Monotonic Reach Control on Polytopes

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Abstract

The paper studies the problem of making an affine system defined on a polytopic state space reach a prescribed facet of the polytope in finite time without first leaving the polytope. The focus is on solvability by continuous piecewise affine feedback, and we formulate a variant of the problem in which trajectories exit in a monotonic sense. This allows to obtain necessary and sufficient conditions for solvability in certain geometric situations. Next, we show that, generically, solvability via arbitrary triangulations is equivalent to monotonic solvability. In contrast with existing simplex-based methods, this provides an avenue for reach control on polytopes that does not depend on the choice of triangulation of the polytope.

I. INTRODUCTION

We study the reach control problem (RCP) for affine systems on polytopes. The problem is to design a feedback to force closed-loop trajectories starting anywhere in a polytopic state space to leave the polytope from a prescribed exit facet of the polytope in finite time [10], [11]. Unlike previous work, here it is not required that trajectories leave the polytope at the first time they reach the exit facet. The problem sits within a family of reachability problems for hybrid systems [3], [2], [16], [9]. A hybrid system is a dynamical system that combines both discrete event and continuous time behavior [16], [9]. In the last decade hybrid systems have received special interest for the reason that they more accurately capture the complex models and specifications faced in industry. Our interest lies on a special subclass of hybrid systems, piecewise affine hybrid systems [20], [21], [1], [2]. A piecewise affine hybrid system consists of a discrete automaton such that each discrete mode is equipped with its own continuous-time affine dynamics defined on a polytope. When the continuous state crosses a facet of a polytope,

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the system is transferred to a new discrete mode. The reachability analysis for piecewise affine hybrid systems at the continuous level reduces to studying RCP for an affine system on a polytope [12].

The most definitive results on RCP are focused on reach control on simplices by affine feedback [12], [19], [5]. Results for polytopes come in one of two forms. Either one must perform a triangulation of the polytope and apply simplex-based reach control methods [12], [19]. Alternatively, one may impose conditions so that the design can be carried out in two independent steps: first one assigns control inputs at the vertices of the polytope guaranteeing propitious closed-loop behavior; second, one selects any triangulation of the polytope and one forms a (continuous) piecewise affine feedback based on the vertex control values of step one. The distinction between these two approaches is that simplex-based methods require stronger conditions on the vector field at the vertices of the triangulation; for instance, closed-loop trajectories can only exit from one exit facet of a given simplex. Instead, the second approach imposes weaker conditions at the vertices; for example, trajectories could exit from more than one exit facet of a given simplex. The penalty for the more relaxed requirements of the second method is that trajectories may not actually achieve the specification to exit the polytope. To guarantee this, an extra, exogenous condition must be added. We study the relative merits of the two approaches, and we find both theoretically and via examples that the two methods are complementary. The investigation highlights that new research is needed to understand triangulation in control problems.

Past research on reach control on polytopes has either required strong sufficient conditions or restrictive assumptions on the system dynamics [11], [15]. This paper initiates a study of the reach control problem in which such restrictions are removed; instead geometric properties of the system are exploited to the best possible extent. In particular, the placement of $\mathcal{O}$, the set of possible equilibria, relative to the polytope $\mathcal{P}$ plays a key role, and in certain cases, clear necessary and sufficient conditions can be obtained which remove the conservativism or restrictiveness of previous work. We then formulate the *monotonic reach control problem* (MRCP) where it is required that trajectories exit the polytopic state space in a monotonic sense relative to a foliation of parallel hyperplanes. This notion of monotonic solvability is shown to be generically equivalent to solvability of RCP by piecewise affine feedback using any choice of triangulation of $\mathcal{P}$. The latter is particularly useful when triangulation is performed by a standalone software not adapted to control problems.

The paper is organized as follows. In the next section we bring into focus the contributions of the paper relative to the literature. In Section III we review the reach control problem on polytopes, particularly summarizing preliminary definitions and results. Section IV includes necessary conditions for RCP on polytopes. In Section V we explore the extent to which existing results for simplices carry over to polytopes. In Section VI we formulate the monotonic reach control problem (MRCP) on polytopes. Section VII shows the relationship between MRCP and solvability of RCP by arbitrary triangulation. In section VIII we present an algorithm for the solvability of MRCP by continuous piecewise affine (PWA) feedback. In Section IX several examples are given illustrating the findings of the paper.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, and the interior is $\mathcal{K}^\circ$. The symbol $\mathbb{U}$ represents a control class such as open-loop controls, continuous state feedback, affine feedback, or piecewise affine feedback. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\mathscr{B}$ denotes the open ball of radius 1 centered at the origin. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. Finally, $T_{\mathcal{P}}(x)$ denotes the Bouligand tangent cone to set $\mathcal{P} \subset \mathbb{R}^n$ at point x.

II. CONTRIBUTIONS

The paper develops control methods for reach control on polytopes. Existing results have focused on PWA feedbacks [11], [12], [15] because of their natural fit with polytopes (any polytope can be triangulated) and because using PWA feedbacks allows to build on the results for simplices using affine feedbacks [12], [19]. More recently there has been a quest to find the largest class of feedbacks to solve RCP [15], [5]. This paper extends these results along several lines. First, [15] aims to find a feedback class that solves RCP whenever open-loop controls do. To carry out this program, one must first identify necessary conditions for solvability by open-loop controls. In the literature necessary conditions are available only for continuous state feedbacks [11], or for open-loop controls, but only restrictive classes of systems [15]. We present three necessary conditions. First, the so-called invariance conditions which guarantee that trajectories do not leave $\mathcal{P}$ from non-exit facets are necessary for solvability by continuous state feedback, but we show by way of a counterexample that this is not so for polytopes; also, we clarify when the invariance conditions are necessary for solvability of RCP on polytopes.

Second, we present a geometric necessary condition for solvability of RCP by continuous state feedbacks associated with circumventing equilibria at the vertices. Finally, the paper presents a third necessary condition regarding a topological obstruction to reach control by continuous state feedbacks. We present the result only for single input systems; the extension to multi-input systems is difficult and is an open area of research.

Armed with a set of necessary conditions, we begin a study of RCP following the point of view of [5]. The invariance conditions are taken as a necessary first computational step for solving RCP by continuous state feedback. Then we ask, given a continuous PWA feedback that satisfies the invariance conditions, what additional conditions are needed to guarantee that either (i) there is no closed-loop equilibrium in the polytope, or (ii) closed-loop trajectories exit the polytope? Whereas for simplices the two questions are the same [12], [19], for polytopes they are distinct. In particular, the statement “if there are no closed-loop equilibria in P, then trajectories exit P” is not known to hold for polytopes. The reason is that equivalence of (i) and (ii) relies on the convexity of the closed-loop vector field. Convexity is preserved with affine feedback (on simplices) but not PWA feedback (on polytopes).

The next contribution of the paper is to formulate a restricted version of RCP called the Monotonic Reach Control Problem (MRCP). It explicitly incorporates a flow condition in the problem statement. We study three geometrically distinct cases based on the position of $\mathcal{O}$: $\mathcal{O} \cap \mathcal{P} = \emptyset$, $\mathcal{O}$ is a face of $\mathcal{P}$, and $\mathcal{O}$ intersects the interior of $\mathcal{P}$. In the first two cases, we completely solve the problem, giving necessary and sufficient conditions. The third case is considerably more difficult, as also illustrated via examples. For this case we focus on single input systems. We propose a method for solving this problem using a b-extremal controller, where b is a control direction of the single input system. The b-extremal solution of MRCP is amenable to an algorithmic solution, and this is the next contribution of the paper. Its soundness is also proved. The most closely related result to our algorithm is the procedure contained in Theorem 4.17 of [12]. Their algorithm is for simplices and multi-input systems, while our algorithm is for polytopes and single-input systems. Comparing only the single-input cases, their algorithm requires solving at most $2^{(n+1)} + 2(n+1)$ linear programming (LP) problems, where n is the dimension of the simplex. Instead, our algorithm requires solving at most $2p+2$ LP problems, where p is the number of vertices of $\mathcal{P}$.

The next contribution of the paper is in the area of triangulations. There are two possible strategies regarding triangulation. Either one constructs a triangulation that has been specially tailored to the control problem [15]. Alternatively, one selects an arbitrary triangulation [11] (or it is selected by a computer program) and one hopes for the best. Tacit to this discussion is that an instance of RCP may be solved with one choice of triangulation but not another. This fact is illustrated via examples. We conjecture and then prove that a beneficial artifact of a flow condition is precisely to allow arbitrary triangulations, and more significantly that MRCP is no loss of generality if invariance to the choice of triangulation is a requirement in the design. Such results on the relationship between triangulation and the outcome of a control design are inexistent in the control literature, despite the attention to PWA systems.

