Monotonic Reach Control on Polytopes

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***Abstract*— The paper studies the problem of making an affine system defined on a polytopic state space reach a prescribed facet of the polytope in finite time without first leaving the polytope. The focus is on solvability by continuous piecewise affine feedback, and we formulate a variant of the problem in which trajectories exit in a monotonic sense. This allows to obtain necessary and sufficient conditions for solvability in certain geometric situations. Next, we show that, generically, solvability via arbitrary triangulations is equivalent to monotonic solvability. In contrast with existing simplex-based methods, this provides an avenue for reach control on polytopes that does not depend on the choice of triangulation of the polytope.**

I. Introduction

We study the reach control problem (RCP) for affine systems on polytopes. The problem is for an affine system defined on a polytopic state space to exit the polytope through a prespecified facet in finite time without first leaving the polytope [4], [5]. The problem sits within a family of reachability problems for hybrid systems [1], [9], [3]. The most definitive results on the problem are focused on reach control on simplices by affine feedback [6], [10], [2]. Results for polytopes come in one of two forms. Either one must perform a triangulation of the polytope and apply simplex-based reach control methods [6], [10]. Alternatively, one may impose conditions so that the design can be carried out in two independent steps: first one assigns control inputs at the vertices of the polytope guaranteeing propitious closed-loop behavior; second, one selects any triangulation of the polytope and one forms a (continuous) piecewise affine feedback based on the vertex control values of step one. We study the relative merits of the two approaches, and we find via examples that the two methods are complementary. The investigation highlights that new research is needed to understand triangulation in control problems.

Past research on reach control on polytopes has either required strong sufficient conditions or restrictive assumptions on the system dynamics [5], [8]. This paper initiates a study of the reach control problem in which such restrictions are removed; instead geometric properties of the system are exploited to the best possible extent. In particular, the placement of $\mathcal{O}$, the set of possible equilibria, relative to the polytope $\mathcal{P}$ plays a key role, and in certain cases, clear necessary and sufficient conditions can be obtained which remove the conservativism or restrictiveness of previous work. We then formulate the *monotonic reach control problem* (MRCP) where it is required that trajectories exit the polytopic state space in a monotonic sense relative to a foliation of parallel hyperplanes. This notion of monotonic solvability is shown to be generically equivalent to solvability of RCP by piecewise affine feedback using any choice of triangulation of $\mathcal{P}$. The latter is particularly useful when triangulation is performed by a standalone software not adapted to control problems.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The symbol $\mathbb{U}$ represents a control class such as open-loop controls, continuous state feedback, affine feedback, etc. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$.

II. Reach Control Problem

Consider an n-dimensional polytope $\mathcal{P} := \text{co}\,\{v_1, \ldots, v_p\}$ with vertex set $V := \{v_1, \ldots, v_p\}$ and facets $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$. The *target set* is the facet $\mathcal{F}_0$ of $\mathcal{P}$. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J = \{1, \ldots, r\}$, and $J(x) = \{j \in J \mid x \in \mathcal{F}_j\}$, where $x \in \mathcal{P}$. For each $v \in V$, define the closed, convex cone $\mathcal{C}(v) := \{\ y \in \mathbb{R}^n \ : \ h_j \cdot y \le 0, \ j \in J(v)\ \}$. (Note that h_0 does not appear since $\mathcal{F}_0$ is the target set). We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{P}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\phi_u(t, x_0)$ be the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the target $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 2.1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \ge 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{P}$ reach and exit the target $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$.

Definition 2.1: A point $x_0 \in \mathcal{P}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{P}$ with control class $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{P}} \mathcal{F}_0$, if there exists a control u of class $\mathbb{U}$ such that property (i) of Problem 2.1 holds. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by control class $\mathbb{U}$ if for every $x_0 \in \mathcal{P}$, $x_0 \xrightarrow{\mathcal{P}} \mathcal{F}_0$ with control of class $\mathbb{U}$.

The following conditions ensure that trajectories only exit $\mathcal{P}$ via $\mathcal{F}_0$ [5].

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Definition 2.2: We say the *invariance conditions* are solvable if for each $v \in V$ there exists $u \in \mathbb{R}^m$ such that $Av + Bu + a \in \mathcal{C}(v)$, $v \in V$. Equivalently,

$$h_j \cdot (Av + Bu + a) \leq 0 \,, \qquad j \in J(v) \,. \tag{2}$$

Let $\mathcal{B} = \text{Im } B$, the image of B. Define the set

$$\mathcal{O} := \{ \, x \in \mathbb{R}^n \; : \; Ax + a \in \mathcal{B} \, \} \,. \tag{3}$$

