An Obstruction to Solvability of the Reach Control Problem Using Affine Feedback ⋆

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Abstract

This paper studies the reach control problem (RCP) using affine feedback on simplices. The contributions of this paper are threefold. First, we identify a new obstruction to solvability of the RCP using affine feedback and provide necessary and sufficient conditions for occurrence of such an obstruction. Second, for two-input systems, these conditions are formulated in terms of scalar linear inequalities. Third, computationally efficient necessary conditions are proposed for checking the obstruction for multi-input systems as feasibility programs in terms of linear inequalities. In contrast to the previous work in the literature, no assumption is imposed on the set of possible equilibria, so the results are applicable to the general RCP.

Key words: Reach control problem, Affine feedback, Dual cones, Cone conditions

1 Introduction

This paper studies the reach control problem (RCP) using affine feedback on simplices. Given an affine system defined on a simplex $\mathcal{S}$, the objective in the RCP is to design a feedback controller such that the trajectories of the closed-loop system leave $\mathcal{S}$ in finite time through a prespecified facet, without first leaving it through other facets. The RCP has been the subject of a great deal of research due to its fundamental importance in controlling a subclass of hybrid systems known as piecewise affine systems (Bemporad et al. (2000); Rodrigues (2004); Habets et al. (2006)). Piecewise affine systems are state-based switched systems where each discrete mode has a corresponding continuous-time affine dynamics. The discrete modes correspond to polytopes in the state space. For piecewise affine systems, reach control is at each mode to design a controller that prevents transitions of the closed-loop system to undesired discrete modes, and guarantees transition to the prespecified desired mode. The RCP has found applications in different fields including biomolecular networks (Belta et al. (2002)), robot motion planning (Belta et al. (2005)), aircraft control (Belta and Habets (2006)), robotic manipulators (Martino and Broucke (2014)), and aggressive maneuvers of mechanical systems (Vukosavljev and Broucke (2014)).

The first approaches to solve the RCP in Habets and van Schuppen (2004); Roszak and Broucke (2006) lead to either conservative sufficient conditions or bilinear inequalities that are NP-hard. It later became evident that the (polytopic) set of possible closed-loop equilibria in the simplex, $\mathcal{O}_{\mathcal{S}}$, plays a crucial role in solvability of the RCP. In particular, several computationally efficient controller synthesis methods were devised by imposing the assumption that $\mathcal{O}_{\mathcal{S}}$ is a *face* of $\mathcal{S}$ (Broucke (2010); Ashford and Broucke (2013); Broucke and Ganness (2014)). The results were extended to polytopes in (Lin and Broucke (2011); Helwa and Broucke (2013)). In Lin and Broucke (2011) the problem was to find a triangulation of the polytope and an associated piecewise affine feedback to solve the RCP assuming the system has $n-1$ inputs. The goal of (Helwa and Broucke (2013)) was to extend the results of (Broucke (2010)) directly to polytopes by formulating the so-called *monotonic RCP*. While all these works regard the RCP, the specific problems solved and the approaches are very different from those of this paper. In this paper we focus on a sub-problem of the RCP regarding the ability to assign a non-vanishing affine function on $\mathcal{O}_{\mathcal{S}}$. We use a numerical, optimization-inspired approach whereas the previous works exploit system structure to arrive at analytical conditions for solvability. Finally, a Lyapunov theory for the RCP based on so-called flow functions (the analog of Lyapunov functions for stability analysis) was presented in Helwa and Broucke (2015). In this work, we do not assume existence of a flow function, and we are interested in necessary conditions for

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solvability rather than the analysis of a given controller.

More closely related to this work, recent research has focused on the existence and structure of the equilibria in the RCP (Semsar-Kazerooni and Broucke (2014)). The notion of reach controllability was introduced to characterize when closed-loop equilibria could be pushed off the simplex using affine feedback. Notions of topological and affine obstructions to solvability arose as necessary conditions to the solvability of RCP (Ornik and Broucke (2015)). The term "obstruction" is used in a similar spirit as in homotopy theory - to extend a continuous (or affine) map on a simplicial complex. The affine obstruction was studied in (Ornik and Broucke (2015)) for the case of two- and three-dimensional systems.

The main contributions of this paper are threefold. First, we formulate necessary and sufficient conditions for existence of a non-vanishing affine extension on $\mathcal{O}_\mathcal{S}$. To the best of our knowledge, this is the first result in the literature to present an obstruction to solvability of the RCP using affine feedback for multi-input systems and for the most general form of $\mathcal{O}_\mathcal{S}$. Second, we propose graphically motivated and computationally efficient necessary and sufficient conditions for checking the obstruction on $\mathcal{O}_\mathcal{S}$ for two-input systems in terms of scalar linear inequalities. Finally, computationally efficient necessary conditions are proposed for checking the obstruction on $\mathcal{O}_\mathcal{S}$ for multi-input systems as feasibility programs in terms of linear inequalities.

2 Problem Formulation

Consider an n-dimensional simplex $\mathcal{S} := \text{co}\{v_0, \dots, v_n\}$, where $v_0, \dots, v_n$ are $n+1$ affinely independent points in $\mathbb{R}^n$. Without loss of generality (w.l.o.g.) we assume $v_0 = 0$. Define $V_\mathcal{S} := \{v_0, \dots, v_n\}$ to be the vertex set of $\mathcal{S}$. Let $\mathcal{F}_0, \dots, \mathcal{F}_n$ denote the facets of $\mathcal{S}$, where each facet is indexed by the vertex it does not contain. We call $\mathcal{F}_0$ the *exit facet*. Let h_j, $j \in \{0, \dots, n\}$, be the unit normal vector of facet $\mathcal{F}_j$ pointing outside of the simplex. Let $\mathbf{0}$ denote the singleton set $\{0\}$. Define $I := \{1, \dots, n\}$ and let $I(x)$ be the minimal index set among $\{0, \dots, n\}$ such that $x \in \text{co}\{v_i \mid i \in I(x)\}$.

We consider an affine control system on $\mathcal{S}$ defined as

$$\dot{x} = Ax + Bu + a, \qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Define $\mathcal{B} := \text{Im}(B)$, the image of B. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at $x_0 \in \mathcal{S}$, under control input u, and evaluated at time instant t. Reach control theory studies the reachability of the exit facet $\mathcal{F}_0$ from any initial point in $\mathcal{S}$.

Reach Control Problem (RCP). Consider the affine system (1) defined on a simplex $\mathcal{S}$. Find an affine feedback $u(x) := Kx + g$, where $K \in \mathbb{R}^{m\times n}$ and $g \in \mathbb{R}^m$, such that for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\delta > 0$ such that

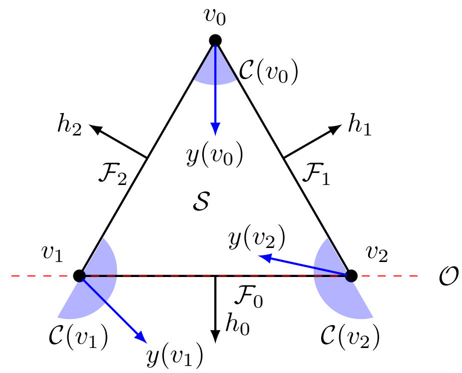

Fig. 1. A simplex $\mathcal{S} = \text{co}\{v_0, v_1, v_2\}$ with vertices $V_\mathcal{S} = \{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$. The facet $\mathcal{F}_i$, $i \in \{0, 1, 2\}$, is the convex hull of all vertices not including v_i. For each facet $\mathcal{F}_i$, the unit normal vector pointing out of $\mathcal{S}$ is shown by h_i. The cones $\mathcal{C}(v_i)$ are illustrated attached at each v_i along with sample vectors $y_i \in \mathcal{C}(v_i)$.

(i) $\phi_u(t, x_0) \in \mathcal{S}, \forall\, t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi_u(t, x_0) \notin \mathcal{S}, \forall\, t \in (T, T+\delta)$.