The final contribution of the paper is via examples. It is currently not well understood why RCP may fail on polytopes: either because an improper triangulation is chosen, because simplex methods fail, because existence of a flow condition fails, or because a topological obstruction appears, among other issues. We catalog all these failures by way of examples, and importantly we show that the proposed solutions to these failures are complementary. MRCP may work when simplex methods fail (Example 9.2). Conversely, simplex methods may work when MRCP fails (Example 4.1), and so forth. In sum, the paper presents a panoply of techniques and ideas for tackling RCP on polytopes.

III. REACH CONTROL PROBLEM

Consider an n-dimensional polytope

$$\mathcal{P} := \text{co}\ \{v_1, \dots, v_p\}$$

with vertex set $V := \{v_1, \dots, v_p\}$ and facets $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_r$. The *target set* is the facet $\mathcal{F}_0$ of $\mathcal{P}$. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \dots, p\}$, $J = \{1, \dots, r\}$, and $J(x) = \{j \in J \mid x \in \mathcal{F}_j\}$. For each $x \in \mathcal{P}$, define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \ j \in J(x) \ \right\}.$$

(Note that h_0 never appears in $J(x)$ since $\mathcal{F}_0$ is the target set). We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a\,, \qquad\qquad x \in \mathcal{P}\,, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\mathrm{rank}(B) = m$. Let $\phi_u(t, x_0)$ be the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the target set $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 3.1 (Reach Control Problem (RCP)): Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{P}$ reach and exit the target $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. Notice that in order for condition (i) to make sense it is assumed that the dynamics (1) are extended to a small neighborhood of $\mathcal{P}$.

Definition 3.1: A point $x_0 \in \mathcal{P}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{P}$ with control class $\mathbb{U}$, denoted by $x_0 \xrightarrow{\mathcal{P}} \mathcal{F}_0$, if there exists a control u of class $\mathbb{U}$ such that property (i) of Problem 3.1 holds. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by control class $\mathbb{U}$ if for every $x_0 \in \mathcal{P}$, $x_0 \xrightarrow{\mathcal{P}} \mathcal{F}_0$ with control of class $\mathbb{U}$.

Definition 3.2: We say the *invariance conditions are solvable* if for each $v \in V$ there exists $u \in \mathbb{R}^m$ such that

$$Av + Bu + a \in \mathcal{C}(v)\,. \tag{2a}$$

Equivalently,

$$h_j \cdot (Av + Bu + a) \leq 0\,, \qquad j \in J(v)\,. \tag{2b}$$

Equation (2a) or (2b) is referred to as the *invariance conditions* either for a specific vertex, or collecting all conditions for all vertices, for a polytope. The relevance of the invariance conditions to RCP is that they ensure that trajectories only exit $\mathcal{P}$ via $\mathcal{F}_0$.

Lemma 3.1 ([11]): Consider the system (1) on $\mathcal{P}$ and let $u(x)$ be a continuous state feedback on $\mathcal{P}$ such that the closed-loop system has unique solutions. Suppose that for some facet $\mathcal{F}_i$ of $\mathcal{P}$ the following conditions hold:

$$h_i \cdot (Ax + Bu(x) + a) \leq 0, \qquad \forall x \in \mathcal{F}_i\,. \tag{3}$$

Then all trajectories originating in $\mathcal{P}$ that leave $\mathcal{P}$ do so via a facet $\mathcal{F}_j$, $j \neq i$.

Let $\mathcal{B} = \text{Im } B$, the image of B. Define the set

$$\mathcal{O} := \{ \, x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B} \, \} \, . \tag{4}$$

Notice that the vector field $Ax + Bu + a$ can vanish at any $x \in \mathcal{O}$ for an appropriate choice of $u \in \mathbb{R}^m$, so $\mathcal{O}$ is the set of all possible equilibrium points of (1). That is, if x_0 is an equilibrium of (1) under feedback control, then $x_0 \in \mathcal{O}$. It can be verified that either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ is an affine space with dimension between m and n. We also define the set of possible equilibrium points of (1) on $\mathcal{P}$ by

$$\mathcal{O}_{\mathcal{P}} := \mathcal{P} \cap \mathcal{O} \, . \tag{5}$$

Since $\mathcal{O}$ is an affine space, either $\mathcal{O}_{\mathcal{P}} = \emptyset$ or $\mathcal{O}_{\mathcal{P}}$ is a κ-dimensional polytope in $\mathcal{P}$. If $\mathcal{O}_{\mathcal{P}} \neq \emptyset$, we define the vertex set of $\mathcal{O}_{\mathcal{P}}$ to be $V_{\mathcal{O}} = \{o_1, \ldots, o_q\}$, where o_i are the vertices of $\mathcal{O}_{\mathcal{P}}$ (not necessarily vertices of $\mathcal{P}$). Also define the index set $I_{\mathcal{O}} = \{1, \ldots, q\}$.

IV. NECESSARY CONDITIONS FOR RCP ON POLYTOPES

In this section we explore necessary conditions for solvability of RCP on polytopes under various classes of controls. Particularly, open-loop controls have received relatively less attention than continuous state feedbacks. First, in [11] it was shown that the invariance conditions (2) are necessary for solvability of RCP on polytopes by continuous state feedback. Here we study if the invariance conditions are necessary for solvability by open-loop controls; the main result appears in Theorem 4.1. Second, we expose a necessary condition, stated in Theorem 4.2, associated with removal of equilibria on vertices. This condition arises directly from the geometric relationship between the space $\mathcal{B}$ and the tangent cones at vertices of $\mathcal{O}_{\mathcal{P}}$. Finally, we study a topological necessary condition, stated in Theorem 4.3, associated with existence of equilibria using continuous state feedback.

We say that a function $\mu : [0, \infty) \to \mathbb{R}^m$ is an *open-loop control* if it is bounded on any compact interval and it is measurable. By Caratheodory's theorem [8] solutions of (1) using open-loop controls exist and are unique.

Despite the positive results reported in [11] for continuous state feedbacks, unfortunately, when one works with general polytopes and open-loop controls, the invariance conditions are

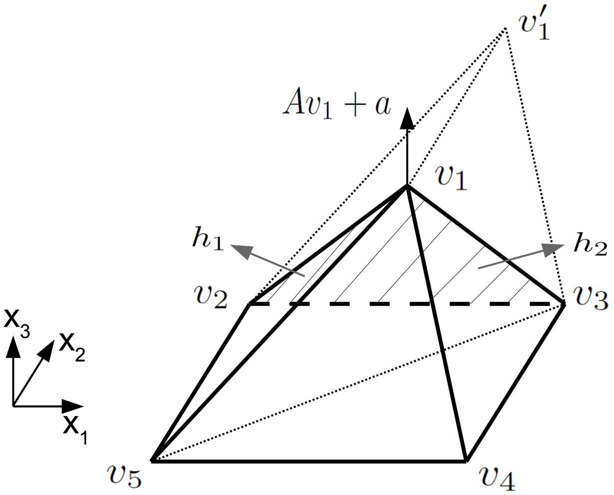


Fig. 1. The invariance conditions are not solvable but RCP is solvable by open-loop controls.

no longer necessary for solvability of RCP. The following counterexample suggested by Zhiyun Lin illustrates this fact.

Example 4.1: Consider the polytope $\mathcal{P} = \text{co } \{v_1, \cdots, v_5\}$ shown in Figure 1. The exit facet is $\mathcal{F}_0 = \text{co } \{v_1, v_2, v_3\}$, depicted as a hatched region in the figure. Suppose that the two hyperplanes containing the facet $\mathcal{F}_1 = \text{co } \{v_1, v_2, v_5\}$ and the facet $\mathcal{F}_2 = \text{co } \{v_1, v_3, v_4\}$ intersect at the line through v_1 and v_1', and this line is parallel to the horizontal hyperplane containing co $\{v_2, v_3, v_4, v_5\}$. For the system, we suppose $\dim(\mathcal{B}) = 2$, $\mathcal{B}$ is parallel to the horizontal hyperplane containing co $\{v_2, v_3, v_4, v_5\}$, (A, B) is controllable, $\mathcal{P} \cap \mathcal{O} = \emptyset$, and $Av_1 + a$ has a strictly positive x_3 component, as shown in the figure. Now we show that for the polytope and the system described above, RCP is solvable by open-loop controls, even if the invariance conditions are not solvable.

First, we show that the invariance conditions are not solvable at v_1. Let $y_1 := Av_1 + a + Bu_1$. Since $Av_1 + a$ has a strictly positive x_3 component, $\mathcal{B}$ is horizontal, and $v_1 \notin \mathcal{O}$, for any choice $u_1 \in \mathbb{R}^2$, y_1 has a strictly positive x_3 component. On the other hand, to achieve the invariance conditions at v_1, y_1 must lie in the closed cone $\{y \in \mathbb{R}^3 \mid h_1 \cdot y \le 0,\ h_2 \cdot y \le 0,\ h_3 \cdot y \le 0\}$, where h_3 is the outward unit normal vector of $\mathcal{F}_3 = \text{co}\ \{v_1, v_4, v_5\}$. In particular, to satisfy the invariance conditions of $\mathcal{F}_1$ and $\mathcal{F}_2$ simultaneously, y_1 must have a non-positive x_3 component.