Notice that the vector field $Ax + Bu + a$ can vanish at any $x \in \mathcal{O}$ for an appropriate choice of $u \in \mathbb{R}^m$, so $\mathcal{O}$ is the set of all possible equilibrium points of (1). We also define the set of possible equilibrium points of (1) on $\mathcal{P}$ by

$$\mathcal{O}_{\mathcal{P}} := \mathcal{P} \cap \mathcal{O} \,. \tag{4}$$

Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_{\mathcal{P}} = \emptyset$ or $\mathcal{O}_{\mathcal{P}}$ is a κ-dimensional polytope in $\mathcal{P}$. If $\mathcal{O}_{\mathcal{P}} \neq \emptyset$, we define the vertex set of $\mathcal{O}_{\mathcal{P}}$ to be $V_{\mathcal{O}} = \{o_1, \ldots, o_q\}$, where o_i are the vertices of $\mathcal{O}_{\mathcal{P}}$ (not necessarily vertices of $\mathcal{P}$). Also define the index set $I_{\mathcal{O}} = \{1, \ldots, q\}$.

In this paper we focus on piecewise affine feedback. Let $\mathbb{T}$ be a triangulation of polytope $\mathcal{P}$. A point $x \in \mathcal{P}$ lies in the interior of precisely one simplex in $\mathbb{T}$ whose vertices are, say, $v_1, \ldots, v_k$. Then $x = \sum_{i=1}^{k} \lambda_i v_i$, where $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. Coefficients $\lambda_1, \ldots, \lambda_k$ are called the *barycentric coordinates* of x. Given a state feedback $u(x)$ on $\mathcal{P}$, we say u is a *piecewise affine feedback* if for any $x \in \mathcal{P}$, $x = \sum_i \lambda_i v_i$ implies $u(x) = \sum_i \lambda_i u(v_i)$, where the λ_i are barycentric coordinates of x. It is easy to show that $u(x)$ is a continuous state feedback on $\mathcal{P}$. If $u(x)$ is a piecewise affine feedback on $\mathcal{P}$, then for each n-dimensional simplex $\mathcal{S}_j \in \mathbb{T}$, there exist $K_j \in \mathbb{R}^{m \times n}$ and $g_j \in \mathbb{R}^m$ such that $u(x)$ takes the form $u(x) = K_j x + g_j$, $x \in \mathcal{S}_j$. We say $\mathbb{T}$ is a *triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$* if $\mathbb{T}$ is a refinement of a subdivision of the point set $V \cup V_{\mathcal{O}}$ such that $\mathcal{O}_{\mathcal{P}}$ is a union of simplices in $\mathbb{T}$.

III. From Simplices to Polytopes

It is known that for simplices, RCP is solvable by affine feedback if and only if two conditions hold: (a) the invariance conditions (2) are solvable, and (b) there is no closed-loop equilibrium in the simplex [6], [10]. The no-equilibrium requirement can also be expressed as a so-called flow condition, which gives an equivalent numerical test. We are interested to obtain the most immediate extension of this result for polytopes. First, we restrict our attention to continuous piecewise affine (PWA) feedback. Assuming PWA feedback, the invariance conditions remain necessary conditions for solvability of RCP on polytopes [5]. Instead, the flow condition is no longer necessary for solvability on polytopes. Indeed the statement that there is no closed-loop equilibrium is no longer equivalent to existence of a flow condition when dealing with general polytopes, because the equivalence relies on the convexity of the closed-loop vector field. Convexity is preserved with affine feedback, but not necessarily with PWA feedback. On the other hand, the flow condition affords useful properties; particularly that trajectories exit the polytope in an orderly way. In this section we begin an exploration of the extent to which results for simplices carry over to polytopes. Guided by these insights, we formulate in Section IV a restricted version of RCP: we incorporate the requirement of a flow condition into the problem statement, and we call this restricted problem *monotonic reach control*.

Suppose we are given a triangulation $\mathbb{T}$ of $\mathcal{P}$ with respect to $\mathcal{O}$ and we are given $u(x)$, a piecewise affine feedback defined on $\mathbb{T}$ which satisfies the invariance conditions of $\mathcal{P}$. Define

$$b_i := Ao_i + Bu(o_i) + a \in \mathcal{B} \cap \mathcal{C}(o_i) \,, \qquad\qquad i \in I_{\mathcal{O}} \,.$$

If we want to exclude closed-loop equilibria in $\mathcal{P}$, then we only need to concentrate on the behavior of the closed-loop vector field in $\mathcal{O}_{\mathcal{P}}$. A basic result of convex analysis says that there are no closed-loop equilibria in $\mathcal{O}_{\mathcal{P}}$ if there is a flow condition on $\mathcal{O}_{\mathcal{P}}$.

Lemma 3.1: Let $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B}\}$ be such that $0 \notin \text{co}\, \{b_1, \ldots, b_q\}$. Then there exists $\beta \in \mathcal{B}$ such that $\beta \cdot b_i < 0$, $i = 1, \ldots, q$.