Two necessary conditions for solvability of the RCP by affine feedback are known (Habets and van Schuppen (2004); Roszak and Broucke (2006)). First, the velocity vector $Ax + Bu(x) + a$ must point inside the cone generated by $\mathcal{S}$ at points in the facets $\mathcal{F}_i$, $i \in I$. This requirement is known as the *invariance conditions* (Roszak and Broucke (2006)). For $x \in \mathcal{S}$, define the closed, convex cone

$$\mathcal{C}(x) := \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in I \setminus I(x)\}. \tag{2}$$

Note that h_0 never appears in (2) and $\mathcal{C}(x) = \mathbb{R}^n$ for $x \in \mathcal{S}^\circ$, where $\mathcal{S}^\circ$ represents the interior of $\mathcal{S}$. Figure 1 illustrates the cones $\mathcal{C}(v_i)$, $i \in \{0, 1, 2\}$, attached at the corresponding vertex v_i to describe allowable directions for the vector field at the vertices. Here, e.g., since $I(v_0) = \{0\}$, we have $\mathcal{C}(v_0) = \{y \in \mathbb{R}^2 \mid h_j \cdot y \leq 0,\ j \in \{1, 2\}\}$. We say that $u(x)$ satisfies the invariance conditions if

$$Ax + Bu(x) + a \in \mathcal{C}(x), \qquad \forall x \in \mathcal{S}. \tag{3}$$

A second necessary condition for the feedback $u(x)$ to solve the RCP is that there are no closed-loop equilibria in $\mathcal{S}$, i.e., $Ax + Bu(x) + a \neq 0$, for all $x \in \mathcal{S}$. It was shown in Habets et al. (2006); Roszak and Broucke (2006) that these two necessary conditions combined form a sufficient condition for solvability of RCP using affine feedback. Closed-loop equilibria of (1) can only appear in the affine space

$$\mathcal{O} := \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}. \tag{4}$$

Therefore, we are interested in the feedback $u(x)$ that denies any equilibria in the set

$$\mathcal{O}_\mathcal{S} := \mathcal{S} \cap \mathcal{O} = \text{co}\{o_1, \dots, o_\kappa\}.$$

The intersection of a simplex $\mathcal{S}$ and an affine space $\mathcal{O}$ is either an empty set or a $\hat{\kappa}$-dimensional (compact and convex) polytope, where $0 \leq \hat{\kappa} \leq n$ and $\hat{\kappa} < \kappa$. We note that

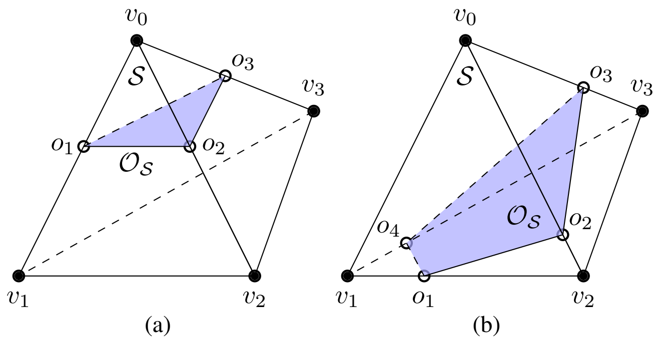

Fig. 2. Two hypothetical scenarios in $\mathbb{R}^3$ with $\mathcal{S} = \text{co}\{v_0, v_1, v_2, v_3\}$; (a) $\mathcal{O}_{\mathcal{S}} = \text{co}\{o_1, o_2, o_3\}$ is a simplex, and (b) $\mathcal{O}_{\mathcal{S}} = \text{co}\{o_1, o_2, o_3, o_4\}$ is a polytope but not a simplex.

$\dim(\mathcal{O}) \geq m$. However, as $\mathcal{O}_{\mathcal{S}}$ might not pass through the interior of $\mathcal{S}$, there is no guarantee that $\dim(\mathcal{O}_{\mathcal{S}}) \geq m$. We define $V_{\mathcal{O}_{\mathcal{S}}} := \{o_1, \ldots, o_\kappa\}$ to be the set of vertices of $\mathcal{O}_{\mathcal{S}}$. Two examples of the set $\mathcal{O}_{\mathcal{S}}$ are shown in Fig. 2. Many papers in the literature study the RCP under the assumption that $\mathcal{O}_{\mathcal{S}}$ is a $\hat{\kappa}$-dimensional *face* of S. Due to the critical role of the set $\mathcal{O}_{\mathcal{S}}$ in the second necessary condition, relaxing the above assumptions and characterizing $\mathcal{O}_{\mathcal{S}}$ serve as major stepping stones for solving the RCP. In this paper, we introduce an obstruction to the RCP on the set of possible equilibria $\mathcal{O}_{\mathcal{S}}$ and, in contrast to the above mentioned papers, study $\mathcal{O}_{\mathcal{S}}$ in its most general form.

For all $x \in \mathcal{O}_{\mathcal{S}}$ the closed-loop vector field satisfies $Ax + Bu(x) + a \in \mathcal{B}$. Therefore, since $\mathcal{O}_{\mathcal{S}} \subseteq \mathcal{S}$, the existence of an affine map $F : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}$ that satisfies

$$F(x) \in \mathcal{C}(x), \qquad \forall x \in \mathcal{O}_{\mathcal{S}}, \tag{5}$$

is a necessary condition for the existence of an affine feedback $u(x)$ that satisfies the invariance conditions (3). It is sufficient to check these invariance conditions at vertices of $\mathcal{O}_{\mathcal{S}}$. Furthermore, if the affine map F is non-vanishing on $\mathcal{O}_{\mathcal{S}}$, then the corresponding affine feedback produces no closed-loop equilibria in $\mathcal{S}$. The following lemma formulates the non-vanishing condition on F as κ inequalities. The proof is omitted due to its similarity to Theorem 6 in (Roszak and Broucke (2006)).

Lemma 1 *Assume that there exists a vertex map $f : V_{\mathcal{O}_{\mathcal{S}}} \to \mathcal{B}$ that is extendible on $\mathcal{O}_{\mathcal{S}} \neq \emptyset$ to an affine map $F : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}$. The following statements are equivalent:*

(1) The affine map F satisfies (5) *and is non-vanishing.*
(2) The vertex map f satisfies

$$\begin{aligned} & f(o_i) \in \mathcal{C}(o_i), \qquad i \in \{1, \ldots, \kappa\}, \\ (\exists\, \xi \in \mathbb{R}^n) \quad & \xi \cdot f(o_i) < 0, \qquad i \in \{1, \ldots, \kappa\}. \end{aligned}$$

It is worth mentioning that the second condition above is equivalent to the statement that the vector field points outside the set $\mathcal{O}_{\mathcal{S}}$. In light of Lemma 1, in this paper we address the following problem as a necessary condition and a stepping stone for solving the RCP.

Problem 2 *Given the set $\mathcal{O}_{\mathcal{S}} \neq \emptyset$, find a vertex map $f : V_{\mathcal{O}_{\mathcal{S}}} \to \mathcal{B}$ such that $f(o_i) \in \mathcal{C}(o_i)$, $i \in \{1, \ldots, \kappa\}$, and that there exists a vector $\xi \in \mathbb{R}^n$ satisfying $\xi \cdot f(o_i) < 0$, $i \in \{1, \ldots, \kappa\}$.*

Based on Lemma 1, if Problem 2 is infeasible then an obstruction to solvability of the RCP using affine feedback exists. For single-input systems, i.e. $m = 1$, an immediate consequence of Theorem 1 in Semsar-Kazerooni and Broucke (2014) is that Problem 2 is solvable if and only if $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$, where

$$\text{cone}(\mathcal{O}_{\mathcal{S}}) := \bigcap_{i=1}^{\kappa} \mathcal{C}(o_i). \tag{6}$$

However, this only becomes a sufficient condition in the case of multi-input systems (see Example 13). In this paper, we focus on solvability of Problem 2 for systems with more than one input. Necessary and sufficient conditions for the obstruction identified in Problem 2 are formulated in the next section.

3 Necessary and Sufficient Conditions for Existence of a Non-vanishing Affine Extension on $\mathcal{O}_{\mathcal{S}}$

Let $B = QR$ be the *QR factorization* of B. Since B has full column rank, the columns of $Q \in \mathbb{R}^{n \times m}$ form an orthonormal basis for $\mathcal{B}$ (Golub and Van Loan (1996), p. 223), such that $Q^T Q = I_{m \times m}$ and $QQ^T b = b$ for all $b \in \mathcal{B}$. Our objective is to find a vertex map f such that $f(o_i) \in \mathcal{B} \cap \mathcal{C}(o_i)$, $i \in \{1, \ldots, \kappa\}$. To this end, define the convex cone $\mathcal{C}_i$, $i \in \{1, \ldots, \kappa\}$, as

$$\begin{aligned} \mathcal{C}_i :&= \{w \in \mathbb{R}^m \mid (Q^T h_j) \cdot w \leq 0,\ j \in I \setminus I(o_i)\} && (7) \\ &= \bigcap_{j \in I \setminus I(o_i)} H_j, && (8) \end{aligned}$$

where H_j is a closed convex cone defined as

$$H_j := \{w \in \mathbb{R}^m \mid (Q^T h_j) \cdot w \leq 0\}, \qquad j \in I. \tag{9}$$

Cones $\mathcal{C}_i$ are analogues of the cones $\mathcal{C}(o_i)$ in the original problem set-up. In fact, as $\mathcal{B}$ is mapped by into $\mathbb{R}^m$ by Q^T, the same mapping transforms $\mathcal{C}(o_i) \cap \mathcal{B}$ into $\mathcal{C}_i$. Cones H_j identify each of the possible constraints impacting $\mathcal{C}_i$.