Second, we show there exist open-loop controls solving RCP on $\mathcal{P}$ by invoking Theorem 6 of [15]. The conditions to apply this theorem are that (i) (A, B) is controllable; (ii) $m = n - 1$; (iii) if $\mathcal{P} \cap \mathcal{O} \neq \emptyset$, then $\mathcal{P} \cap \mathcal{O}$ is a κ-dimensional face of $\mathcal{P}$; and (iv) certain sets denoted $\mathcal{A}^-$ and $\mathcal{A}^+$ describing states that cannot reach $\mathcal{F}_0$ by open-loop controls are empty. In our case, conditions (i) and (ii) hold by assumption, and (iii) holds trivially since $\mathcal{P} \cap \mathcal{O} = \emptyset$. Only (iv) requires a small computation aided by the fact that $\mathcal{P} \cap \mathcal{O} = \emptyset$. We omit the details. We conclude $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by open-loop controls, from which Theorem 9 of [15] gives $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by discontinuous piecewise affine feedback.

To make the previous discussion more concrete, consider the system

$$\dot{x} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ \frac{1}{10} & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u\,.$$

Polytope $\mathcal{P}$ is as in Figure 1 with $v_1 = (1/2, 1, 1)$, $v_2 = (0, 1, 0)$, $v_3 = (1, 1, 0)$, $v_4 = (1, 0, 0)$, and $v_5 = (0, 0, 0)$. It can be verified that the system and polytope satisfy all the conditions described above, including that the invariance conditions of $\mathcal{P}$ are not solvable at v_1. Now we follow the procedure in Section 4 of [15] to construct a discontinuous piecewise affine feedback solving RCP. First, using the Basic triangulation Algorithm of [15], we triangulate $\mathcal{P}$ into $\mathcal{S}_1 = \text{co}\ \{v_1, v_2, v_3, v_5\}$ and $\mathcal{S}_2 = \text{co}\ \{v_1, v_3, v_4, v_5\}$ as shown in Figure 1. Let $\mathcal{F} := \mathcal{S}_1 \cap \mathcal{S}_2 = \text{co}\ \{v_1, v_3, v_5\}$. We split the control objective as $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback and $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback. For simplices, RCP is solvable by affine feedback if and only if the invariance conditions of the simplex are solvable and there is no closed-loop equilibrium in the simplex [12], [19]. Since $\mathcal{P} \cap \mathcal{O} = \emptyset$, neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ can have equilibria, so we must only construct affine feedbacks satisfying the invariance conditions of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. For the vertices of $\mathcal{S}_2$ we select control values $u_1 = (-5, 10)$, $u_3 = (-12, 10)$, $u_4 = (-12, 12)$, and $u_5 = (5, 12)$. Then we construct the unique affine feedback $u(x)$ on $\mathcal{S}_2$ satisfying $u(v_i) = u_i$, $v_i \in \mathcal{S}_2$ [11]. Similarly, we construct the affine feedback on $\mathcal{S}_1$ that achieves $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$. We conclude by Theorem 9 of [15] that the following discontinuous piecewise affine feedback solves

RCP on $\mathcal{P}$.

$$u(x) = \begin{cases} \begin{bmatrix} -22 & 5 & 6 \\ 0 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 5 \\ 12 \end{bmatrix}, & x \in \mathcal{S}_1 \\ \begin{bmatrix} -17 & 0 & -\frac{3}{2} \\ 0 & -2 & 0 \end{bmatrix} x + \begin{bmatrix} 5 \\ 12 \end{bmatrix}, & x \in \mathcal{S}_2 \setminus \mathcal{S}_1 \,. \end{cases}$$

This feedback has a discontinuity along $\mathcal{F}$, but it does not have sliding modes. Finally, we note that on $\mathcal{F}$ the controller for $\mathcal{S}_1$ is selected because $\mathcal{S}_1$ has the shortest path to reach $\mathcal{F}_0$ (see step $3(c)$ of Algorithm 2 in [14]).

To overcome the obstacle identified in the previous example, we identify a suitable class of polytopes for which the invariance conditions remain necessary conditions. To that end, an n-dimensional polytope $\mathcal{P}$ is said to be *simple* if each k-dimensional face of $\mathcal{P}$ is contained in exactly $n-k$ facets.

Remark 4.1: If $\mathcal{P}$ is a simple polytope, then $\mathcal{P}$ has the following properties [4]:

(i) Each vertex of $\mathcal{P}$ is contained in exactly n edges.

(ii) Let $\mathcal{F}$ be a facet of $\mathcal{P}$ and v a vertex of $\mathcal{P}$ in $\mathcal{F}$. Then there are exactly $n-1$ edges in $\mathcal{F}$ containing v.

Theorem 4.1: Let $\mathcal{P}$ be a simple polytope. If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by a open-loop controls, then the invariance conditions (2) are solvable.

Proof: Define $\mathcal{Y}(x) := \big\{Ax + Bw + a \mid w \in \mathbb{R}^m\big\}$. Let $x_0 \in \mathcal{P} \setminus \mathcal{F}_0$. By assumption there exists $\mu(t)$ and a time $T > 0$ such that $\phi_\mu(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$. Since $\mu(t)$ is an open-loop control, there exists $c > 0$ such that $\|\mu(t)\| \leq c$, for all $t \in [0, T]$. Consider the set $\mathcal{Y}_c(x) := \big\{Ax + Bw + a \mid w \in \mathbb{R}^m, \|w\| \leq c\big\}$. It is easy to show that both $x \mapsto \mathcal{Y}_c(x)$ and $x \mapsto \mathcal{Y}(x)$ are upper semicontinuous. Now take a sequence $\{t_i \mid t_i \in (0, T]\}$ with $t_i \to 0$. Since $\{y \in \mathcal{Y}_c(x) \mid x \in \mathbb{R}^n\}$ is bounded on $\mathcal{P}$, there exists $M > 0$ such that $\|\phi_\mu(t_i, x_0) - x_0\| \leq M t_i$. Therefore $\{\frac{\phi_\mu(t_i, x_0) - x_0}{t_i}\}$ is a bounded sequence and there exists a convergent subsequence (with indices relabeled) such that $\lim_{i \to \infty} \frac{\phi_\mu(t_i, x_0) - x_0}{t_i} =: v$. Since $\phi_\mu(t_i, x_0) \in \mathcal{P}$, by the definition of the Bouligand tangent cone, $v \in T_{\mathcal{P}}(x_0)$ [7]. Now we show $v \in \mathcal{Y}(x_0)$.

We have

$$\frac{\phi_\mu(t_i, x_0) - x_0}{t_i} = \frac{1}{t_i} \int_0^{t_i} \big[A\phi_\mu(\tau, x_0) + B\mu(\tau) + a\big] d\tau \,. \tag{6}$$

Let $y_0 := Ax_0 + B\mu(0) + a \in \mathcal{Y}(x_0)$ and $y(\tau) := A\phi_\mu(\tau, x_0) + B\mu(\tau) + a \in \mathcal{Y}(\phi_\mu(\tau, x_0))$.

By the upper semicontinuity of $x \mapsto \mathcal{Y}(x)$, given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x_0 - \phi_\mu(\tau, x_0)\| < \delta$, then $\|y_0 - y(\tau)\| < \epsilon$. This implies, for i sufficiently large and for all $\tau \in [0, t_i]$, $\|y_0 - y(\tau)\| < \epsilon$. This can be rewritten as: for i sufficiently large and $\forall \tau \in [0, t_i]$, $y(\tau) = y_0 + p(\tau)$ for some function $p(\tau)$ satisfying $\|p(\tau)\| < \epsilon$. Thus, for i sufficiently large $\frac{1}{t_i}\int_0^{t_i} y(\tau)d\tau = y_0 + \frac{1}{t_i}\int_0^{t_i} p(\tau)d\tau$. However $\left\|\frac{1}{t_i}\int_0^{t_i} p(\tau)d\tau\right\| \leq \frac{1}{t_i}\int_0^{t_i}\|p(\tau)\|d\tau < \epsilon$. We conclude, for i sufficiently large, $\frac{1}{t_i}\int_0^{t_i}\big[A\phi_\mu(\tau, x_0) + B\mu(\tau) + a\big]d\tau \in \mathcal{Y}(x_0) + \epsilon\mathscr{B}$. Using (6), for i sufficiently large we have $\frac{\phi_\mu(t_i, x_0) - x_0}{t_i} \in \mathcal{Y}(x_0) + \epsilon\mathscr{B}$. Since $\mathcal{Y}(x_0)$ is a closed subset of $\mathbb{R}^n$, $v \in \mathcal{Y}(x_0) + \epsilon\overline{\mathscr{B}}$, and since ϵ is arbitrary, $v \in \mathcal{Y}(x_0)$. We conclude that $v \in \mathcal{Y}(x_0) \cap T_{\mathcal{P}}(x_0) \neq \emptyset$, $x_0 \in \mathcal{P} \setminus \mathcal{F}_0$. Since $T_{\mathcal{P}}(x_0) = \mathcal{C}(x_0)$, $x_0 \in \mathcal{P} \setminus \mathcal{F}_0$, it follows that the invariance conditions are solvable at $x_0 \in \mathcal{P} \setminus \mathcal{F}_0$.