The condition that $0 \notin \text{co}\, \{b_1, \ldots, b_q\}$ can be related to the existence of closed-loop equilibria in $\mathcal{P}$.

Theorem 3.2: Consider the system (1) defined on a polytope $\mathcal{P}$. Let $\mathbb{T}$ be a triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$, and let $u(x)$ be a piecewise affine feedback defined on $\mathbb{T}$. If $0 \notin \text{co}\, \{b_1, \ldots, b_q\}$, then the closed-loop system has no equilibrium in $\mathcal{P}$.

In [2], two geometric sufficient conditions were obtained for constructing a flow condition on simplices. The first condition was that $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, where $\text{cone}(\mathcal{S})$ is the tangent cone to simplex $\mathcal{S}$ at the vertex not containing the exit facet $\mathcal{F}_0$. The second condition is that there is a set of linearly independent vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$, where it is assumed that $v_1, \ldots, v_q$ are the vertices of $\mathcal{O}_{\mathcal{P}}$. To relate these results to the more general setting of polytopes, we need to translate the geometric conditions to analogous conditions for polytopes, and secondly, we need to relate those translated geometric conditions to the statement that there are no equilibria in $\mathcal{O}_{\mathcal{P}}$ via Theorem 3.2.

First, we introduce a condition analogous to the statement for a simplex $\mathcal{S}$ that $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$. Define

$$\text{cone}(\mathcal{O}_{\mathcal{P}}) := \bigcap_{o \in V_{\mathcal{O}}} \mathcal{C}(o) \,.$$

In particular, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ is the cone of directions in $\mathcal{B}$ that simultaneously satisfy the union of all invariance conditions at all vertices of $\mathcal{O}_{\mathcal{P}}$.

Lemma 3.3: Suppose $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$. Then there exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$ such that $0 \notin \text{co}\, \{b_1, \ldots, b_q\}$.

Next, consider the condition for a simplex $\mathcal{S}$ that there is a linearly independent set of vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$. Removing the restriction that vertices of $\mathcal{O}_{\mathcal{P}}$ are vertices of $\mathcal{S}$, we have the following analogous condition for polytopes.

Lemma 3.4: Suppose there exists a linearly independent set of vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$. Then $0 \notin$

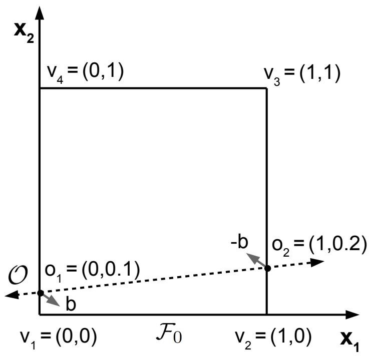

Fig. 1. $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) = \mathbf{0}$

co $\{b_1, \ldots, b_q\}$.

Based on Theorem 3.2, both of the previous conditions imply there is no closed-loop equilibrium in $\mathcal{P}$, assuming $\mathcal{P}$ is triangulated with respect to $\mathcal{O}$. But this is not enough to deduce that RCP is solved. The primarily obstacle is that convexity of the closed-loop vector field is lost using general PWA controls. Consequently, the no closed-loop equilibrium condition is no longer equivalent to the existence of a flow condition. For this reason we'll bring in a stronger condition, to be developed in the next section. It is interesting to note that the first condition $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$ is not merely one of the sufficient conditions, but is also necessary for solvability by continuous state feedback. We present here a result for single-input systems; we conjecture that the result also holds for multi-input systems, though the proof will be considerably more complex.

Theorem 3.5: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_{\mathcal{P}} \neq \emptyset$. If $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) = \mathbf{0}$, then RCP is not solvable by continuous state feedback.

Example 3.1: Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -10 \\ 1 & -10 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u\,.$$

The polytope $\mathcal{P} = \text{co}\ \{v_1, \ldots, v_4\}$ is shown in Figure 1. It can be verified that $\mathcal{O}_{\mathcal{P}} = \text{co}\ \{o_1, o_2\}$, where $o_1 = (0, 0.1)$ and $o_2 = (1, 0.2)$. Set $\mathcal{O}$ is shown in Figure 1. It can be easily checked that the invariance conditions are solvable. However, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) = \mathbf{0}$, as seen in the figure. By Theorem 3.5, RCP is not solvable by continuous state feedback.

We conclude this section by showing that solvability of RCP on polytopes by any class of controls is inextricably linked to what happens on $\mathcal{O}_{\mathcal{P}}$. In particular, if RCP is not solved by some piecewise affine feedback, it is because some trajectory encircles $\mathcal{O}_{\mathcal{P}}$, approaches $\mathcal{O}_{\mathcal{P}}$, or remains on $\mathcal{O}_{\mathcal{P}}$.