Lemma 3 *Let $y \in \mathcal{B}$. Then $Q^T y \in \mathcal{C}_i$ if and only if $y \in \mathcal{B} \cap \mathcal{C}(o_i)$.*

PROOF. ($\Longrightarrow$) Since $y \in \mathcal{B}$ we can write $QQ^T y = y$. Assume $Q^T y \in \mathcal{C}_i$. For $j \in I \setminus I(o_i)$, (7) yields $(Q^T h_j) \cdot (Q^T y) \leq 0 \iff h_j^T QQ^T y \leq 0 \iff h_j \cdot y \leq 0$. Considering (2), $y \in \mathcal{B} \cap \mathcal{C}(o_i)$. ($\Longleftarrow$) The result follows from arguments similar to the first part of the proof. □

If $Q^T h_j \neq 0$ then H_j represents a closed half space in $\mathbb{R}^m$.

If $Q^T h_j = 0$ then $H_j = \mathbb{R}^m$. Since $\mathcal{C}_i$ is the intersection of a finite number of half spaces it is a convex polyhedral cone, and it can be written as the convex hull of a finite set of rays. For each $i \in \{1, \dots, \kappa\}$, let the unit vectors corresponding to these rays be $v_{i,r}$, $r \in \{1, \dots, r_i\}$. Therefore, any vector $w \in \mathcal{C}_i$ can be written as the conic combination of $v_{i,r}$, $r \in \{1, \dots, r_i\}$, i.e., $w = \sum_{r=1}^{r_i} \lambda_r v_{i,r}$, for some $\lambda_r \geq 0$. Next, for $i \in \{1, \dots, \kappa\}$, define

$$\mathcal{N}_i := \bigcup_{r=1}^{r_i} \{h \in \mathbb{R}^m \mid h \cdot v_{i,r} < 0\}. \tag{10}$$

Each $\mathcal{N}_i$ represents the set of all vectors $h \in \mathbb{R}^m$ for which there exists a vector $w \in \mathcal{C}_i$ such that $h \cdot w < 0$. In other words, each $\mathcal{N}_i$ contains all the vectors h for which there exists an element w which is contained in $\mathcal{C}_i$ and in the half-space *opposite* from h. This definition may be inelegant; however, we will soon show that it can be naturally interpreted in relation to the standard notion of dual cones. Note that $\mathcal{N}_i$ is a (not necessarily convex) *blunt cone* (i.e., a cone that does not contain 0) because $h \in \mathcal{N}_i$ implies $\lambda h \in \mathcal{N}_i$ for any $\lambda > 0$, and $0 \notin \mathcal{N}_i$. Theorem 4 formulates a necessary and sufficient condition for the obstruction described by Problem 2 in terms of the intersection of $\mathcal{N}_i$, $i \in \{1, \dots, \kappa\}$.

Theorem 4 *There exists a solution to Problem 2 if and only if*

$$\bigcap_{i=1}^{\kappa} \mathcal{N}_i \neq \emptyset. \tag{11}$$

PROOF. ($\Longrightarrow$) Suppose Problem 2 is solvable, i.e., there exist $f(o_i) \in \mathcal{B} \cap \mathcal{C}(o_i)$, $i \in \{1, \dots, \kappa\}$, and $\xi \in \mathbb{R}^n$ such that $\xi \cdot f(o_i) < 0$ for all $i \in \{1, \dots, \kappa\}$. By Lemma 3, $Q^T f(o_i) \in \mathcal{C}_i$ and we can write $Q^T f(o_i) = \sum_{r=1}^{r_i} \lambda_r v_{i,r}$, for some $\lambda_r \geq 0$. Since $f(o_i) \in \mathcal{B}$ we have $QQ^T f(o_i) = f(o_i)$. Therefore, for $i \in \{1, \dots, \kappa\}$, $\xi \cdot f(o_i) = \xi^T QQ^T f(o_i) = (Q^T \xi) \cdot (\sum_{r=1}^{r_i} \lambda_r v_{i,r}) < 0$. Since $\lambda_r \geq 0$ for all $r \in \{1, \dots, r_i\}$, there exists r^* such that $(Q^T \xi) \cdot v_{i,r^*} < 0$. Therefore, $Q^T \xi \in \mathcal{N}_i$, for all $i \in \{1, \dots, \kappa\}$, and (11) is satisfied.

($\Longleftarrow$) Suppose (11) holds. Since $\mathcal{N}_i$ does not contain 0, there exists a non-zero vector $h^* \in \mathbb{R}^m$ such that for each $i \in \{1, \dots, \kappa\}$ there exists a vector v_{i,r_i^*} such that $h^* \cdot v_{i,r_i^*} < 0$. Let $f(o_i) := Qv_{i,r_i^*}$, $i \in \{1, \dots, \kappa\}$. Clearly, $f(o_i) \in \mathcal{B} = \text{Im}(Q)$. Since $Q^T f(o_i) = Q^T Q v_{i,r_i^*} = v_{i,r_i^*} \in \mathcal{C}_i$, by Lemma 3, $f(o_i) \in \mathcal{B} \cap \mathcal{C}(o_i)$. Next, let $\xi = Qh^*$. Then for $i \in \{1, \dots, \kappa\}$, $\xi \cdot f(o_i) = (Qh^*) \cdot (Qv_{i,r_i^*}) = {h^*}^T Q^T Q v_{i,r_i^*} = h^* \cdot v_{i,r_i^*} < 0$. □

To the best of our knowledge, Theorem 4 is the first result in the literature that provides a necessary and sufficient condition for solvability of Problem 2 for multi-input systems and for the most general form of the set $\mathcal{O}_\mathcal{S}$. However, since the cones $\mathcal{N}_i$ are not necessarily convex, condition (11) in Theorem 4 leads to a non-convex feasibility problem. Nonetheless, assuming that (11) is satisfied and a vector $h^* \in \bigcap_{i=1}^{\kappa} \mathcal{N}_i$ is known, Problem 2 can be formulated as a computationally efficient feasibility program in terms of linear inequalities, as shown in the next corollary.

Corollary 5 *Suppose there exists a vector $h^* \in \bigcap_{i=1}^{\kappa} \mathcal{N}_i$. The vertex map $f(o_i) = Qw_i$, $i \in \{1, \dots, \kappa\}$, is a solution to Problem 2, where w_i always exists and is a solution of the following feasibility program:*

$$\begin{array}{ll} \text{find} & w_i \in \mathcal{C}_i \\ \text{subject to} & h^* \cdot w_i < 0. \end{array} \tag{12}$$

PROOF. Suppose $h^* \in \bigcap_{i=1}^{\kappa} \mathcal{N}_i$. Considering (10), for each $i \in \{1, \dots, \kappa\}$, there exists $r_i^* \in \{1, \dots, r_i\}$ such that $h^* \cdot v_{i,r_i^*} < 0$. Therefore, optimization program (12) is always feasible since clearly $v_{i,r_i^*} \in \mathcal{C}_i$. Since $Q^T Q w_i = w_i \in \mathcal{C}_i$, by Lemma 3, $f(o_i) = Qw_i \in \mathcal{B} \cap \mathcal{C}(o_i)$. For $i \in \{1, \dots, \kappa\}$, observe that $h^* \cdot w_i = {h^*}^T Q^T Q w_i = (Qh^*) \cdot (Qw_i) < 0$. Define $\xi := Qh^* \in \mathbb{R}^n$. Therefore, $\xi \cdot f(o_i) < 0$, which completes the proof. □

The necessary and sufficient condition in Theorem 4 is based on the cones $\mathcal{N}_i$, which are not necessarily convex. Instead, it is appealing to reformulate the necessary and sufficient condition (11) in terms of the more standard notion of the dual cones, which are convex. This new dual formulation simplifies the presentation of the results in the following sections of the paper. To this end, note that the *dual cone* of a cone $C \in \mathbb{R}^n$ is defined as (Boyd and Vandenberghe (2004))

$$C^* := \{y \in \mathbb{R}^n \mid y \cdot c \geq 0, \ \forall \, c \in C\}. \tag{13}$$

Since every vector in $\mathcal{C}_i$ is a conic combination of the vectors $v_{i,r}$, $r \in \{1, \dots, r_i\}$, the dual cone of $\mathcal{C}_i$, $i \in \{1, \dots, \kappa\}$, is written as

$$\mathcal{C}_i^* := \{y \in \mathbb{R}^m \mid y \cdot v_{i,r} \geq 0, \ \forall \, r = 1, \dots, r_i\}. \tag{14}$$

Corollary 6 *There exists a solution to Problem 2 if and only if*

$$\bigcup_{i=1}^{\kappa} \mathcal{C}_i^* \neq \mathbb{R}^m. \tag{15}$$

PROOF. Considering (10) and (14), it is easy to see that $\mathcal{N}_i = \mathbb{R}^m \setminus \mathcal{C}_i^*$. Therefore,

$$\bigcap_{i=1}^{\kappa} \mathcal{N}_i = \bigcap_{i=1}^{\kappa} (\mathbb{R}^m \setminus \mathcal{C}_i^*) = \mathbb{R}^m \setminus \bigcup_{i=1}^{\kappa} \mathcal{C}_i^*. \tag{16}$$

Thus, conditions (11) and (15) are equivalent. □

Corollary 6 presents necessary and sufficient conditions for the obstruction identified in Problem 2 as a compact and plausible cone condition. According to Elbassioni and Tiwary (2011), however, determining if the union of a set of polyhedral cones covers $\mathbb{R}^m$ is an NP-complete problem. Therefore, Corollary 6 (and analogously Theorem 4) cannot be efficiently solved using optimization software. In the rest of the paper, we present computationally efficient conditions for solvability of Problem 2. In particular, (i) for two-input systems, necessary and sufficient conditions are presented in Section 4 in terms of easily verifiable convexity relations, and (ii) for general systems, necessary conditions are presented in Section 5 as feasibility programs in terms of linear inequalities. The results of Sections 4 and 5 can be easily programmed and solved using available optimization software.