Now consider $v_i \in \mathcal{F}_0$. If $v_i \in \mathcal{O}$, then the invariance conditions are solvable by selecting $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = 0$. Instead suppose $v_i \notin \mathcal{O}$. Suppose by the way of contradiction that $\mathcal{Y}(v_i) \cap \mathcal{C}(v_i) = \emptyset$. Then $\mathcal{Y}(v_i)$ and $\mathcal{C}(v_i)$ are non-empty disjoint polyhedral convex sets in $\mathbb{R}^n$. By Corollary 19.3.3 in [18], they are strongly separated. So, there exists $\epsilon > 0$ such that $\inf_{y \in \mathcal{Y}(v_i), z \in \mathcal{C}(v_i)} \|y - z\| > \epsilon$. By the upper semicontinuity of $x \mapsto \mathcal{Y}(x)$, there exists $\delta > 0$ such that if $\|x - v_i\| < \delta$, then $\mathcal{Y}(x) \subset \mathcal{Y}(v_i) + \frac{\epsilon}{2}\mathscr{B}$. As $\mathcal{P}$ is a simple polytope, by Remark 4.1(i) $v_i \in \mathcal{F}_0$ is the intersection of exactly n edges, and by Remark 4.1(ii) $n-1$ of these edges are contained in $\mathcal{F}_0$. Let $\overline{v_i v_j}$ be the edge that is not contained in $\mathcal{F}_0$. By definition of a simple polytope, $\overline{v_i v_j}$ is the intersection of exactly $n-1$ facets. Since $\overline{v_i v_j}$ is not contained in $\mathcal{F}_0$, the $n-1$ facets are the restricted facets at v_i. We conclude $\forall x \in [v_i, v_j)$, $\mathcal{C}(x) = \mathcal{C}(v_i)$. Let $\bar{x} \in (v_i, v_j) \cap \{x \in \mathbb{R}^n : \|x - v_i\| < \delta\}$. We have $\mathcal{C}(\bar{x}) = \mathcal{C}(v_i)$, and $\mathcal{Y}(\bar{x}) \subset \mathcal{Y}(v_i) + \frac{\epsilon}{2}\mathscr{B}$. Therefore, $\mathcal{C}(\bar{x}) \cap \mathcal{Y}(\bar{x}) = \emptyset$, a contradiction. ■

Example 4.2: We return to Example 4.1 and identify the defect to be that $\mathcal{P}$ is not simple - vertex v_1 is contained in four facets. This makes the solvability of the invariance conditions at v_1 not necessary for solvability of RCP by open-loop controls.

We explore a second necessary condition for solvability of RCP, this time for general polytopes and for continuous state feedbacks. This rather intuitive necessary condition targets the appearance of equilibria at vertices of $\mathcal{O}_{\mathcal{P}}$. We know that equilibria in $\mathcal{P}$ can only appear in $\mathcal{O}_{\mathcal{P}}$. Moreover, for a vertex $o_i \in V_{\mathcal{O}}$ and for any $u_i \in \mathbb{R}^m$, $Ao_i + a + Bu_i \in \mathcal{B}$. That is, velocity vectors at any $o_i \in V_{\mathcal{O}}$ are only drawn from $\mathcal{B}$. From [11] it follows that if RCP is solvable by continuous state feedback, then $\mathcal{B} \cap \mathcal{C}(o_i) \neq \emptyset$ for $o_i \in V_{\mathcal{O}}$. The next result says that, moreover, the zero vector cannot be the only element of $\mathcal{B} \cap \mathcal{C}(o_i)$.

Theorem 4.2: If $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by a continuous state feedback, then $\mathcal{B} \cap \mathcal{C}(o_i) \neq \mathbf{0}$ for all $o_i \in V_{\mathcal{O}}$.

Proof: Let $u(x)$ be a continuous state feedback that achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$, and define $y(x) := Ax + a + Bu(x)$. Clearly, $y(x) \neq 0$ for all $x \in \mathcal{P}$. Suppose by the way of contradiction that for $o_i \in V_{\mathcal{O}}$, $\mathcal{B} \cap \mathcal{C}(o_i) = \mathbf{0}$. Since $y(o_i) \neq 0$ and necessarily $y(o_i) \in \mathcal{B}$, we get $y(o_i) \notin \mathcal{C}(o_i)$. That is, $y(x)$ violates the invariance conditions at o_i. By Proposition 3.1 of [11], RCP is not solved using $u(x)$, a contradiction. ∎

Finally, we introduce a third necessary condition for solvability of RCP on $\mathcal{P}$ by continuous state feedback. Define

$$\text{cone}(\mathcal{O}_{\mathcal{P}}) := \bigcap_{o \in V_{\mathcal{O}}} \mathcal{C}(o)\,.$$

In particular, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ is the cone of directions in $\mathcal{B}$ that simultaneously satisfy the union of all invariance conditions at all vertices of $\mathcal{O}_{\mathcal{P}}$.

In the following theorem it is shown that for single input systems the condition $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$ is necessary for solvability of RCP by continuous state feedback. This condition appears to provide the analogous condition for polytopes as a seemingly related condition for simplices for existence of affine feedbacks. Given a simplex $\mathcal{S}$, let $\text{cone}(\mathcal{S})$ be the tangent cone to $\mathcal{S}$ at the vertex not containing the exit facet $\mathcal{F}_0$. In [5] it is shown that if the invariance conditions are solvable and $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. For polytopes, while we will indirectly use $\mathcal{B} \cap \text{cone}(\mathcal{P}) \neq \mathbf{0}$ as a sufficient condition for solvability by PWA (see Remark 6.1), the similarity with the simplex condition is more in form and less in content. This is because the simplex condition $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$ is used as a sufficient condition for solvability of RCP by affine feedback, but it is not necessary. Instead, for polytopes, and therefore also for simplices, and for single input systems, the condition $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$ is necessary. The distinction is that while the simplex condition describes a propitious geometric situation when a control direction points into the simplex, the polytope condition is much deeper - it regards a topological obstruction to reach control by continuous state feedback. Finding the multi-input version of this condition is challenging and remains an open problem in the area.

Theorem 4.3: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_{\mathcal{P}} \neq \emptyset$. If RCP is solvable by continuous state feedback, then $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$.

Proof: Let $u(x)$ be a continuous state feedback that achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$, and define $y(x) :=$

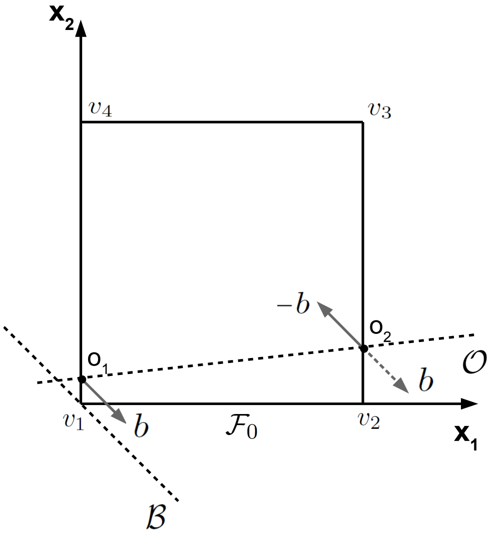


Fig. 2. $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) = \mathbf{0}$

$Ax + a + Bu(x)$. Since the invariance conditions (3) are necessary for solvability by continuous state feedback, $u(x)$ satisfies (3). Let $\mathcal{O}_\mathcal{P} = \text{co}\ \{o_1, \ldots, o_q\}$ and suppose by way of contradiction $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) = \mathbf{0}$. If $q = 1$, then $\mathcal{B} \cap \mathcal{C}(o_1) = \mathbf{0}$. This contradicts Theorem 4.2. Instead suppose $q > 1$ and w.l.o.g. $0 \neq b := Ao_1 + Bu(o_1) + a \in \mathcal{B} \cap \mathcal{C}(o_1)$. Then there exists $k \in \{2, \ldots, q\}$ such that $b \notin \mathcal{C}(o_k)$. Consider the segment $\overline{o_1 o_k}$. Since $\overline{o_1 o_k} \subset \mathcal{O}$, $y(x) \in \mathcal{B}$ for all $x \in \overline{o_1 o_k}$. Thus there exists a continuous function $c : \mathbb{R}^n \to \mathbb{R}$ such that $y(x) = c(x)b$ for $x \in \overline{o_1 o_k}$, with $c(o_1) > 0$ and $c(o_k) \leq 0$. By the Intermediate Value Theorem, there exists $x^* \in \overline{o_1 o_k} \subset \mathcal{P}$ such that $c(x^*) = 0$. The closed-loop system has an equilibrium in $\mathcal{P}$, a contradiction. ■