Lemma 3.6: Suppose there exists $x_0 \in \mathcal{P}$ and an open-loop control $u(t)$ such that the associated (unique) solution $\phi_u(t, x_0)$ of (1) satisfies $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \geq 0$. Define $\mathcal{A} = \text{co}\ \{\phi_u(t, x_0) \mid t \geq 0\}$. Then $\overline{\mathcal{A}} \cap \mathcal{O}_{\mathcal{P}} \neq \emptyset$.

IV. Monotonic Reach Control Problem

The previous section identified issues concerning existence of equilibria on $\mathcal{O}_{\mathcal{P}}$, necessary conditions for solvability of RCP, and the relationship between failure to solve RCP and behavior of trajectories with respect to $\mathcal{O}_{\mathcal{P}}$. However, clear necessary and sufficient conditions are not obtained. This is because a no-equilibrium condition (in addition to solvability of invariance conditions) is not known to be sufficient to solve RCP on polytopes. Instead, we study a more restrictive form of the problem which does lead to the natural analog of results for simplices. These necessary and sufficient conditions for solvability are examined under various assumptions on the placement of $\mathcal{O}_{\mathcal{P}}$. We also make comparisons with the main results for simplices to better understand the limits of those results when dealing with polytopes.

Problem 4.1 (Monotonic Reach Control Problem (MRCP)): Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

(ii) There exists $\xi \in \mathbb{R}^n$ such that for all $x \in \mathcal{P}$, $\xi \cdot (Ax + Bu(x) + a) < 0$.

The new condition (ii) is called a *flow condition*, and the problem is called "monotonic" because trajectories flow through the polytope in a common sense with respect to a foliation of parallel hyperplanes with normal vector ξ. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically if properties (i)-(ii) of Problem 4.1 hold.

Now we investigate necessary and sufficient conditions for solvability of MRCP under assumptions on the placement of $\mathcal{O}$ with respect to $\mathcal{P}$. The first result when $\mathcal{O}_{\mathcal{P}} = \emptyset$ is based on the following technical lemma.

Lemma 4.1: Consider the system (1) defined on a compact, convex set $\mathcal{A}$. If $\mathcal{A} \cap \mathcal{O} = \emptyset$, then there exists $\beta \in$ Ker (B^T) such that $\beta \cdot (Ax + Bu + a) < 0$, for all $x \in \mathcal{A}$ and $u \in \mathbb{R}^m$.

Theorem 4.2: Consider the system (1) defined on a polytope $\mathcal{P}$, and suppose $\mathcal{O}_{\mathcal{P}} = \emptyset$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback if and only if the invariance conditions (2) are solvable.

In [2] necessary and sufficient conditions for solvability of RCP on simplices were obtained based on the assumption that $\mathcal{O}_{\mathcal{P}}$ is a face of the simplex. The same assumption for polytopes makes possible a straightforward generalization to polytopes for solvability of MRCP.

Assumption 4.1: Polytope $\mathcal{P}$ and system (1) satisfy the following condition: $\mathcal{O}_{\mathcal{P}}$ is a κ-dimensional face of $\mathcal{P}$, where $0 \leq \kappa \leq n$. In particular, $\mathcal{O}_{\mathcal{P}} = \text{co}\ \{v_1, \ldots, v_q\}$, where v_i is a vertex of $\mathcal{P}$. Let $V_{\mathcal{O}} := \{v_1, \ldots, v_q\}$.

Theorem 4.3: Consider the system (1) and suppose Assumption 4.1 holds. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback if and only if

(a) The invariance conditions (2) are solvable.

(b) There exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ such that $0 \notin \text{co}\ \{b_1, \ldots, b_q\}$.

Proof: $(\Longrightarrow)$ Let $y(x) := Ax + Bu(x) + a$, where $u(x)$ is the PWA feedback achieving $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically.

Since $u(x)$ is a continuous state feedback, the invariance conditions are solvable [5]. Now suppose that condition (b) does not hold. This implies $0 \in \text{co}\ \{y(v_1), \ldots, y(v_p)\}$. On the other hand, by assumption that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y(v_i) < 0$ for $i \in I$. This implies $\mathbf{0}$ and co $\{y(v_1), \ldots, y(v_p)\}$ are strongly separated, a contradiction.