4 Two-input Systems

This section is focused on two-input systems, i.e., systems for which $\mathcal{B}$ is a 2-dimensional sub-space of $\mathbb{R}^n$. This allows us to present a graphical representation of Corollary 6, and propose computationally efficient necessary and sufficient conditions for solving Problem 2 for two-input systems. To this end, consider (8) where $\mathcal{C}_i$ is defined as the intersection of a set of closed convex cones H_j. According to Theorem 2 in Sandgren (1954), for $i \in \{1, \dots, \kappa\}$ we can write

$$\mathcal{C}_i^* = \text{co}\left\{ \bigcup_{j \in I \setminus I(o_i)} H_j^* \right\} = \text{co}\{H_j^* \mid j \in I \setminus I(o_i)\}, \quad (17)$$

where H_j^* is the dual cone of H_j. The set H_j^* is either a ray or the singleton $\mathbf{0}$, as proved in the following lemma.

Lemma 7 *If $Q^T h_j \neq 0$ then H_j^* is a ray given by $H_j^* := \{-\alpha Q^T h_j \mid \alpha \geq 0\}$. Otherwise, $H_j^* = \mathbf{0}$.*

PROOF. Considering (9) and (13), for all $j \in I$, we can write

$$\begin{aligned} H_j^* =& \{y \mid y \cdot w \geq 0, \ \forall w \in H_j\} \quad (18) \\ =& \{y \mid y \cdot w \geq 0, \ \forall w \text{ s.t. } -(Q^T h_j) \cdot w \geq 0\}. \end{aligned}$$

If $Q^T h_j = 0$ then $H_j^* = \{y \mid y \cdot w \geq 0, \ \forall w \in \mathbb{R}^m\} = \mathbf{0}$. Next, assume $Q^T h_j \neq 0$. Any vector αy^*, where $y^* = -Q^T h_j$, lies in H_j^* if and only if $\alpha \geq 0$. We claim that any vector y that is not collinear with y^* does not lie in H_j^*. W.l.o.g, let $y := \beta y^* + \beta_\perp y_\perp^*$, where $y^* \cdot y_\perp^* = 0$, $y_\perp^* \neq 0$, $\beta, \beta_\perp \in \mathbb{R}$, and $\beta_\perp \neq 0$. Equation (9) yields $y_\perp^* \in H_j$ and $-y_\perp^* \in H_j$. However, $y \cdot y_\perp^* = \beta_\perp |y_\perp^*|^2$ and $y \cdot (-y_\perp^*) = -\beta_\perp |y_\perp^*|^2$. Therefore, either $y \cdot y_\perp^*$ or $y \cdot (-y_\perp^*)$ is less than zero. Hence, by (18), $y \notin H_j^*$. □

W.l.o.g., by reordering indices, assume $Q^T h_j \neq 0$, $j \in \{1, \dots, n'\}$, where $0 \leq n' \leq n$. Now assume the system has two inputs, i.e., assume $m = 2$. W.l.o.g., by reordering indices $j \in \{2, \dots, n'\}$, assume the rays H_j^*, $j \in \{1, \dots, n'\}$,

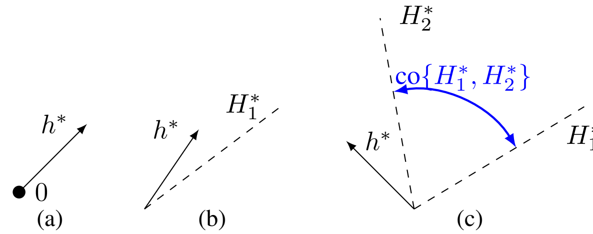

Fig. 3. The graphic corresponding to the proof of part *(i)* of Lemma 8. The set $\text{co}\{\mathbf{0} \cup \left(\bigcup_{1 \leq j \leq n'} H_j^*\right)\}$ is equal to (a) the singleton $\mathbf{0}$ (if $n' = 0$), (b) the ray H_1^* (if $n' = 1$), or (c) the cone $\text{co}\{H_1^*, H_2^*\}$ (if $n' = 2$).

are arranged in clockwise order (see Fig. 4 for an example). Let $H_0^* := H_{n'}^*$ and define $-H_j^* := \{y \mid -y \in H_j^*\}$, $j \in \{0, \dots, n'\}$. The following lemma presents three special cases where the obstruction described by Problem 2 does not exist.

Lemma 8 *Suppose the affine system has two inputs. Problem 2 is solvable if one of the following conditions is satisfied.*

(i) $n' \leq 2$
(ii) $\exists j^* \in \{0, \dots, n'-1\}$ *such that* $H_{j^*+1}^* = -H_{j^*}^*$
(iii) $\exists j^* \in \{0, \dots, n'-1\}$ *such that* $H_j^* \subseteq \text{co}\{H_{j^*}^*, H_{j^*+1}^*\}$, $\forall j \in \{1, \dots, n'\}$

PROOF. *(i)* Assume $n' \leq 2$. Considering (17), we have $\mathcal{C}_i^* \subseteq \text{co}\{\mathbf{0} \cup (\bigcup_{1 \leq j \leq n'} H_j^*)\}$, $i \in \{1, \dots, \kappa\}$. Hence, $\bigcup_{i=1}^{\kappa} \mathcal{C}_i^* \subseteq \text{co}\{\mathbf{0} \cup (\bigcup_{1 \leq j \leq n'} H_j^*)\}$. Considering Fig. 3, the set $\text{co}\{\mathbf{0} \cup (\bigcup_{1 \leq j \leq n'} H_j^*)\}$ is equal to the singleton $\mathbf{0}$ if $n' = 0$, the ray H_1^* if $n' = 1$, or the cone $\text{co}\{H_1^*, H_2^*\} \neq \mathbb{R}^m$ if $n' = 2$. Therefore, there always exists a vector $h^* \notin \text{co}\{\mathbf{0} \cup (\bigcup_{1 \leq j \leq n'} H_j^*)\}$ and, by Corollary 6, Problem 2 is solvable.

(ii) Assume there exists $j^* \in \{0, \dots, n'-1\}$ such that $H_{j^*+1}^* = -H_{j^*}^*$. Since the rays are arranged in clockwise order, all rays H_j^*, $j \in \{1, \dots, n'\}$, lie in the same side of the line $H_{j^*}^* \cup H_{j^*+1}^*$, in a closed half-plane $\mathcal{P}$ (see Fig. 4). Considering (17), the cones $\mathcal{C}_i^*$, $i \in \{1, \dots, \kappa\}$, are subsets of $\mathcal{P}$. Hence by Corollary 6, Problem 2 is solvable.

(iii) Assume there exists $j^* \in \{0, \dots, n'-1\}$ such that $H_j^* \subseteq \text{co}\{H_{j^*}^*, H_{j^*+1}^*\}$, $\forall j \in \{1, \dots, n'\}$. Hence, considering Lemma 7, each vector $-Q^T h_j$, $j \in \{1, ..., n'\}$, can be written as a conic combination of the two vectors $-Q^T h_{j^*}$ and $-Q^T h_{j^*+1}$. Recall that the indices were reordered w.l.o.g. such that $Q^T h_j = 0$, $j \in \{n'+1, ..., n\}$. Furthermore, according to (17) and Lemma 7, any vector in $\mathcal{C}_i^*$ can be written as a conic combination of the vectors $-Q^T h_j$, $j \in I \setminus I(o_i)$. Therefore, any vector in $\mathcal{C}_i^*$ can be written as a conic combination of the two vectors $-Q^T h_{j^*}$

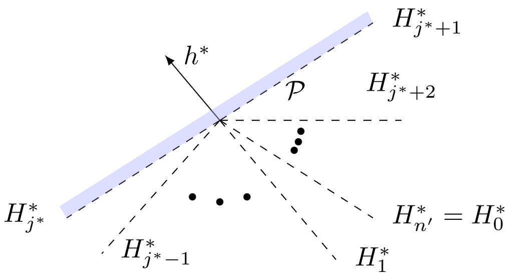

Fig. 4. The graphic corresponding to the proof of part *(ii)* of Lemma 8.