Example 4.3: Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -10 \\ 1 & -10 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u\,.$$

The polytope $\mathcal{P}$, shown in Figure 2, has vertices $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,1)$, and $v_4 = (0,1)$. The exit facet is $\mathcal{F}_0 = \text{co}\ \{v_1, v_2\}$. We find by direct computation that $\mathcal{O} = \{x \mid x_1 - 10x_2 = -1\}$, depicted as a dashed line in Figure 2. It can be verified that $\mathcal{O}_\mathcal{P} = \text{co}\ \{o_1, o_2\}$, where $o_1 = (0, 0.1)$ and $o_2 = (1, 0.2)$, and also the invariance conditions are solvable. The subspace $\mathcal{B}$ is shown in the figure attached at v_1. We know that in $\mathcal{O}_\mathcal{P}$ the only velocity vectors available to the closed-loop system are vectors in $\mathcal{B}$. To achieve the invariance conditions at

$o_1 \in \mathcal{O}_{\mathcal{P}}$, the velocity vector $b \in \mathcal{B}$, shown in Figure 2, must be selected. On the other hand, b violates the invariance conditions at o_2. At o_2 the velocity vector $-b \in \mathcal{B}$ must be selected to achieve the invariance conditions. It is clear that there does not exist a non-zero vector in $\mathcal{B}$ that satisfies the invariance conditions at both o_1 and o_2 simultaneously. That is, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) = \mathbf{0}$. By Theorem 4.3, RCP is not solvable by continuous state feedback.

V. FROM SIMPLICES TO POLYTOPES

It is known that for simplices, RCP is solvable by affine feedback if and only if two conditions hold: (a) the invariance conditions (2) are solvable, and (b) there is no closed-loop equilibrium in the simplex [12], [19]. The no-equilibrium requirement can also be expressed as a so-called *flow condition*, which gives an equivalent numerical test [19]. We are interested to obtain the most immediate extension of this result for polytopes. First, we restrict our attention to continuous piecewise affine (PWA) feedback. Assuming PWA feedback, the invariance conditions remain necessary conditions for solvability of RCP on polytopes [11]. Instead, the flow condition is no longer necessary for solvability on polytopes. Indeed the statement that there is no closed-loop equilibrium is no longer equivalent to existence of a flow condition when dealing with general polytopes, because the equivalence relies on the convexity of the closed-loop vector field. Convexity is preserved with affine feedback, but it may not be with PWA feedback. On the other hand, the flow condition affords useful properties; particularly that trajectories exit the polytope in an orderly way. In this section we begin an exploration of the extent to which results for simplices carry over to polytopes. Guided by these insights, we formulate in Section VI a restricted version of RCP: we incorporate the requirement of a flow condition into the problem statement, and we call this restricted problem *monotonic reach control*.

Let $\mathbb{T}$ be a triangulation of polytope $\mathcal{P}$. A point $x \in \mathcal{P}$ lies in the interior of precisely one simplex in $\mathbb{T}$ whose vertices are, say, $v_1, \ldots, v_k$. Then $x = \sum_{i=1}^{k} \lambda_i v_i$, where $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. Coefficients $\lambda_1, \ldots, \lambda_k$ are called the *barycentric coordinates* of x. Given a state feedback $u(x)$ on $\mathcal{P}$, we say u is a *piecewise affine feedback* if for any $x \in \mathcal{P}$,

$$x = \sum_i \lambda_i v_i \qquad \Longrightarrow \qquad u(x) = \sum_i \lambda_i u(v_i) \,,$$

where the λ_i are barycentric coordinates of x. It is easy to show that $u(x)$ is a continuous state feedback on $\mathcal{P}$ [17].

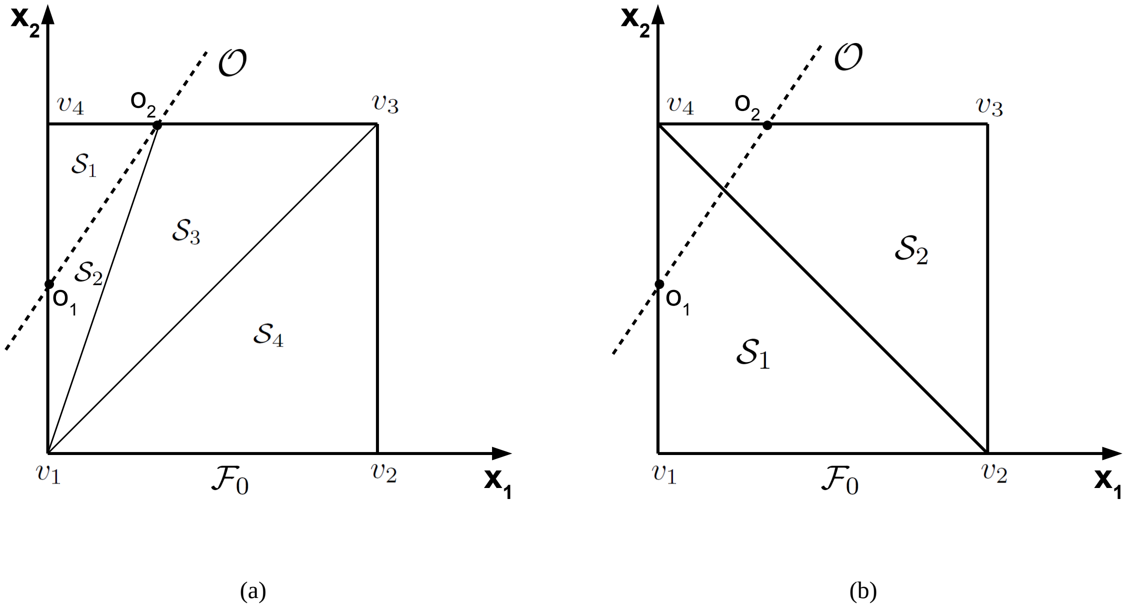


Fig. 3. Two triangulations of $\mathcal{P}$. (a) $\mathbb{T}$ is a triangulation with respect to $\mathcal{O}$, and (b) $\mathbb{T}$ is not.

Remark 5.1: If $u(x)$ is a piecewise affine feedback on $\mathcal{P}$, then for each n-dimensional simplex $\mathcal{S}_k \in \mathbb{T}$, there exist $K_k \in \mathbb{R}^{m\times n}$ and $g_k \in \mathbb{R}^m$ such that $u(x)$ takes the form

$$u(x) = K_k x + g_k\,, \qquad\qquad x \in \mathcal{S}_k\,.$$

We say $\mathbb{T}$ is a *triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$* if $\mathbb{T}$ is a refinement of a subdivision of the point set $V \cup V_{\mathcal{O}}$ such that $\mathcal{O}_{\mathcal{P}}$ is a union of simplices in $\mathbb{T}$.

Example 5.1: Consider the polytope in Figure 3. In Figure 3(a) $\mathcal{O}_{\mathcal{P}} = co\{o_1, o_2\}$ is a 1-dimensional simplex in $\mathbb{T}$, so we say $\mathbb{T}$ is a triangulation with respect to $\mathcal{O}$. In Figure 3(b) $\mathcal{O}_{\mathcal{P}}$ cannot be expressed as a union of simplices in $\mathbb{T}$, so $\mathbb{T}$ is not a triangulation with respect to $\mathcal{O}$.

Suppose we are given a triangulation $\mathbb{T}$ of $\mathcal{P}$ with respect to $\mathcal{O}$ and we are given $u(x)$, a piecewise affine feedback defined on $\mathbb{T}$ which satisfies the invariance conditions of $\mathcal{P}$. Define

$$b_i := Ao_i + Bu(o_i) + a \in \mathcal{B} \cap \mathcal{C}(o_i)\,, \qquad\qquad i \in I_{\mathcal{O}}\,. \tag{7}$$

If we want to exclude closed-loop equilibria in $\mathcal{P}$, then we only need to concentrate on the behavior of the closed-loop vector field in $\mathcal{O}_{\mathcal{P}}$. A basic result of convex analysis says that there are no closed-loop equilibria in $\mathcal{O}_{\mathcal{P}}$ if there is a flow condition on $\mathcal{O}_{\mathcal{P}}$.