($\Longleftarrow$) For each vertex $v_i \in V \setminus \mathcal{O}_{\mathcal{P}}$, select a control $u_i \in \mathbb{R}^m$ to satisfy the invariance conditions (2). For $v_i \in V_{\mathcal{O}}$, select $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. Form a triangulation $\mathbb{T}$ of $\mathcal{P}$. Using the method of [5], one can find unique K_j and g_j corresponding to the affine feedback $u(x) = K_j x + g_j$ on each n-dimensional simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = u_i$, $i = 1, \ldots, p$ and $y(v_i) = b_i$, $i = 1, \ldots, q$. We obtain the piecewise affine closed-loop system $\dot{x} = (A + BK_j)x + (a + Bg_j) =: y(x)$, $x \in \mathcal{P}$. We show a flow condition holds on $\mathcal{P}$. First, a flow condition holds for the closed loop vector field $y(x) := (A + BK_i)x + Bg_i + a$ on $\mathcal{O}_{\mathcal{P}}$. By Lemma 3.1, there exists $\beta_1 \in \mathcal{B}$ such that $\beta_1 \cdot y(v_i) = \beta_1 \cdot b_i < 0$, $i = 1, \ldots, q$. Next let $\mathcal{P}' :=$ co $\{v_i \mid v_i \in V \setminus V_{\mathcal{O}}\}$. Note that $\mathcal{P}' \cap \mathcal{O} = \emptyset$, so according to Lemma 4.1, there exists $\beta_2 \in \text{Ker}\ (B^T)$ such that for all $x \in \mathcal{P}'$, $\beta_2 \cdot (Ax + Bu(x) + a) < 0$. Define $\beta = \alpha\beta_1 + (1-\alpha)\beta_2$ for some $\alpha \in (0, 1)$. Consider $v_i \in V_{\mathcal{O}}$. Using the fact that $\beta_2^T b_i = 0$, we have $\beta \cdot y(v_i) = \alpha\beta_1 \cdot y(v_i) < 0$. Next consider $v_i \in V \setminus V_{\mathcal{O}}$. We have

$$\beta \cdot (Av_i + Bu_i + a) = \alpha\beta_1 \cdot (Av_i + Bu_i + a) + (1-\alpha)\beta_2 \cdot (Av_i + a)\,.$$

The term $\beta_1 \cdot (Av_i + Bu_i + a)$ is a constant of unknown sign, whereas we know $\beta_2 \cdot (Av_i + a) < 0$. Therefore it is possible to select α sufficiently small so that $\beta \cdot (Av_i + Bu_i + a) < 0$ for all $v_i \in V \setminus V_{\mathcal{O}}$. We conclude that for all $v_i \in V$, $\beta \cdot y(v_i) < 0$.

Now let $x \in \mathcal{P}$, and without loss of generality, suppose $x = \sum_{i=1}^{k} \lambda_i v_i$, where λ_i are the barycentric coordinates of x such that $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since $y(x)$ is affine on simplices of $\mathbb{T}$, we have $y(x) = \sum_{i=1}^{k} \lambda_i y(v_i)$. Therefore, for $x \in \mathcal{P}$, $\beta \cdot y(x) = \sum_{i=1}^{k} \lambda_i \beta \cdot y(v_i) < 0$. Since $\mathcal{P}$ is compact, by a standard argument all trajectories exit $\mathcal{P}$, and by the invariance conditions, they do so through $\mathcal{F}_0$. Thus, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback. ∎

Remark 4.1: Lemmas 3.3 and 3.4 provide sufficient geometric conditions for condition (b) of Theorem 4.3. These provide the analog to the results for simplices appearing in [2].

Finally, we consider the general case when $\mathcal{O}_{\mathcal{P}} \cap \mathcal{P}^{\circ} \neq \emptyset$. To illustrate the approach we study only single-input systems. Starting from Theorem 3.5, we create a monotonic flow by "pushing" the vector $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ onto each of the vertices of $\mathcal{P}$ while preserving the invariance conditions. We show that if MRCP is solvable, then it is solvable by this b-extremal solution. This then leads to a design procedure for constructing the appropriate controls.

Let $y \in \mathbb{R}^n$ and define the index set

$$I_y := \{i \in I \mid y \in \mathcal{C}(v_i)\}\,.$$

That is, I_y is the index set of vertices for which the velocity vector y satisfies the invariance conditions of that vertex. By Theorem 3.5, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$ is a necessary condition for solvability of RCP when $m = 1$, so we assume we have such a $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$. For the indices $i \notin I_b$, let $\overline{u}_i$ be such that $\overline{y}_i := Av_i + B\overline{u}_i + a \in \mathcal{C}(v_i)$ contains the maximal b component. Since $b \notin \mathcal{C}(v_i)$ and $m = 1$, the maximum exists and is unique, and it corresponds to one or more invariance conditions evaluating to zero at v_i. Given a triangulation $\mathbb{T}$ of $\mathcal{P}$, let $\overline{u}(x)$ denote any PWA feedback such that $\overline{u}(v_i) = \overline{u}_i$, $i \notin I_b$.

Proposition 4.4: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_{\mathcal{P}} \neq \emptyset$. Suppose $\mathbb{T}$ is a triangulation and $u(x)$ an associated PWA control such that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $u(x)$. Then there exists $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ and $\overline{u}(x)$ as above such that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $\overline{u}(x)$.