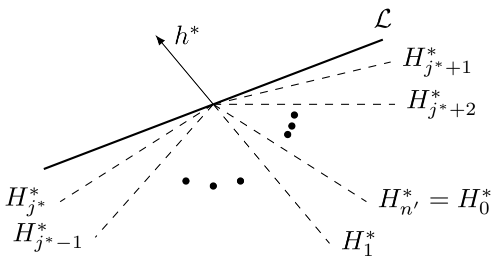

Fig. 5. The graphic corresponding to the proof of part *(iii)* of Lemma 8 and the second part of Theorem 11.

and $-Q^T h_{j^*+1}$, which yields $\mathcal{C}^*_i \subseteq \mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$. Pick a vector $h^* \notin \mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$. Clearly, $h^* \notin \mathcal{C}^*_i$, $i \in \{1, \dots, \kappa\}$ (see Fig. 5). Therefore, by Corollary 6, Problem 2 is solvable. □

The results of the following Lemma are used to address the case where none of the conditions in Lemma 8 is satisfied.

Lemma 9 *The following statements hold for two-input systems.*

(i) *The dual cones $\mathcal{C}^*_i$, $i \in \{1, \dots, \kappa\}$, are convex and their boundaries are among H^*_j, $j \in I$.*

(ii) *If $\cup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\} = \mathbb{R}^2$, then $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i = \mathbb{R}^2$ if and only if each $\mathrm{co}\{H^*_j, H^*_{j+1}\}$, $j \in \{0, \dots, n'-1\}$, is contained in some $\mathcal{C}^*_i$.*

PROOF. *(i)* This can be verified by (17) and Theorem 2 in Gerstenhaber (1951).

(ii) ($\Longleftarrow$) Assume each $\mathrm{co}\{H^*_j, H^*_{j+1}\}$, $j \in \{0, \dots, n'-1\}$, is contained in some $\mathcal{C}^*_i$. Then, using the assumption in part (ii) of the lemma, $\mathbb{R}^2 = \cup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\} \subseteq \bigcup_{i=1}^{\kappa} \mathcal{C}^*_i$, i.e. $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i = \mathbb{R}^2$.

($\Longrightarrow$) Let $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i = \mathbb{R}^2$. Now assume there exists $j^* \in \{0, \dots, n'-1\}$ such that $\mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\} \nsubseteq \mathcal{C}^*_i$, $i \in \{1, \dots, \kappa\}$. Since the boundaries of $\mathcal{C}^*_i$ are among H^*_j (part (i) of the lemma) and considering the fact that $H^*_{j^*}$

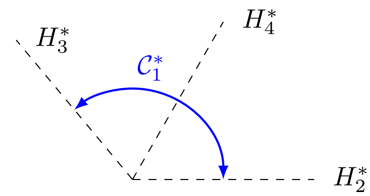

Fig. 6. In this example $H^*_4 \subseteq \mathcal{C}^*_1$, but $1 \notin J_4$, because in this hypothetical scenario $I(o_1) = \{0, 4\}$ and $I = \{1, 2, 3, 4\}$.

and $H^*_{j^*+1}$ are in consecutive (clockwise) order, we can conclude that the intersection of $\mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$ with $\mathcal{C}^*_i$, $i \in \{1, \dots, \kappa\}$, is either $\mathbf{0}$ or any of the rays $H^*_{j^*}$ or $H^*_{j^*+1}$. Hence, there exists a vector $h^* \in \mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$ such that $h^* \notin \mathcal{C}^*_i$, $i \in \{1, \dots, \kappa\}$, which is in contradiction with $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i = \mathbb{R}^2$. □

Based on Lemma 9, if the union of convex hulls $\cup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\}$ covers the whole space, then $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i = \mathbb{R}^2$ if and only if each convex hull $\mathrm{co}\{H^*_j, H^*_{j+1}\}$ is contained in some $\mathcal{C}^*_i$. Furthermore, according to Corollary 6, Problem 2 is infeasible if and only if the union of the cones $\mathcal{C}^*_i$ covers $\mathbb{R}^2$. In other words, if $\cup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\} = \mathbb{R}^2$, Problem 2 is infeasible if and only if each pair of rays H^*_j and H^*_{j+1} is contained in some $\mathcal{C}^*_i$. Thus, we need to know each ray H^*_j, $j \in \{1, \dots, n'\}$, is contained in which cones $\mathcal{C}^*_i$, $i \in \{1, \dots, \kappa\}$. To this end, for $j \in \{1, \dots, n'\}$, define

$$J_j := \{i \in \{1, \dots, \kappa\} \mid j \in \; I \setminus I(o_i)\}. \tag{19}$$

Considering (17), if $i \in J_j$ then $H^*_j \subseteq \mathcal{C}^*_i$. However, there may exist j^* and i^* such that $H^*_{j^*} \subseteq \mathcal{C}^*_{i^*}$ but $i^* \notin J_{j^*}$. For example, assume $I = \{1, 2, 3, 4\}$ and $I(o_1) = \{0, 4\}$. Let the rays H^*_2, H^*_3, and H^*_4 be as shown in Fig. 6. Here, $H^*_4 \subseteq \mathcal{C}^*_1$, but $1 \notin J_4$. Therefore, we should update the sets J_j by adding new indices i^* that satisfy $H^*_j \subseteq \mathcal{C}^*_{i^*}$. Note that if $\mathcal{C}^*_i$ contains the rays $H^*_{j_1}$ and $H^*_{j_2}$, then by convexity $\mathcal{C}^*_i$ contains any ray $H^*_{j_3} \subseteq \mathrm{co}\{H^*_{j_1}, H^*_{j_2}\}$, where $j_1, j_2, j_3 \in \{1, \dots, n'\}$. Based on this fact, Algorithm 10 updates the sets J_j and stores them in new sets $\widetilde{J}_j$. The members of each set $\widetilde{J}_j$ are indices $i \in \{1, \dots, \kappa\}$ such that $H^*_j \subseteq \mathcal{C}^*_i$. If $\widetilde{J}_j \cap \widetilde{J}_{j+1} \neq \emptyset$ for all $j \in \{0, \dots, n'-1\}$, then there exists some i^* such that $H^*_j, H^*_{j+1} \subseteq \mathcal{C}^*_{i^*}$ and thus the union of the cones $\mathcal{C}^*_i$ covers $\mathbb{R}^2$. The following theorem presents computationally efficient necessary and sufficient conditions for checking the obstruction associated with Problem 2 for two-input systems.

Theorem 11 *Suppose the affine system has two inputs and none of the conditions in Lemma 8 is satisfied. Let the index sets $\widetilde{J}_j$ for all $j \in \{0, \dots, n'\}$ be computed as in Algorithm 10. Problem 2 is solvable if and only if there exists $j^* \in \{0, \dots, n'-1\}$ such that $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} = \emptyset$.*

Algorithm 10

$\widetilde{J}_j := J_j,\, j \in \{1,\dots,n'\}$
for $j_1, j_2, j_3 \in \{1,\dots,n'\}$
  if $H^*_{j_3} \subseteq \mathrm{co}\{H^*_{j_1}, H^*_{j_2}\}$
    $\widetilde{J}_{j_3} := \widetilde{J}_{j_3} \cup (J_{j_1} \cap J_{j_2})$
  end if
end for
$\widetilde{J}_0 := \widetilde{J}_{n'}$

PROOF. ($\Longleftarrow$) Suppose there exists $j^* \in \{0,\dots,n'-1\}$ such that $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} = \emptyset$. We study the following two cases separately: *(i)* $H^*_{j^*} = H^*_{j^*+1}$ and *(ii)* $H^*_{j^*} \neq H^*_{j^*+1}$.

(i) Assume $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} = \emptyset$ and $H^*_{j^*} = H^*_{j^*+1}$. We claim

$$J_{j'} = \emptyset,\ \forall\, j' \in \{1,\dots,n'\} \text{ s.t. } H^*_{j^*} = H^*_{j^*+1} = H^*_{j'}. \tag{20}$$

Suppose not. Let $J_{j'} \neq \emptyset$. Since $H^*_{j^*} = H^*_{j^*+1} = \mathrm{co}\{H^*_{j'}, H^*_{j'}\}$, Algorithm 10 guarantees that $J_{j'}$ is added to $\widetilde{J}_{j^*}$ and $\widetilde{J}_{j^*+1}$. This yields $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} \neq \emptyset$, a contradiction. Therefore, (20) holds. By (19), observe that (20) implies

$$\nexists\, i \in \{1,\dots,\kappa\} \text{ s.t. } j' \in I \setminus I(o_i),\ H^*_{j^*} = H^*_{j^*+1} = H^*_{j'}. \tag{21}$$

Next, for $l_1, l_2 \in \{1,\dots,n'\}$, we claim

$$H^*_{j^*} = H^*_{j^*+1} \subseteq \mathrm{co}\{H^*_{l_1}, H^*_{l_2}\} \implies J_{l_1} \cap J_{l_2} = \emptyset. \tag{22}$$