Lemma 5.1: Let $\{b_1, \dots, b_q \mid b_i \in \mathcal{B}\}$ be such that $0 \notin \text{co}\ \{b_1, \dots, b_q\}$. Then there exists $\beta \in \mathcal{B}$ such that

$$\beta \cdot b_i < 0\,, \qquad\qquad i = 1, \dots, q\,.$$

Proof: Let $\mathcal{W}_1 := \mathbf{0}$ and $\mathcal{W}_2 := \text{co}\ \{b_1, \dots, b_q\}$. Note that both $\mathcal{W}_1$ and $\mathcal{W}_2$ are compact, convex sets, and by assumption $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. By Corollary 11.4.2 in [18], there exists a hyperplane $\mathcal{H}$ separating $\mathcal{W}_1$ and $\mathcal{W}_2$ strongly. Let ξ be the normal vector to $\mathcal{H}$ pointing to the side containing $\mathcal{W}_1$. Then, $\xi \cdot b_i < 0$ for $i = 1, \dots, q$. Now let $\xi = \beta + \eta$, where $\beta \in \mathcal{B}$ and $\eta \in \text{Ker}\ (B^T)$. Then we have

$$\xi \cdot b_i = \beta \cdot b_i < 0\,, \qquad\qquad i = 1, \dots, q\,,$$

as desired. ■

The condition that $0 \notin \text{co}\ \{b_1, \dots, b_q\}$ can be related to the existence of closed-loop equilibria in $\mathcal{P}$.

Theorem 5.2: Consider the system (1) defined on a polytope $\mathcal{P}$. Let $\mathbb{T}$ be a triangulation of $\mathcal{P}$ with respect to $\mathcal{O}$, $u(x)$ be a piecewise affine feedback defined on $\mathbb{T}$, and b_i be as in (7). If $0 \notin \text{co}\ \{b_1, \dots, b_q\}$, then the closed-loop system has no equilibrium in $\mathcal{P}$.

Proof: Let $x \in \mathcal{O}_{\mathcal{P}}$, and without loss of generality, suppose $x = \sum_{i=1}^{k} \lambda_i o_i$, where λ_i are the barycentric coordinates of x such that $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Let $\beta \in \mathcal{B}$ be as in Lemma 5.1. Since $\mathcal{O}_{\mathcal{P}}$ is a union of simplices in $\mathbb{T}$, and $y(x) := Ax + Bu(x) + a$ is affine on each simplex, we have

$$\beta \cdot y(x) = \beta \cdot \left(\sum_{i=1}^{k} \lambda_i y(o_i) \right) = \sum_{i=1}^{k} \lambda_i (\beta \cdot b_i) < 0\,.$$

Thus, $y(x) \neq 0$ for all $x \in \mathcal{O}_{\mathcal{P}}$. Since $y(x) \neq 0$ for all $x \in \mathcal{P} \setminus \mathcal{O}_{\mathcal{P}}$, the result is obtained. ■

The previous theorem gives a general condition in order that the closed-loop system has no equilibrium in $\mathcal{P}$. In [5], two geometric sufficient conditions were presented to guarantee that there are no closed-loop equilibria in a given simplex. The first condition was that $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$, where, as mentioned before, $\text{cone}(\mathcal{S})$ is the tangent cone to simplex $\mathcal{S}$ at the vertex not containing the exit facet $\mathcal{F}_0$. The second condition was that there is a set of linearly independent vectors $\{b_1, \dots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$, where it is assumed that $v_1, \dots, v_q$ are the vertices of

$\mathcal{S} \cap \mathcal{O}$. We would like to translate these two geometric conditions for simplices to the more general setting of polytopes. This is a straightforward exercise whose outcome is Lemmas 5.3 and 5.4 below.

First, the condition $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) \neq \mathbf{0}$, introduced in Section IV, has analogies with the statement for a simplex $\mathcal{S}$ that $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq \mathbf{0}$ (see the discussion before Theorem 4.3).

Lemma 5.3: Suppose $\mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P}) \neq \mathbf{0}$. Then there exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$ such that $0 \notin \text{co }\{b_1, \ldots, b_q\}$.

Proof: Select any $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{P})$ and set $b_i = b$ for $i \in I_\mathcal{O}$. Since $\text{cone}(\mathcal{O}_\mathcal{P}) \subset \mathcal{C}(o_i)$, we have $b_i \in \mathcal{B} \cap \mathcal{C}(o_i)$ for all $o_i \in V_\mathcal{O}$. Clearly $0 \notin \text{co }\{b_1, \ldots, b_q\}$. ■

Next, consider the condition for a simplex $\mathcal{S}$ that there is a linearly independent set of vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$. Removing the restriction that vertices of $\mathcal{O}_\mathcal{P}$ are vertices of $\mathcal{P}$, we have the following analogous condition for polytopes.

Lemma 5.4: Suppose there exists a linearly independent set of vectors $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(o_i)\}$. Then $0 \notin \text{co }\{b_1, \ldots, b_q\}$.

Proof: If $0 \in \text{co }\{b_1, \ldots, b_q\}$, then $\{b_1, \ldots, b_q\}$ are linearly dependent. ■

The previous two conditions provide the analogy for polytopes of the related geometric conditions for simplices. Then based on Theorem 5.2, both of the previous conditions imply there is no closed-loop equilibrium in $\mathcal{P}$, assuming $\mathcal{P}$ is triangulated with respect to $\mathcal{O}$. Unfortunately, in contrast with the situation for simplices, a no-equilibrium condition is not enough to deduce that RCP is solved. The primarily obstacle is that convexity of the closed-loop vector field is lost using general PWA controls. Consequently, the no closed-loop equilibrium condition is no longer equivalent to the existence of a flow condition (which guarantees that trajectories exit). For this reason we bring in the flow condition explicitly in the problem statement; this approach is developed in Section VI.

We conclude this section by showing that solvability of RCP on polytopes by any class of controls is inextricably linked to what happens on $\mathcal{O}_\mathcal{P}$. In particular, if RCP is not solved by some control strategy, it is because some trajectory encircles $\mathcal{O}_\mathcal{P}$, approaches $\mathcal{O}_\mathcal{P}$, or remains on $\mathcal{O}_\mathcal{P}$.

Lemma 5.5: Suppose there exists $x_0 \in \mathcal{P}$ and an open-loop control $u(t)$ such that the associated (unique) solution $\phi_u(t, x_0)$ of (1) satisfies $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \geq 0$. Define $\mathcal{A} = \text{co }\{\phi_u(t, x_0) \mid t \geq 0\}$. Then $\overline{\mathcal{A}} \cap \mathcal{O}_\mathcal{P} \neq \emptyset$.

Proof: The set $\overline{\mathcal{A}}$ is by construction convex and compact. Suppose by way of contradiction that $\overline{\mathcal{A}} \cap \mathcal{O}_{\mathcal{P}} = \emptyset$. Since $\overline{\mathcal{A}}$ is convex and compact, the image of $\overline{\mathcal{A}}$ under the affine map $x \mapsto Ax + a$, denoted $\mathcal{W}_1$ is also convex and compact. Also, $\mathcal{W}_1 \cap \mathcal{B} = \emptyset$. For suppose there is a point $x \in \overline{\mathcal{A}}$ such that $Ax + a \in \mathcal{B}$. Then $x \in \mathcal{O}_{\mathcal{P}}$, a contradiction. Thus, $\mathcal{W}_1 \cap \mathcal{B} = \emptyset$. Note that both $\mathcal{W}_1$ and $\mathcal{B}$ are convex sets, and that $\mathcal{W}_1$ is bounded. By Corollary 11.4.2 in [18], there exists a hyperplane $\mathcal{H}$ separating $\mathcal{B}$ and $\mathcal{W}_1$ strongly. This implies $\mathcal{B}$ is parallel to $\mathcal{H}$ since $\mathcal{B}$ is a subspace. Let $\xi \in \mathbb{R}^n$ be the normal vector to $\mathcal{H}$ pointing to the side containing $\mathcal{B}$. Then $\xi \in$ Ker (B^T) and

$$\xi \cdot (Ax + Bu + a) = \xi \cdot (Ax + a) < 0\,, x \in \overline{\mathcal{A}}\,, u \in \mathbb{R}^m\,.$$

As $\overline{\mathcal{A}}$ is compact and $\xi \cdot (Ax + a)$ is continuous in x, there is $\epsilon > 0$ such that $\xi \cdot (Ax + Bu + a) = \xi \cdot (Ax + a) < -\epsilon$ for all $x \in \overline{\mathcal{A}}$. In particular,

$$\xi \cdot (A\phi_u(t, x_0) + Bu(t) + a) < -\epsilon\,, \qquad \forall\ t \geq 0\,.$$