Corollary 4.5: Let $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ and let $\overline{u}_i$, $i \notin I_b$ be defined as above. Also let $\overline{y}_i := Av_i + B\overline{u}_i + a$, $i \notin I_b$. If for every $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$, $0 \in \text{co}\ \{\overline{y}_i \mid i \notin I_b\}$, then MRCP is not solvable by PWA feedback.

Proposition 4.4 suggests a design procedure to synthesize a PWA control $\overline{u}(x)$ to achieve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. The procedure is simply to inject the largest possible $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ component in any vertex with $i \notin I_b$, and to use a sufficiently large b component for vertices with $i \in I_b$. An example of the procedure is given in Section VI.

V. Arbitrary Triangulation and Monotonic Reach Control

In the previous section we studied MRCP and we saw that the effect of the flow condition is to allow a solution that does not depend on the choice of triangulation. This is a useful feature if the triangulation is performed by a standalone software not adapted to control problems. An intuition emerges that the role of the flow condition is precisely to provide this invariance to triangulation. In this section we explore the extent to which this intuition is correct. For this we formulate a version of RCP under arbitrary triangulations. By *arbitrary triangulation* of $\mathcal{P}$ we mean any triangulation of $\mathcal{P}$ with the property that if $v \in \mathcal{P}$ is a vertex of a simplex belonging to $\mathbb{T}$, then v is a vertex of $\mathcal{P}$. We show that MRCP by PWA feedback and RCP under arbitrary triangulations are equivalent in a generic sense.

Problem 5.1 (RCP by Arbitrary Triangulations): Consider the system (1) defined on $\mathcal{P}$. Find a control assignment u_i, $i \in I$, such that for an arbitrary triangulation $\mathbb{T}$ of $\mathcal{P}$, the associated PWA feedback $u(x)$ with $u(v_i) = u_i$, $i \in I$, achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$.

A set of $p > n$ points in $\mathbb{R}^n$ are in *general position* if no $n+1$ of them lie in a common affine hyperplane. The convex hull of any set of points in general position in $\mathbb{R}^n$ is called a *generic polytope*. For a generic polytope, all (proper) faces are simplices. A polytope whose faces are simplices is called a *simplicial polytope* [11].

Theorem 5.1: Consider the system (1) defined on a generic polytope $\mathcal{P}$. MRCP is solvable by PWA feedback if and only if RCP by arbitrary triangulations is solvable.

Proof: ($\Longrightarrow$) By the same argument as at the end of the proof of Theorem 4.3, for any choice of triangulation and associated PWA feedback, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$.

($\Longleftarrow$) Suppose u_i, $i \in I$, is a control assignment such that for any triangulation $\mathbb{T}$ of $\mathcal{P}$, the associated PWA feedback $u(x)$ with $u(v_i) = u_i$, achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$. Let $y_i := Av_i + Bu_i + a$, $i \in I$. We claim $0 \notin \text{co}\,\{y_1, \ldots, y_p\}$. Suppose not. By Caratheodory's Theorem and w.l.o.g. there exist $\alpha_1, \ldots, \alpha_k$ with $1 \leq k \leq n+1$ such that $0 = \sum_{i=1}^{k} \alpha_i y_i$ with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Let $\overline{x} = \sum_{i=1}^{k} \alpha_i v_i \in \mathcal{P}$. Since $\{v_1, \ldots, v_k\}$ are in general position, one can apply the placing triangulation [7] to the ordered point set $V = \{v_1, \ldots, v_p\}$ such that $\mathcal{S} := \text{co}\,\{v_1, \ldots, v_k\}$ is a simplex of the resulting triangulation $\mathbb{T}$. Let $u(x)$ be the PWA feedback associated with $\mathbb{T}$ such that $u(v_i) = u_i$. Since $u(x)$ is affine on $\mathcal{S}$, $A\overline{x} + Bu(\overline{x}) + a = \sum_{i=1}^{k} \alpha_i \big(Av_i + Bu(v_i) + a\big) = \sum_{i=1}^{k} \alpha_i y_i = 0$. That is $\overline{x}$ is an equilibrium of the closed-loop system, so RCP is not solved, a contradiction. We conclude $0 \notin \text{co}\,\{y_1, \ldots, y_p\}$. By a standard argument of convex analysis, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y_i < 0$, $i \in I$. By the same argument as at the end of the proof of Theorem 4.3, we have $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. ■

VI. Examples

We give several examples motivating why new research is needed on reach control for polytopes and illustrating some of the findings of the paper.