Suppose not. Then Algorithm 10 guarantees that $J_{l_1} \cap J_{l_2} \neq \emptyset$ is added to $\widetilde{J}_{j^*}$ and $\widetilde{J}_{j^*+1}$. This yields $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} \neq \emptyset$, a contradiction. Therefore, (22) holds. By (19), observe that

$$J_{l_1} \cap J_{l_2} = \emptyset \implies \nexists\, i \in \{1,\dots,\kappa\} \text{ s.t. } l_1, l_2 \in I \setminus I(o_i). \tag{23}$$

The sets $\mathcal{C}^*_i$, $i \in \{1,\dots,\kappa\}$, are m'-dimensional cones, where $0 \le m' \le 2$. By (17), a 1-dimensional cone $\mathcal{C}^*_i$ is the union of some linearly dependent rays H^*_j, $j \in I \setminus I(o_i)$. Since by (21), no index $j' \in \{1,\dots,n'\}$ such that $H^*_{j'} = H^*_{j^*} = H^*_{j^*+1}$ appears in a set $I \setminus I(o_i)$, $i \in \{1,\dots,\kappa\}$, the ray $H^*_{j^*} = H^*_{j^*+1}$ cannot be a subset of a 1-dimensional cone. Furthermore, a 2-dimensional cone $\mathcal{C}^*_i$ is the convex hull of the union of rays $H^*_{j_1}$ and $H^*_{j_2}$, $j_1, j_2 \in I \setminus I(o_i)$. Combining (22) and (23), however, if $H^*_{j^*} = H^*_{j^*+1}$ lies in the convex hull of two rays $H^*_{l_1}$ and $H^*_{l_2}$, then there is no index $i \in \{1,\dots,\kappa\}$ such that $l_1, l_2 \in I \setminus I(o_i)$. Therefore, the ray $H^*_{j^*} = H^*_{j^*+1}$ cannot be a subset of a 2-dimensional cone either. Hence, $H^*_{j^*} = H^*_{j^*+1} \nsubseteq \bigcup_{i=1}^{\kappa} \mathcal{C}^*_i$ and, according to Corollary 6, Problem 2 is solvable.

(ii) Assume $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} = \emptyset$ and $H^*_{j^*} \neq H^*_{j^*+1}$. Pick any vector h^* in the relative interior of $\mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$ (note that since condition *(ii)* in Lemma 8 is not satisfied, the rays $H^*_{j^*}$ and $H^*_{j^*+1}$ are linearly independent and the relative interior is non-empty). We prove by contradiction that $h^* \notin \bigcup_{i=1}^{\kappa} \mathcal{C}^*_i$. Assume $h^* \in \mathcal{C}^*_{i^*}$ for some

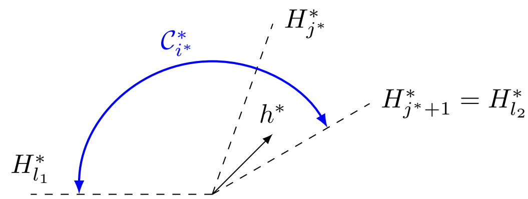

Fig. 7. The graphic corresponding to the proof of the first part of Theorem 11.

$i^* \in \{1,\dots,\kappa\}$. Since $h^* \in \mathcal{C}^*_{i^*}$ and $h^* \notin H^*_j$, $j \in \{1,\dots,n'\}$, by (17), $\mathcal{C}^*_{i^*}$ is a 2-dimensional cone. Using (17), there exist $l_1, l_2 \in \{1,\dots,n'\} \setminus I(o_{i^*})$ such that $h^* \in \mathrm{co}\{H^*_{l_1}, H^*_{l_2}\} \subseteq \mathcal{C}^*_{i^*}$, where $H^*_{l_1}$ and $H^*_{l_2}$ are two linearly independent rays. Notice that $i^* \in J_{l_1} \cap J_{l_2}$. Since $h^* \in \mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\} \cap \mathrm{co}\{H^*_{l_1}, H^*_{l_2}\} \neq \emptyset$ (see Fig. 7) and $H^*_{j^*}$ and $H^*_{j^*+1}$ are in consecutive (clockwise) order, we have $\mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\} \subseteq \mathrm{co}\{H^*_{l_1}, H^*_{l_2}\}$. Therefore, Algorithm 10 guarantees that i^* is added to the sets $\widetilde{J}_{j^*}$ and $\widetilde{J}_{j^*+1}$. This contradicts $\widetilde{J}_{j^*} \cap \widetilde{J}_{j^*+1} = \emptyset$. Hence, $h^* \notin \bigcup_{i=1}^{\kappa} \mathcal{C}^*_i$ and, according to Corollary 6, Problem 2 is solvable.

($\Longrightarrow$) First, we claim that $\bigcup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\} = \mathbb{R}^m$. Suppose not. Then there exists a ray $h^* \nsubseteq \mathrm{co}\{H^*_j, H^*_{j+1}\}$, $j \in \{0,\dots,n'-1\}$. Let $H^*_{j^*}$ and $H^*_{j^*+1}$ be the rays that have the smallest angular distance to h^* on each side (see Fig. 5). Since h^* and the relative interior of $\mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$ have no points in common, there exists a line $\mathcal{L}$ that separates them. Since the rays are arranged in clockwise order, all the rays H^*_j, $j \in \{1,\dots,n'\}$, lie in the same side of the line $\mathcal{L}$, i.e., all the rays lie in an open half-plane. Therefore, $H^*_j \subseteq \mathrm{co}\{H^*_{j^*}, H^*_{j^*+1}\}$, $\forall j \in \{1,\dots,n'\}$, and condition *(iii)* in Lemma 8 is satisfied, which is a contradiction.

Now, suppose Problem 2 is solvable. According to Corollary 6, $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i \neq \mathbb{R}^m$. Suppose by way of contradiction $\widetilde{J}_j \cap \widetilde{J}_{j+1} \neq \emptyset$ for all $j \in \{0,\dots,n'-1\}$. Therefore, for each $j \in \{0,\dots,n'-1\}$, there exists $i^* \in \{1,\dots,\kappa\}$ such that $j, j+1 \in I \setminus I(o_{i^*})$. By (17), $\mathrm{co}\{H^*_j, H^*_{j+1}\} \subseteq \mathcal{C}_{i^*}$. Therefore, $\bigcup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\} \subseteq \bigcup_{i=1}^{\kappa} \mathcal{C}^*_i$, which is a contradiction because $\bigcup_{i=1}^{\kappa} \mathcal{C}^*_i \neq \mathbb{R}^m$ and we showed in the previous argument that $\bigcup_{j=0}^{n'-1} \mathrm{co}\{H^*_j, H^*_{j+1}\} = \mathbb{R}^m$. □

Remark 12 (Computational complexity) *It was assumed that the rays H^*_j, $j \in \{1,\dots,n'\}$, are arranged in clockwise order. The time complexity of efficient sorting algorithms is known to be in $O(n \log n)$ Knuth (1998). Lemma 8 requires verification of three conditions: condition (i) is trivially checked in constant time. Conditions (ii) and (iii) require at most $O(n)$ and $O(n^2)$ operations, each of which can be formulated as scalar linear equalities or inequalities. Algorithm 10 contains, at worst, $O(n^3)$ repetitions of a set union and intersection operation. Thus, the total complexity of Algorithm 10 may vary depending on the data structure used in the implementation, but it can certainly be bounded* *by $O(n^4)$. Theorem 11 has the same computational complexity.*

Next, we use the results of this section to solve Problem 2 for an affine system with two inputs. Example 13, which is four-dimensional as by Ornik and Broucke (2015) lower-dimensional examples will not be particularly illustrative, is important for two reasons. First, it shows that the necessary condition proposed in Semsar-Kazerooni and Broucke (2014) for single-input systems, i.e., $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$, where $\text{cone}(\mathcal{O}_{\mathcal{S}})$ is defined in (6), is not a necessary condition for solvability of Problem 2 for multi-input systems. Second, as mentioned earlier, since $\mathcal{O}_{\mathcal{S}}$ is not a face of $\mathcal{S}$, previous results in the literature cannot address the solvability of the RCP in Example 13, without resorting to some triangulation of $\mathcal{S}$.