Integrating both sides of this equation we get

$$\xi \cdot \phi_u(t, x_0) < \xi \cdot x_0 - \epsilon t\,, \qquad \forall\ t \geq 0\,. \tag{8}$$

This contradicts the compactness of $\mathcal{P}$. We conclude $\overline{\mathcal{A}} \cap \mathcal{O}_{\mathcal{P}} \neq \emptyset$. ■

VI. MONOTONIC REACH CONTROL PROBLEM

The previous section identified issues concerning existence of equilibria on $\mathcal{O}_{\mathcal{P}}$ and the relationship between failure to solve RCP and behavior of trajectories with respect to $\mathcal{O}_{\mathcal{P}}$. However, clear necessary and sufficient conditions for solvability are not obtained. This is because a no-equilibrium condition (in addition to solvability of invariance conditions) is not known to be sufficient to solve RCP on polytopes. Instead, we study a more restrictive form of the problem which does lead to the natural analog of results for simplices. These necessary and sufficient conditions for solvability are examined under various assumptions on the placement of $\mathcal{O}_{\mathcal{P}}$. We also make comparisons with the main results for simplices to better understand the limits of those results when dealing with polytopes.

Problem 6.1 (Monotonic Reach Control Problem (MRCP)): Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for every $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

(ii) There exists $\xi \in \mathbb{R}^n$ such that for all $x \in \mathcal{P}$, $\xi \cdot (Ax + Bu(x) + a) < 0$.

The new condition (ii) is called a *flow condition*, and the problem is called "monotonic" because trajectories flow through the polytope in a common sense with respect to a foliation of parallel hyperplanes with normal vector ξ. We write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically if properties (i)-(ii) of Problem 6.1 hold.

Now we investigate necessary and sufficient conditions for solvability of MRCP under assumptions on the placement of $\mathcal{O}$ with respect to $\mathcal{P}$. The first result when $\mathcal{O}_{\mathcal{P}} = \emptyset$ is based on the following technical lemma.

Lemma 6.1: Consider the system (1) defined on a compact, convex set $\mathcal{A}$. If $\mathcal{A} \cap \mathcal{O} = \emptyset$, then there exists $\beta \in$ Ker (B^T) such that

$$\beta \cdot (Ax + Bu + a) < 0\,, \qquad \forall x \in \mathcal{A}\,, \quad \forall u \in \mathbb{R}^m\,.$$

Proof: Since $\mathcal{A}$ is compact and convex, the image of $\mathcal{A}$ under the affine map $x \mapsto Ax + a$, denoted $\mathcal{W}_1 = A(\mathcal{A}) + a$ is also compact and convex. Also, $\mathcal{W}_1 \cap \mathcal{B} = \emptyset$. For suppose not. Then there is a point $x \in \mathcal{A}$ such that $Ax + a \in \mathcal{B}$. Then $x \in \mathcal{O}$, by definition, which contradicts $\mathcal{A} \cap \mathcal{O} = \emptyset$. Note that both $\mathcal{W}_1$ and $\mathcal{B}$ are convex sets, and that $\mathcal{W}_1$ is bounded. By Corollary 11.4.2 in [18], there exists a hyperplane $\mathcal{H}$ separating $\mathcal{B}$ and $\mathcal{W}_1$ strongly. This implies $\mathcal{B}$ is parallel to $\mathcal{H}$ since $\mathcal{B}$ is a subspace. Let β be the normal vector to $\mathcal{H}$ pointing to the side containing $\mathcal{B}$. Then, $\beta \in$ Ker (B^T) and $\beta \cdot (Ax + a) < 0$ for all $x \in \mathcal{A}$. Since $\beta \cdot B = 0$, the result follows. ■

Theorem 6.2: Consider the system (1) defined on a polytope $\mathcal{P}$, and suppose $\mathcal{O}_{\mathcal{P}} = \emptyset$. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback if and only if the invariance conditions (2) are solvable.

Proof: ($\Longrightarrow$) Follows from the necessity of the invariance conditions [11]. ($\Longleftarrow$) Select the control $u_i \in \mathbb{R}^m$ for each vertex $v_i \in V$ to satisfy the invariance conditions (2). Form a triangulation $\mathbb{T}$ of $\mathcal{P}$. Using the method of [11], one can find unique K_i and g_i corresponding to the affine feedback $u(x) = K_j x + g_j$ on each simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = u_i$, $i = 1, \ldots, p$. We obtain the piecewise affine closed-loop system $\dot{x} = (A + BK_j)x + (a + Bg_j)$. (Note that since $\mathcal{P} \cap \mathcal{O} = \emptyset$, the closed-loop system has no equililbria in $\mathcal{P}$.) By Lemma 6.1 there exists $\beta \in$ Ker (B^T) such that

$$\beta \cdot (Ax + Bu(x) + a) = \beta \cdot (Ax + a) < 0\,, \qquad\qquad \forall x \in \mathcal{P}\,.$$

By the same argument expressed in (8) all trajectories exit $\mathcal{P}$. Moreover, by the convexity of $Ax + Bu(x) + a$ on each simplex of $\mathbb{T}$, conditions (2) imply (3). By Lemma 3.1 trajectories in $\mathcal{P}$ exit through $\mathcal{F}_0$. Thus, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback. ■

In [5] necessary and sufficient conditions for solvability of RCP on simplices were obtained based on the assumption that $\mathcal{O}_{\mathcal{P}}$ is a face of the simplex. The same assumption for polytopes makes possible a straightforward generalization to polytopes for solvability of MRCP.

Assumption 6.1: Polytope $\mathcal{P}$ and system (1) satisfy the following condition: $\mathcal{O}_{\mathcal{P}}$ is a κ-dimensional face of $\mathcal{P}$, where $0 \le \kappa \le n$. In particular,

$$\mathcal{O}_{\mathcal{P}} = \text{co}\ \{v_1, \ldots, v_q\}\,,$$

where v_i is a vertex of $\mathcal{P}$, and let $V_{\mathcal{O}} := \{v_1, \ldots, v_q\}$.

Theorem 6.3: Consider the system (1) defined on $\mathcal{P}$ and suppose Assumption 6.1 holds. Then $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback if and only if

(i) The invariance conditions (2) are solvable.

(ii) There exists $\{b_1, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ such that $0 \notin$ co $\{b_1, \ldots, b_q\}$.

Proof: ($\Longrightarrow$) Let $y(x) := Ax + Bu(x) + a$, where $u(x)$ is the PWA feedback achieving $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. Since $u(x)$ is a continuous state feedback, the invariance conditions are solvable [11]. Now suppose that condition (ii) does not hold. This implies $0 \in$ co $\{y(v_1), \ldots, y(v_p)\}$. On the other hand, by the assumption that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y(v_i) < 0$ for $i \in I$. This implies **0** and co $\{y(v_1), \ldots, y(v_p)\}$ are strongly separated, a contradiction.

($\Longleftarrow$) For each vertex $v_i \in V \setminus \mathcal{O}_{\mathcal{P}}$, select a control $u_i \in \mathbb{R}^m$ to satisfy the invariance conditions (2). For $v_i \in V_{\mathcal{O}}$, select $u_i \in \mathbb{R}^m$ such that $Av_i + Bu_i + a = b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. Form a triangulation $\mathbb{T}$ of $\mathcal{P}$. Using the method of [11], one can find unique K_j and g_j corresponding to the affine feedback $u(x) = K_j x + g_j$ on each n-dimensional simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = u_i$,

$i = 1, \ldots, p$ and $y(v_i) = b_i$, $i = 1, \ldots, q$. We obtain the piecewise affine closed-loop system

$$\dot{x} = (A + BK_j)x + (a + Bg_j) =: y(x)\,, \qquad x \in \mathcal{P}\,.$$

We show a flow condition holds on $\mathcal{P}$. First, by Lemma 5.1, a flow condition holds for the closed loop vector field $y(x) := (A + BK_i)x + Bg_i + a$ at vertices of $\mathcal{O}_{\mathcal{P}}$. That is, there exists $\beta_1 \in \mathcal{B}$ such that

$$\beta_1 \cdot y(v_i) = \beta_1 \cdot b_i < 0\,, \qquad\qquad i = 1, \ldots, q\,.$$

Next let $\mathcal{P}' :=$ co $\{v_i \mid v_i \in V \backslash V_{\mathcal{O}}\}$. Note that because $\mathcal{O}_{\mathcal{P}}$ is a face of $\mathcal{P}$, $\mathcal{P}' \cap \mathcal{O} = \emptyset$. According to Lemma 6.1, there exists $\beta_2 \in$ Ker (B^T) such that for all $x \in \mathcal{P}'$, $\beta_2 \cdot (Ax + Bu(x) + a) < 0$. Define

$$\beta = \alpha\beta_1 + (1 - \alpha)\beta_2$$

for some $\alpha \in (0, 1)$. Consider $v_i \in V_{\mathcal{O}}$. Using the fact that $\beta_2 \cdot b_i = 0$, we have

$$\beta \cdot y(v_i) = \alpha\beta_1 \cdot y(v_i) < 0\,.$$

Next consider $v_i \in V \setminus V_{\mathcal{O}}$. We have

$$\beta \cdot (Av_i + Bu_i + a) = \alpha\beta_1 \cdot (Av_i + Bu_i + a) + (1 - \alpha)\beta_2 \cdot (Av_i + a)\,.$$

The term $\beta_1 \cdot (Av_i + Bu_i + a)$ is a constant of unknown sign, whereas we know $\beta_2 \cdot (Av_i + a) < 0$. Therefore it is possible to select α sufficiently small so that $\beta \cdot (Av_i + Bu_i + a) < 0$ for all $v_i \in V \setminus V_{\mathcal{O}}$. We conclude that for all $v_i \in V$, $\beta \cdot y(v_i) < 0$.

