mathcal{F}$ or $x \in \neg\mathcal{F} \cap \mathcal{O}$ and there exists $y \in \mathcal{B}_{v^-} \cap \mathcal{O} \cup \mathcal{F}$. Now consider $\mathcal{P}_3$. Since $\mathcal{P}_3 \subset \mathcal{P}$, $\{x \in \mathcal{P}_3 \mid \beta^T x < \beta^T v^-\} = \emptyset$. Also if $x \in \mathcal{B}_{v^-}$, then either $x \in \mathcal{F}$, which implies by assumption that $x \in \mathcal{F}_3$; otherwise $x \in \neg\mathcal{F}_3 \cap \neg\mathcal{F} \cap \mathcal{O}$, and there exists $y \in \mathcal{B}_{v^-} \cap \mathcal{O} \cap \mathcal{F} \subset \mathcal{B}_{v^-} \cap \mathcal{O} \cap \mathcal{F}_3$. Thus condition (a) for $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}_3$ holds. The conclusion follows.

We can now put together the ideas of Propositions 11 and 12 to solve the synthesis problem when $\mathcal{F}$ is not a facet of $\mathcal{P}$ and there exists a vertex of $\mathcal{F}$ satisfying $v_* \in \mathcal{F} \cap \mathcal{P}^+$. Proposition 11 tells us how to obtain a polytope $\mathcal{P}_1$ for which $\mathcal{F}$ is its exit facet. Proposition 12 is a tool to obtain other polytopes, say $\mathcal{P}_2$ and $\mathcal{P}_3$ which drive trajectories into $\mathcal{P}_1$. Naturally this requires that $\mathcal{P}_2$ and $\mathcal{P}_3$ have a non-zero intersection with $\mathcal{P}_1$. To accommodate this in the simplest possible manner without involving the details of $\mathcal{F}$, we introduce an important new construct for synthesis of piecewise affine controllers. Rather than using a subdivision of $\mathcal{P}$ we begin the design with a cover of $\mathcal{P}$, which later will be refined to a subdivision for control synthesis. A *cover* of $\mathcal{V}$ is a finite collection $\mathbb{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ of n-dimensional polytopes such that the vertices of each $\mathcal{P}_i$ are drawn from $\mathcal{V}$ and $\mathrm{conv}(\mathcal{V})$ is the union of $\mathcal{P}_1, \ldots, \mathcal{P}_k$. Informally, a cover is a subdivision except that the sub-polytopes can intersect on their interiors.

Cover of $\mathcal{P}$ with respect to $\mathcal{F}$:

(1) Select v_* a vertex of $\mathcal{F}$ such that $v_* \in \mathcal{F} \cap \mathcal{P}^+$.
(2) Construct any hyperplane that goes through points v^- and v_*, and partitions $\mathcal{P}$ into two n-dimensional sub-polytopes $\mathcal{P}_2$ and $\mathcal{P}_3$.
(3) Define $\mathcal{P}_1 = \mathrm{conv}(\mathcal{F}, \mathcal{P}_2 \cap \mathcal{P}_3)$.
(4) Define the cover $\mathbb{P} := \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$.

For the example in Figure 4, the algorithm gives $\mathcal{P}_1 = \mathrm{conv}\{v_1, v_2, v_3, v_6, v_5\}$, $\mathcal{P}_2 := \mathrm{conv}\{v_1, v_2, v_3, v_5\}$, and $\mathcal{P}_3 := \mathrm{conv}\{v_1, v_3, v_4, v_5\}$.

Theorem 13 *Suppose that $\mathcal{F}$ is not a facet of $\mathcal{P}$ and there exists v_*, a vertex of $\mathcal{F}$, such that $v_* \in \mathcal{F} \cap \mathcal{P}^+$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by open-loop controls, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by piecewise affine feedback.*

PROOF. $\mathcal{P}_1$ is an n-dimensional polytope in $\mathcal{P}$ for which $\mathcal{F}$ is a facet. Also, $v_* \in \mathcal{F} \cap \mathcal{P}^+$, so by Proposition 11, $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$. Next, let $\mathcal{F}_{23} = \mathcal{P}_2 \cap \mathcal{P}_3$ and notice that $v_{23}^- = v^-$ and $v_{23}^+ = v_*$. By a minor adaptation of the argument for Proposition 12, $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{F}_{23}$ and $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}_{23}$, except at those points in $\mathcal{B}_{v^-} \cap \mathcal{F}$ which are not in $\mathcal{F}_{23}$.