Example 13 *Consider a simplex $S = \text{co}\{v_0, \ldots, v_4\} \subseteq \mathbb{R}^4$, where $v_0^T = [0\ 0\ 0\ 0]$, and for $j \in I := \{1, 2, 3, 4\}$, define $v_j := e_j$, the j^{th} Euclidean basis vector. Therefore, $h_j := -e_j$, $j \in I$. Let the parameters of the affine system* (1) *be*

$$A = \begin{bmatrix} -8 & -8 & 2 & -6 \\ 4 & 16 & 0 & 4 \\ 4 & 10 & 0 & 4 \\ -4 & 2 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}, \ a = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Based on (4)*, it is easy to verify that*

$$\mathcal{O} = \left\{ x \in \mathbb{R}^4 \ \middle|\ \begin{bmatrix} -8 & 16 & 4 & 0 \\ 0 & -6 & -2 & -2 \end{bmatrix} x = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\},$$

and $\mathcal{O}_{\mathcal{S}} = \text{co}\{o_1, o_2, o_3\}$, where $o_1 = (0, 0, 0, \frac{1}{2})$, $o_2 = (\frac{1}{4}, 0, \frac{1}{2}, 0)$, and $o_3 = (\frac{1}{3}, \frac{1}{6}, 0, 0)$. Note that $I(o_1) = \{0, 4\}$, $I(o_2) = \{0, 1, 3\}$, and $I(o_3) = \{0, 1, 2\}$. The QR factorization of B is computed as

$$Q = \begin{bmatrix} 0.64 & -0.64 & -0.43 & 0 \\ -0.41 & -0.41 & 0 & 0.82 \end{bmatrix}^T, \ R = \begin{bmatrix} -4.69 & -2.35 \\ 0 & -1.22 \end{bmatrix}.$$

The cones $\mathcal{C}_i$, $i \in \{1, 2, 3\}$, are shown in Fig. 8(a) and it is easy to see that their intersection is the singleton $\mathbf{0}$. Using Lemma 3, we conclude that $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$. Next, we use Theorem 11 to solve Problem 2. This will show that the necessary condition proposed in Semsar-Kazerooni and Broucke (2014) for single-input systems, i.e., $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$, is no longer necessary for two-input systems. Note that $Q^T h_j \neq 0$ for all $j \in I$ and the rays H_j^*, $j \in I$, are already in clockwise order as illustrated in Fig. 8(b). It is easy to see that $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 3\}$, and $J_4 = \{2, 3\}$. Next, Algorithm 10 yields $\widetilde{J}_1 = \{1\}$, $\widetilde{J}_2 = \{1, 2\}$, $\widetilde{J}_3 = \{1, 2, 3\}$, and $\widetilde{J}_0 = \widetilde{J}_4 = \{2, 3\}$. Since $\widetilde{J}_1 \cap \widetilde{J}_4 = \emptyset$, by Theorem 11, Problem 2 is solvable. This is also in accordance with Corollary 6, since $\bigcup_{i=1}^3 \mathcal{C}_i^* \neq \mathbb{R}^2$ as shown in Fig. 8(b). Next, we use Corollary 5 to compute a vertex map $f : V_{\mathcal{O}_{\mathcal{S}}} \to \mathcal{B}$ that solves Problem 2. Consider the vector $h^* = [1.5 \quad 0.42]^T \in \text{co}\{H_1^*, H_4^*\}$.

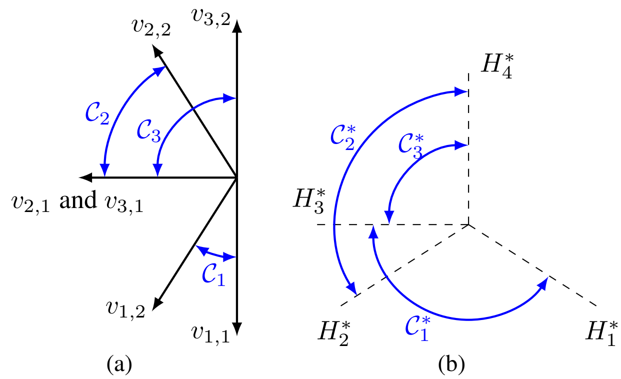

Fig. 8. The graphics corresponding to Example 13. The rays are shown with dashed lines, the vectors are shown with solid arrows, and the cones are illustrate by blue arcs. (a) the cones $\mathcal{C}_i$, $i \in \{1, 2, 3\}$, and (b) the corresponding dual cones $\mathcal{C}_i^*$, $i \in \{1, 2, 3\}$. $v_{i,1}, v_{i,2}$ are the boundary rays for cones $\mathcal{C}_i$.

Clearly, $h^* \notin \bigcup_{i=1}^3 \mathcal{C}_i^*$ and, by (16)*, $h^* \in \bigcap_{i=1}^3 \mathcal{N}_i$. Solving* (12)*, we find the following feasible vectors: $w_1 = [-33.6 \quad -120.46]^T \in \mathcal{C}_1$, $w_2 = [-67.71 \quad 35.1]^T \in \mathcal{C}_2$, and $w_3 = [-90.59 \quad 47.59]^T \in \mathcal{C}_3$. The corresponding closed-loop vertex map $f(o_i) = Qw_i$, $i \in \{1, 2, 3\}$, is then computed as $f(o_1) = (27.69, 70.67, 14.33, -98.36)$, $f(o_2) = (-57.64, 28.98, 28.87, 28.66)$, and $f(o_3) = (-77.37, 38.51, 38.63, 38.86)$. We can use the vertex map f to find an affine feedback $u(x) = Kx + g$ such that the closed-loop vector field is non-vanishing on $\mathcal{O}_{\mathcal{S}}$ and satisfies $Ax + Bu(x) + a \in \mathcal{C}(x)$, $x \in \mathcal{O}_{\mathcal{S}}$. To this end, the vertex map f is extendible on $\mathcal{O}_{\mathcal{S}}$ to the affine map $F = K_F x + g_F$, where*

$$K_F = \begin{bmatrix} -233.52 & 2.8 & 1.48 & 55.37 \\ 112.11 & 0.86 & -0.09 & 139.34 \\ 115.21 & 1.36 & 0.14 & 28.66 \\ 113.41 & 6.34 & 0.61 & -196.71 \end{bmatrix}, \ g_F = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Note that $K_F = A + BK = A + QRK$ and $g_F = Bg + a = QRg + a$. Hence, the affine feedback controller gains $K = R^{-1}Q^T(K_F - A)$ and $g = R^{-1}Q^T(g_F - a)$ are computed as

$$K = \begin{bmatrix} 114.31 & -2.15 & 0.38 & -86.03 \\ -117.41 & -4.34 & -0.61 & 196.71 \end{bmatrix}, \ g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is easy to verify that the closed-loop affine system defined over simplex S satisfies the conditions in Corollary 9 in Roszak and Broucke (2006) with $\xi = -[5.37 \quad 5.83 \quad 0.87 \quad 5.27]^T$. Therefore, the RCP is solvable even though $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$.

5 Computationally Efficient Necessary Conditions for Existence of a Non-vanishing Affine Extension on $\mathcal{O}_{\mathcal{S}}$

In this section, we present necessary conditions for solvability of Problem 2 as a feasibility program in terms of linear inequalities. Feasibility problems subject to linear inequalities can easily be programmed and solved using available optimization software. In contrast to Section 4, the necessary conditions are applicable to all systems regardless of the number of inputs.

Let $\mathcal{I}$ be an $(m+1)$-subset of $\{1,\ldots,\kappa\}$, i.e. $\mathcal{I}$ is a subset of $\{1,\ldots,\kappa\}$ with $m+1$ elements. For each $i\in\mathcal{I}$, let $E_{\mathcal{I}}(o_i)\subseteq I(o_i)$ be the set of non-zero *exclusive members* of $I(o_i)$ in the set $\bigcup_{i\in\mathcal{I}} I(o_i)$, defined as $E_{\mathcal{I}}(o_i):=\{l\in I(o_i)\mid l\notin I(o_j),\ \forall j\in\mathcal{I},\ j\neq i\}\setminus\{0\}$. Next, define $S_{\mathcal{I}}$ to be the set of non-zero *shared vertices* in $\bigcup_{i\in\mathcal{I}} I(o_i)$ given by

$$S_{\mathcal{I}}:=\Big(\bigcup_{i\in\mathcal{I}} I(o_i)\Big)\setminus\Big(\bigcup_{i\in\mathcal{I}} E_{\mathcal{I}}(o_i)\cup\{0\}\Big). \tag{24}$$

The following lemma is used in the proof of the main result of this section.