Example 6.1: In the first example we show that using simplex-based methods for reach control, RCP is solvable for one triangulation but not for another. However, MRCP is solvable using any triangulation, thereby illustrating Theorem 5.1. Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u\,.$$

The polytope is shown in Figure 2. We find $\mathcal{O} = \{x \mid -3x_1 + 2x_2 = 1\}$, depicted as a dashed line in Figure 2. It can be easily verified that $\mathcal{O}_{\mathcal{P}} = \text{co}\,\{o_1, o_2\}$, where $o_1 = (0, 1/2)$ and $o_2 = (1/3, 1)$. The control objective is to achieve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by PWA feedback using existing simplex methods in the literature. Suppose we triangulate $\mathcal{P}$ as in Figure 2(a). Then the control objective splits as $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback. We study the invariance conditions of $\mathcal{S}_1$ at v_4. We have $Av_4 + a = (1, 0)$, so for any choice of control the invariance conditions of $\mathcal{S}_1$ are always violated at v_4. Consequently, $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ is not achievable. Instead, suppose we triangulate $\mathcal{P}$ as in Figure 2(b). The control objective is again $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback. We choose the control values at the vertices to be $u_1 = 4$, $u_2 = 0$, $u_3 = 0$, $u_4 = 2$. The corresponding velocity vector, y_i, at each vertex v_i is shown in Figure 2(b). Based on these selected control values at the vertices, one

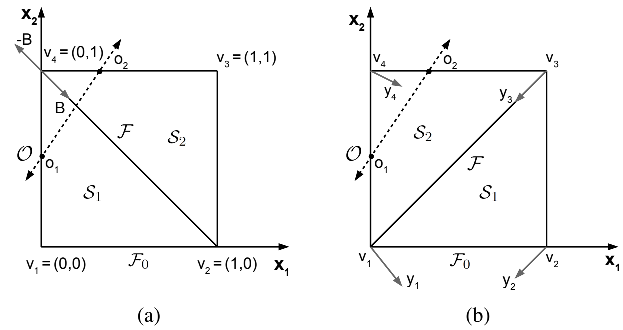

Fig. 2. Two triangulations of $\mathcal{P}$ for Example 6.1.

can construct a PWA feedback such that the control objective based on existing simplex methods is achieved [5].

Now we show MRCP is solvable. We choose the same control values at the vertices as above. Let $\xi = (0, 1)$. It can be verified that $\xi \cdot (Av_i + Bu_i + a) < 0$, for $i = 1, \ldots, 4$. Triangulate $\mathcal{P}$ using any triangulation $\mathbb{T}$ and construct the associated PWA feedback $u(x)$ based on the control values at the vertices [5]. For instance, if the second triangulation shown in Figure 2(b) is selected, then we obtain the following PWA control law:

$$u(x) = \begin{cases} \begin{bmatrix} -4 & 0 \end{bmatrix} x + 4\,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} -2 & -2 \end{bmatrix} x + 4\,, & x \in \mathcal{S}_2\,. \end{cases}$$

Note that generally the two control laws will not be the same. Since the invariance conditions of $\mathcal{P}$ are satisfied, we get $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. Notice the result holds even if the first triangulation were selected.

Example 6.2: In the previous example simplex methods could be used to solve RCP, although there was an advantage to the solution via MRCP, since it was valid for any triangulation. Now we consider an example where simplex methods fail for any choice of triangulation, but MRCP is solvable. Consider the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -3 & -2 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u\,.$$

The polytope is shown in Figure 3. The vertices of $\mathcal{P}$ are: $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 1)$, $v_4 = (0, 0, 0)$, and $v_5 = (0, 1, 0)$. First, we check if the problem is solvable using simplex methods. There are two possible triangulations of $\mathcal{P}$, shown in Figure 3. For the first triangulation the control objective is $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F} = \mathcal{S}_1 \cap \mathcal{S}_2$ by affine feedback. We examine the invariance conditions of $\mathcal{S}_1$ at v_4. We have $Av_4 + Bu_4 + a = (0, 1, u_4)$. The normal vectors to facets $\mathcal{F}_3$ and $\mathcal{F}_1$ in $\mathcal{S}_1$ are $h_3 = (0, 0, -1)$, and $h_1 = (-0.5774, 0.5774, 0.5774)$ respectively. The invariance conditions of $\mathcal{S}_1$ at v_4 yield $h_3 \cdot (Av_4 + Bu_4 + a) \leq 0$ and $h_1 \cdot (Av_4 + Bu_4 + a) \leq 0$. That is, $u_4 \geq 0$ and $u_4 \leq -1$. Thus, RCP is not solvable by simplex methods using this triangulation. Now we try the second triangulation in Figure 3. We examine the invariance conditions of $\mathcal{S}_1$ at v_3. In this case, we have $Av_3 + Bu_3 +$

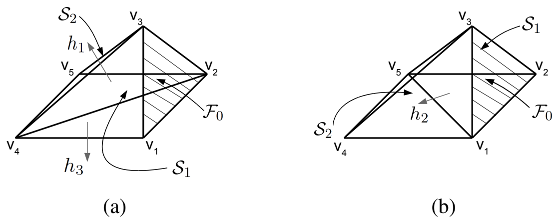

Fig. 3. Two triangulations of $\mathcal{P}$ for Example 6.2.