Theorem 9 gives a piecewise affine control $u(x) = F_{\sigma_1(x)} x + g_{\sigma_1(x)}$, $x \in \mathcal{P}_1$, that achieves $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$. Also, it gives $u(x) = F_{\sigma_2(x)} x + g_{\sigma_2(x)}, x \in \mathcal{P}_2$ and $u(x) = F_{\sigma_3(x)} x + g_{\sigma_3(x)}, x \in \mathcal{P}_3$, that achieve $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{F}_{23}$ and $\mathcal{P}_3 \xrightarrow{\mathcal{P}_3} \mathcal{F}_{23}$, respectively (except at points in $\mathcal{B}_{v^-} \cap \mathcal{F}$ which are not in $\mathcal{F}_{23}$). Since $\mathcal{F}_{23} \subset \mathcal{P}_1$, it means that the controllers can drive all the states not in $\mathcal{P}_1$ to $\mathcal{P}_1$. Thus, the following controller

$$u(x) = \begin{cases} F_{\sigma_1(x)} x + g_{\sigma_1(x)} & x \in \mathcal{P}_1 \\ F_{\sigma_2(x)} x + g_{\sigma_2(x)} & x \in \mathcal{P}_2 \setminus \mathcal{P}_1 \\ F_{\sigma_3(x)} x + g_{\sigma_3(x)} & x \in \mathcal{P}_3 \setminus \mathcal{P}_1 \end{cases}$$

achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

Finally, we are left with the case when $\mathcal{F}$ is not a facet of $\mathcal{P}$, all vertices of $\mathcal{P}$ satisfying Lemma 7 are in $\bar{\mathcal{F}}$ but none of them is in $\mathcal{F}$, and moreover there are no vertices of $\mathcal{F}$

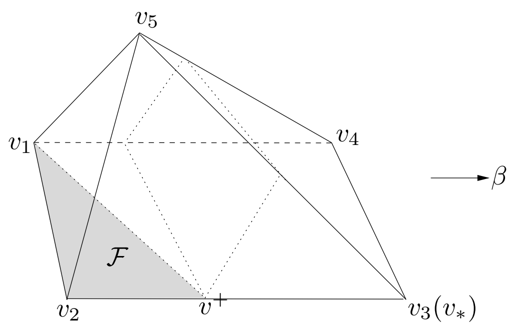

Fig. 5. $\mathcal{F} \subset \bar{\mathcal{F}} = \mathrm{conv}\{v_1, v_2, v_3, v_4\}$ and all v_* are in $\bar{\mathcal{F}}$ but none of them is in $\mathcal{F}$.

in $\mathcal{P}^+$. See Figure 5. Fortunately, this case can be easily handled by our previous results, by observing that $\mathcal{F}$ and $\mathcal{P}^+$ are strongly separated so we can split $\mathcal{P}$ into a sub-polytope which contains $\mathcal{F}$ and satisfies Theorem 13 and another sub-polytope that does not contain $\mathcal{F}$ but must be able to reach it. We have the following straightforward extension of Theorem 13 and main result of this section.

Theorem 14 *If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by open-loop controls, then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by piecewise affine feedback.*

PROOF. We only consider the case excluded by Corollary 10 and Theorem 13 as described above. Consider the hyperplane $\mathcal{B}_{v^+}$ that partitions $\mathcal{P}$ into two sub-polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{F} \subset \mathcal{P}_1$ and v_+ is a vertex of $\mathcal{F}$ satisfying $v^+ \in \mathcal{F} \cap \mathcal{P}_1^+$ (see Figure 5 for an example). From Theorem 13, we have that $\mathcal{P}_1 \xrightarrow{\mathcal{P}_1} \mathcal{F}$ and from the assumption $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ and Theorem 6 it can be verified that $\mathcal{P}_2 \xrightarrow{\mathcal{P}_2} \mathcal{B}_{v^+} \cap \mathcal{P}$.

6 Triangulation with respect to $\mathcal{O}$

So far we have studied reachability problems and control synthesis under the assumption $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. In order to solve the general problem when $\overset{o}{\mathcal{P}} \cap \mathcal{O} \neq \emptyset$ we want to partition $\mathcal{P}$ along $\mathcal{O}$ and apply the results of the previous sections. A complication is that when we split $\mathcal{P}$ along $\mathcal{O}$ to form two polytopes, $\mathcal{P}_1$ and $\mathcal{P}_2$, one of the two target sets $\mathcal{P}_i \cap \mathcal{F}$ may no longer be an $(n-1)$-dimensional polytope. We assume in the following that when we say $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$, there does not exist a full-dimensional set of states in $\mathcal{P}$ that must reach a lower-dimensional (less than $n-1$) subset in $\mathcal{F}$ in order to achieve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$.

Now we would like to propose a partition method which splits $\mathcal{P}$ along $\mathcal{O}$ into two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$. Each subpolytope $\mathcal{P}_i$ will then have two possible target sets: $\mathcal{F} \cap \mathcal{P}_i$ and $\mathcal{O} \cap \mathcal{P}$. This second target allows some trajectories to cross over from one side of $\mathcal{O}$ to the other before reaching $\mathcal{F}$. One could try to make a subdivision according to which target the points in $\mathcal{P}_i$ can reach. However, this approach will generally require new techniques not already developed in the paper. We illustrate with an example.

Example 15 *Consider the 2D example as in Fig. 6. Suppose there are two target sets $\mathcal{F}_1$ and $\mathcal{F}_2$ where $\mathcal{F}_2 \subset$*

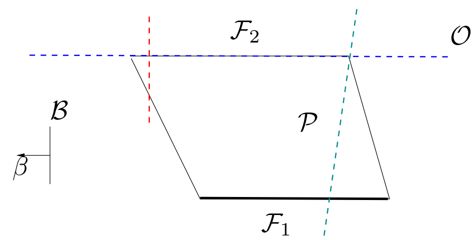

Fig. 6.