Lemma 14 *Each ray H_j^*, $j\in I\setminus S_{\mathcal{I}}$, is contained in all, except (potentially) one, cones $\mathcal{C}_i^*$, $i\in\mathcal{I}$.*

PROOF. Equation (24) yields $I\setminus S_{\mathcal{I}}=(I\setminus\bigcup_{i\in\mathcal{I}} I(o_i))\cup(\bigcup_{i\in\mathcal{I}} E_{\mathcal{I}}(o_i))$. First, assume $j\in I\setminus\bigcup_{i'\in\mathcal{I}} I(o_{i'})$. Since $I\setminus\bigcup_{i'\in\mathcal{I}} I(o_{i'})\subseteq I\setminus I(o_i)$, $i\in\mathcal{I}$, (17) yields $H_j^*\subseteq\mathcal{C}_i^*$, $i\in\mathcal{I}$. Therefore, each ray H_j^*, $j\in I\setminus\bigcup_{i'\in\mathcal{I}} I(o_{i'})$, is contained in all $m+1$ cones $\mathcal{C}_i^*$, $i\in\mathcal{I}$. Second, assume $j\in E_{\mathcal{I}}(o_{i'})$, $i'\in\mathcal{I}$. Since

$$E_{\mathcal{I}}(o_{i'})\subseteq I\setminus I(o_i),\ i'\in\mathcal{I},\ i\in\mathcal{I}\setminus\{i'\},$$

(17) yields $H_j^*\subseteq\mathcal{C}_i^*$, $i\in\mathcal{I}\setminus\{i'\}$. Therefore, each ray H_j^*, $j\in E_{\mathcal{I}}(o_{i'})$, $i'\in\mathcal{I}$, is contained in all cones $\mathcal{C}_i^*$, $i\in\mathcal{I}$, except potentially $\mathcal{C}_{i'}^*$. This completes the proof. □

Given the set $\mathcal{I}$, let the cone $M_{\mathcal{I}}$ be defined as

$$M_{\mathcal{I}}:=\{y\in\mathbb{R}^n\mid h_j\cdot y\le 0,\ j\in I\setminus S_{\mathcal{I}}\}. \tag{25}$$

The next result shows that if $\mathcal{B}$ is not of sufficiently high dimension, and there are no non-zero vectors lying in $\mathcal{B}\cap M_{\mathcal{I}}$, then the obstruction described by Problem 2 exists.

Theorem 15 *Assume $1\le m<\kappa$ and let $\mathcal{I}$ be an $(m+1)$-subset of $\{1,\ldots,\kappa\}$. If Problem 2 is solvable, then $\mathcal{B}\cap M_{\mathcal{I}}\neq\mathbf{0}$.*

PROOF. By way of contradiction, assume Problem 2 is solvable, but $\mathcal{B}\cap M_{\mathcal{I}}=\mathbf{0}$. Define the cone $\mathcal{M}_{\mathcal{I}}$ as

$$\begin{aligned}\mathcal{M}_{\mathcal{I}}:=&\{w\in\mathbb{R}^m\mid(Q^Th_j)\cdot w\le 0,\ j\in I\setminus S_{\mathcal{I}}\}\\=&\bigcap_{j\in I\setminus S_{\mathcal{I}}} H_j,\end{aligned}$$

where H_j is defined in (9). Similar to Lemma 3, it is easy to show that if $y\in\mathcal{B}$, then $Q^Ty\in\mathcal{M}_{\mathcal{I}}$ if and only if $y\in\mathcal{B}\cap M_{\mathcal{I}}$. By assumption, $\mathcal{B}\cap M_{\mathcal{I}}=\mathbf{0}$. Therefore, $\mathcal{M}_{\mathcal{I}}=\mathbf{0}$, and $\mathcal{M}_{\mathcal{I}}^*=\mathbb{R}^m$, where $\mathcal{M}_{\mathcal{I}}^*$ is the dual cone of $\mathcal{M}_{\mathcal{I}}$. Using Theorem 2 in Sandgren (1954), we can write

$$\mathcal{M}_{\mathcal{I}}^*=\mathrm{co}\Big\{\bigcup_{j\in I\setminus S_{\mathcal{I}}} H_j^*\Big\}=\mathrm{co}\{H_j^*\mid j\in I\setminus S_{\mathcal{I}}\}.$$

Based on Theorem 3.2 in Beck and Robins (2007), the m-dimensional cone $\mathcal{M}_{\mathcal{I}}^*$ can be triangulated into *simplicial cones*, where each simplicial cone is defined as the convex hull of precisely m linearly independent rays H_j^*, $j\in I\setminus S_{\mathcal{I}}$. Therefore,

$$\mathcal{M}_{\mathcal{I}}^*=\bigcup_{\mathcal{J}_m}\mathrm{co}\Big\{\bigcup_{j\in\mathcal{J}_m} H_j^*\Big\}, \tag{26}$$

where $\mathcal{J}_m$ is an m-subset [1] of $I\setminus S_{\mathcal{I}}$. Evidently, not all m-subsets of $I\setminus S_{\mathcal{I}}$ might consist of linearly independent rays, however, their union is equal to $\mathcal{M}_{\mathcal{I}}^*$. Consider the rays H_j^*, $j\in\mathcal{J}_m\subseteq I\setminus S_{\mathcal{I}}$. Now, let us note $|\mathcal{I}|=m+1\ge 2$. Hence, from Lemma 14, there exists at least one $i^*\in\mathcal{I}$ such that $H_j^*\subseteq\mathcal{C}_{i^*}^*$, $j\in\mathcal{J}_m$. Since dual cones are convex, we also have $\mathrm{co}\{\bigcup_{j\in\mathcal{J}_m} H_j^*\}\subseteq\mathcal{C}_{i^*}^*$. Now, (26) yields

$$\mathcal{M}_{\mathcal{I}}^*=\bigcup_{\mathcal{J}_m}\mathrm{co}\Big\{\bigcup_{j\in\mathcal{J}_m} H_j^*\Big\}\subseteq\bigcup_{i=1}^{\kappa}\mathcal{C}_i^*.$$

Since $\mathcal{M}_{\mathcal{I}}^*=\mathbb{R}^m$, we have $\bigcup_{i=1}^{\kappa}\mathcal{C}_i^*=\mathbb{R}^m$. Therefore, according to Corollary 6, Problem 2 is unsolvable, which is a contradiction. □

Based on Theorem 15, if Problem 2 is solvable, the cone condition $\mathcal{B}\cap M_{\mathcal{I}}\neq\mathbf{0}$ must hold for any $(m+1)$-subset of $\{1,\ldots,\kappa\}$. The number of $(m+1)$-subsets of κ elements is given by the binomial coefficient. Therefore, Theorem 15 presents $\binom{\kappa}{m+1}$ sets of necessary conditions for solvability of Problem 2. These conditions rely on determining an intersection of up to n half-spaces in $\mathbb{R}^m$, which was shown in Brown (1978) to be equivalent to the multi-dimensional convex hull problem. This problem can be solved in time polynomial in n. One of the algorithms was given in Skiena (2008), with the worst case complexity of $O(n^{\lfloor m/2\rfloor+1})$.

We finish this section with an interesting special case where the familiar condition $\mathcal{B}\cap\mathrm{cone}(\mathcal{O}_{\mathcal{S}})\neq\mathbf{0}$ (Semsar-Kazerooni and Broucke (2014)) emerges from Theorem 15 as both a necessary and sufficient condition.

Assumption 16 *Any shared vertex index $j\in I$ is shared by all $I(o_i)$, $i=1,\ldots,\kappa$.*

Corollary 17 *Suppose Assumption 16 holds and $1\le m<\kappa$. Problem 2 is solvable if and only if $\mathcal{B}\bigcap\mathrm{cone}(\mathcal{O}_{\mathcal{S}})\neq\mathbf{0}$.*

[1] Since the convex hull of rays H_j^*, $j\in I\setminus S_{\mathcal{I}}$, covers the whole space $\mathbb{R}^m$, the cardinality of the set $I\setminus S_{\mathcal{I}}$ is greater than or equal to m.

PROOF. ($\Longrightarrow$) From (6) we can write

$$\begin{aligned}\operatorname{cone}(\mathcal{O}_{\mathcal{S}}) &= \bigcap_{i=1}^{\kappa} \mathcal{C}(o_i) = \bigcap_{i=1}^{\kappa} \{y \mid h_j \cdot y \le 0,\ j \in I \setminus I(o_i)\} \\ &= \{y \mid h_j \cdot y \le 0,\ j \in [I \setminus I(o_1)] \cup \cdots \cup [I \setminus I(o_\kappa)]\} \\ &= \{y \mid h_j \cdot y \le 0,\ j \in I \setminus [I(o_1) \cap \cdots \cap I(o_\kappa)]\}.\end{aligned} \tag{27}$$

Suppose Assumption 16 holds. Then for any set $\mathcal{I}$ defined as an $(m+1)$-subset of $\{1, \ldots, \kappa\}$, the set of non-zero shared vertices $S_{\mathcal{I}}$ can be written as $S_{\mathcal{I}} = I(o_1) \cap \cdots \cap I(o_\kappa) \setminus \{0\}$. Thus, $I \setminus [I(o_1) \cap \cdots \cap I(o_\kappa)] = I \setminus S_{\mathcal{I}}$. Therefore, (25) and (27) yield $\operatorname{cone}(\mathcal{O}_{\mathcal{S}}) = M_{\mathcal{I}}$. The result follows from Theorem 15.

($\Longleftarrow$) Problem 2 is trivially solvable by setting $f(o_i) = y$ for any $0 \neq y \in \mathcal{B} \cap \operatorname{cone}(\mathcal{O}_{\mathcal{S}})$. □

6 Conclusions

In this paper, we introduced a new obstruction to solvability of the RCP on a simplex using affine feedback, and we provide necessary and sufficient conditions for occurrence of the obstruction. These conditions can be formulated as scalar linear inequalities for two-input systems. Finally, we proposed computationally efficient necessary conditions for checking the obstruction for multi-input systems as feasibility programs in terms of linear inequalities.

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