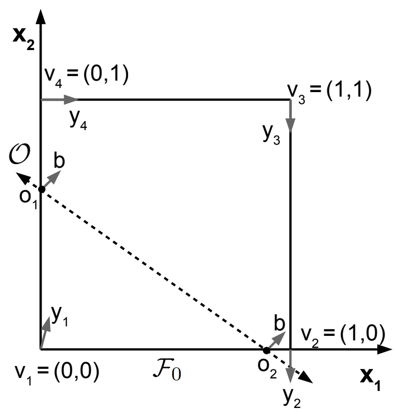

Fig. 4. Polytope for Example 6.3

$a = (-1, 0, -1 + u_3)$, $h_2 = (-0.7071, -0.7071, 0)$, and $h_2 \cdot (Av_3 + Bu_3 + a) > 0$ for all $u_3 \in \mathbb{R}$. Again, RCP is not solvable by simplex methods using this triangulation.

Now we study solvability of MRCP. Select control values $u_1 = 0$, $u_2 = 0$, $u_3 = -20$, $u_4 = 0$, and $u_5 = 0$. It can be verified they satisfy the invariance conditions of $\mathcal{P}$. Let $\xi = (-1, -0.01, 1)$. One can then verify that $\xi \cdot (Av_i + Bu_i + a) < 0$ for all $i = 1, \ldots, 5$. Triangulate $\mathcal{P}$ using any triangulation $\mathbb{T}$ and construct the associated PWA feedback $u(x)$ based on the control values at the vertices [5]. For instance, if the first triangulation in Figure 3 is selected, then we obtain the following control law:

$$u(x) = \begin{cases} \begin{bmatrix} 0 & 0 & -20 \end{bmatrix} x \,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} 0 & 0 & -20 \end{bmatrix} x \,, & x \in \mathcal{S}_2 \,. \end{cases}$$

Since the invariance conditions of $\mathcal{P}$ are satisfied, we get $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically.

Example 6.3: In this example we study a case when $\mathcal{O}_\mathcal{P} \cap \mathcal{P}^\circ \neq \emptyset$, and we use the procedure suggested by Proposition 4.4 to solve MRCP. Consider the system

$$\dot{x} = \begin{bmatrix} 2.1 & 2.5 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

The polytope is shown in Figure 4. We find $\mathcal{O} = \{x \mid 1.1x_1 + 1.5x_2 = 1\}$ depicted as a dashed line in Figure 4. Clearly, $\mathcal{P}^\circ \cap \mathcal{O} \neq \emptyset$. We find that $\mathcal{O}_\mathcal{P} = \text{co}\,\{o_1, o_2\}$, where $o_1 = (0, 2/3)$ and $o_2 = (0.90909, 0)$. It can be verified that the invariance conditions are solvable at the vertices of $\mathcal{P}$. Also, as seen in the figure, $b := (1,1) \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$. Our procedure to solve MRCP is to push a maximal amount of b at the vertices v_i, $i \notin I_b$. We have $I_b = \{1\}$, and the extremal control values are $\overline{u}(v_2) = -2.1$, $\overline{u}(v_3) = -4.6$, $\overline{u}(v_4) = -2$. As seen in the figure, certain invariance conditions evaluate to zero at these vertices. Next we must select $\overline{u}(v_1)$ with a b component sufficiently large such that a flow condition on $\mathcal{P}$ is obtained. If we choose $\overline{u}(v_1) = 0.0075$ then the invariance conditions hold at v_1 and moreover, a flow condition based on $\xi = (-2.01, 0.01)$ can be verified to hold at all vertices. Finally, we form a triangulation $\mathbb{T}$ of $\mathcal{P}$ consisting of two simplices $\mathcal{S}_1 = \text{co}\,\{v_1, v_2, v_3\}$ and $\mathcal{S}_2 = \text{co}\,\{v_1, v_3, v_4\}$. The piecewise affine feedback is

$$u(x) = \begin{cases} \begin{bmatrix} -2.1075 & -2.5 \end{bmatrix} x + 0.0075 \,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} -2.6 & -2.0075 \end{bmatrix} x + 0.0075 \,, & x \in \mathcal{S}_2 \,. \end{cases}$$

The example shows that the method of pushing b works even if b does not point to the exit facet $\mathcal{F}_0$. Also, we did not need to push a large amount of b at v_1. It turns out that a small push ($c_1 b,\ c_1 = 0.0075$) is enough to construct a flow condition on $\mathcal{P}$. This small push is important. If we select $c_1 = 0$, it can be verified that $0.090909(Av_1 + B\overline{u}(v_1) + a) + 0.90909(Av_2 + B\overline{u}(v_2) + a) = 0$, so a flow condition cannot be achieved on $\mathcal{P}$.

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