$\mathcal{O}$. It can be checked that $\mathcal{P} \to \mathcal{F}_1 \cup \mathcal{F}_2$, but neither $\mathcal{P} \to \mathcal{F}_1$ or $\mathcal{P} \to \mathcal{F}_2$ holds. If we were to apply Algorithm 1 of [11] to cut off the failure set for reaching $\mathcal{F}_1$, we would obtain the region on the left-side of the (red) dotted line (parallel to $\mathcal{B}$). However, the approximate failure set to reach $\mathcal{F}_1$ cannot reach $\mathcal{F}_2$, no matter how small is ϵ, without crossing into the region that can reach $\mathcal{F}_1$. Thus, if one insists on a true subdivision, the reachability problem would not be solvable using our feedback methods. On the other hand, Reach$_\epsilon(\mathcal{P}, \mathcal{F}_1)$ and Reach$_\epsilon(\mathcal{P}, \mathcal{F}_2)$ is a cover for $\mathcal{P}$, where Reach$_\epsilon(\mathcal{P}, \mathcal{F}_1)$ is the right-side of the red line and Reach$_\epsilon(\mathcal{P}, \mathcal{F}_2)$ is the left-side of the green line.

To overcome the issue in the above example, we subdivide $\mathcal{P}$ along $\mathcal{O}$ and then use a cover in each subpolytope according to two possible target sets. Let $\mathrm{Reach}_\epsilon(\mathcal{P}, \mathcal{F})$ denote a closed polyhedral ϵ-approximation of the set of states which can reach $\mathcal{F}$ from $\mathcal{P}$, as obtained, for example, by Algorithm 1 of [11]. (Note that these computations are explicit due to the simplicity of the reachable sets for hypersurface systems).

Cover of $\mathcal{P}$ with respect to $\mathcal{O}$: (Let $\epsilon > 0$ be sufficiently small.)

(a) Divide $\mathcal{P}$ along $\mathcal{O}$ to obtain $\mathcal{P}_1$ and $\mathcal{P}_2$.
(b) If $\dim(\mathcal{P}_i \cap \mathcal{F}) = n-1$, compute $\mathcal{Q}_{i1} := \mathrm{Reach}_\epsilon(\mathcal{P}_i, \mathcal{P}_i \cap \mathcal{F})$, $i = 1, 2$. Otherwise $\mathcal{Q}_{i1} = \emptyset$.
(c) If $\dim(\mathcal{P}_i \cap \mathcal{Q}_{j1}) = n-1$, compute $\mathcal{Q}_{i2} := \mathrm{Reach}_\epsilon(\mathcal{P}_i, \mathcal{P}_i \cap \mathcal{Q}_{j1})$, $i = 1, 2$, $j \neq i$. Otherwise $\mathcal{Q}_{i2} = \emptyset$.
(d) Define the cover $\mathbb{P} := \{\mathcal{Q}_{11}, \mathcal{Q}_{12}, \mathcal{Q}_{21}, \mathcal{Q}_{22}\}$.

Theorem 16 *$\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by open-loop controls if and only if $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}$ by piecewise affine feedback.*

The main idea of the result is that when $\mathcal{P}$ is partitioned along $\mathcal{O}$, there are only two types of points in each subpolytope: those that reach $\mathcal{F}$ while remaining in the subpolytope, or those that cross over to the other polytope to then reach $\mathcal{F}$. The essential idea rests on the following technical lemma. It's proof as well as that of Theorem 16 may be found in [12].

Lemma 17 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $(n-1)$-dimensional polytopes on the boundary of $\mathcal{P}$ but not on a common hyperplane and assume $\overset{o}{\mathcal{P}} \cap \mathcal{O} = \emptyset$. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_1 \cup \mathcal{F}_2$, then there exists $\epsilon > 0$ sufficiently small such that $Reach_\epsilon(\mathcal{P}, \mathcal{F}_1) \cup Reach_\epsilon(\mathcal{P}, \mathcal{F}_2) = \mathcal{P}$.*

7 Conclusion

We have presented methods of triangulation, subdivision, and covers for a control problem for affine hypersurface systems. Some unique features of this work are: (1) We do not impose what class of controls should be used to implement the reachability specifications. Because of the structure of hypersurface systems, we then derive that piecewise affine feedbacks are a sufficiently rich class. (2) We place emphasis on triangulation and subdivision, guided by the the principle that these cannot be performed independently of control synthesis. In particular, by proper triangulation we establish greedy dynamic programming algorithms that are guaranteed to outperform dynamic programming algorithms based on random triangulations: our algorithm always finds a solution when one exists via open-loop control. (3) We introduce a technique of covers which overcomes the technical problems with taking subdivisions. Fortunately, it naturally leads to synthesis of piecewise affine feedbacks.

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