Now let $x \in \mathcal{P}$, and without loss of generality, suppose $x = \sum_{i=1}^{k} \lambda_i v_i$, where λ_i are the barycentric coordinates of x such that $\lambda_i > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since $y(x)$ is affine on simplices of $\mathbb{T}$, we have $y(x) = \sum_{i=1}^{k} \lambda_i y(v_i)$. Therefore, for $x \in \mathcal{P}$,

$$\beta \cdot y(x) = \sum_{i=1}^{k} \lambda_i \beta \cdot y(v_i) < 0\,.$$

By the same argument expressed in (8) all trajectories exit $\mathcal{P}$, and by Lemma 3.1 they do so through $\mathcal{F}_0$. Thus, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by piecewise affine feedback. ■

Remark 6.1: Lemmas 5.3 and 5.4 provide sufficient geometric conditions for condition (ii) of

Theorem 6.3. These provide the analog to the results for simplices appearing in [5].

Finally, we consider the general case when $\mathcal{O}_{\mathcal{P}} \cap \mathcal{P}^{\circ} \neq \emptyset$. To illustrate the approach we study only single-input systems. Starting from Theorem 4.3, we create a monotonic flow by "pushing" a vector $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ onto each of the vertices of $\mathcal{P}$ while preserving the invariance conditions. We show that if MRCP is solvable, then it is solvable by this b-extremal solution. This then leads to a design procedure for constructing the appropriate controls, to be developed in Section VIII.

Let $y \in \mathbb{R}^n$ and define the index set

$$I_y := \{ i \in I \mid y \in \mathcal{C}(v_i) \} .$$

That is, I_y is the index set of vertices for which the velocity vector y satisfies the invariance conditions of that vertex. By Theorem 4.3, $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) \neq \mathbf{0}$ is a necessary condition for solvability of RCP when $m = 1$, so we assume we have such a $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$. For the indices $i \notin I_b$, let $\overline{u}_i$ be such that $\overline{y}_i := Av_i + B\overline{u}_i + a \in \mathcal{C}(v_i)$ contains the maximal b component. Since $b \notin \mathcal{C}(v_i)$ and $m = 1$, the maximum exists and is unique, and it corresponds to one or more invariance conditions evaluating to zero at v_i. Given a triangulation $\mathbb{T}$ of $\mathcal{P}$, let $\overline{u}(x)$ denote any PWA feedback such that $\overline{u}(v_i) = \overline{u}_i$, $i \notin I_b$.

The following result tells us that a b-extremal controller, in the sense just described, can always be selected to solve MRCP, if it is solvable by PWA feedback. The second condition (ii) below, presently less meaningful, will be seen to provide a useful tool in the algorithmic solution of MRCP, to be developed in Section VIII.

Theorem 6.4: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_{\mathcal{P}} \neq \emptyset$. Suppose $\mathbb{T}$ is a triangulation and $u(x)$ an associated PWA control such that $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $u(x)$. Then there exist $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$, $\overline{u}(x)$, and $\overline{y}_i$ as above such that:

(i) $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $\overline{u}(x)$,
(ii) $0 \notin \text{co } \{ b, \ \overline{y}_i \mid i \notin I_b \}$.

Proof: Since $u(x)$ is a continuous state feedback solving $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$, $u(x)$ satisfies the invariance conditions [11]. Let $\mathcal{O}_{\mathcal{P}} = \text{co } \{ o_1, \ldots, o_q \}$. Define $b = Ao_1 + Bu(o_1) + a \in \mathcal{B}$. Then we must have $Ao_i + Bu(o_i) + a = \alpha_i b$ with $\alpha_i > 0$ for $i = 1, \ldots, q$. Otherwise, by the same argument as in the proof of Theorem 4.3, there is an equilibrium in $\mathcal{P}$ using $u(x)$. We conclude $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$. Since $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $u(x)$, there exists $\xi \in \mathbb{R}^n$ such that

$$\xi \cdot (Ax + Bu(x) + a) < 0\,, \qquad x \in \mathcal{P}\,.$$

In particular, $\xi \cdot (Ao_1 + Bu(o_1) + a) = \xi \cdot b < 0$. Now define $\overline{u}(x) := u(x) + w(x)$, where $w(x)$ is determined by $Bw(x) = c(x)b$, such that the positive PWA function $c(x) \geq 0$ associated with $\mathbb{T}$ arises from assigning the maximal b component at vertices v_i, $i \notin I_b$, without violating invariance conditions. Also set $c(v_i) = 0$, $i \in I_b$. Then the invariance conditions still hold, and for $x \in \mathcal{P}$,

$$\xi \cdot (Ax + B\overline{u}(x) + a) = \xi \cdot (Ax + Bu(x) + a) + \xi \cdot c(x)b < 0\,. \tag{9}$$

We conclude $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically using $\overline{u}(x)$. By equation (9) and the fact that $\xi \cdot b < 0$, we obtain (ii). ■

Theorem 6.4 suggests a design procedure to synthesize a PWA control $\overline{u}(x)$ to achieve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. The procedure is simply to inject the largest possible $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ component in any vertex with $i \notin I_b$, and to use a sufficiently large b component for vertices with $i \in I_b$. We present an algorithm in Section VIII.

VII. ARBITRARY TRIANGULATION AND MRCP

In the previous section we studied MRCP and, while the results are centered on PWA feedback, they did not depend on the particular choice of triangulation. Indeed, the effect of the flow condition is to allow a solution that does not depend on the choice of triangulation. This is a useful feature if the triangulation is performed by a standalone software not adapted to control problems. An intuition emerges that the role of the flow condition is precisely to provide this invariance to triangulation. In this section we explore the extent to which this intuition is correct. For this we formulate a version of RCP under arbitrary triangulations. By *arbitrary triangulation* of $\mathcal{P}$ we mean any triangulation of $\mathcal{P}$ with the property that if $v \in \mathcal{P}$ is a vertex of a simplex belonging to $\mathbb{T}$, then v is a vertex of $\mathcal{P}$. We show that MRCP by PWA feedback and RCP under arbitrary triangulations are equivalent in a generic sense.

Problem 7.1 (RCP by Arbitrary Triangulations): Consider the system (1) defined on $\mathcal{P}$. Find a control assignment u_i, $i \in I$, such that for an arbitrary triangulation $\mathbb{T}$ of $\mathcal{P}$, the associated

PWA feedback $u(x)$ with $u(v_i) = u_i$, $i \in I$, achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$.

A set of $p > n$ points in $\mathbb{R}^n$ are in *general position* if no $n+1$ of them lie in a common affine hyperplane. The convex hull of any set of points in general position in $\mathbb{R}^n$ is called a *generic polytope* [6]. For a generic polytope, all (proper) faces are simplices. A polytope whose faces are simplices is called a *simplicial polytope*.

Theorem 7.1: Consider the system (1) defined on a generic polytope $\mathcal{P}$. MRCP by PWA feedback is solvable if and only if RCP by arbitrary triangulations is solvable.

Proof: ($\Longrightarrow$) By the same argument as at the end of the proof of Theorem 6.3, for any choice of triangulation and associated PWA feedback, $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$.

($\Longleftarrow$) Suppose u_i, $i \in I$, is a control assignment such that for any triangulation $\mathbb{T}$ of $\mathcal{P}$, the associated PWA feedback $u(x)$ with $u(v_i) = u_i$, achieves $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$. Let $y_i := Av_i + Bu_i + a$, $i \in I$. We claim $0 \notin$ co $\{y_1, \ldots, y_p\}$. Suppose not. By Caratheodory's Theorem [18] and w.l.o.g. there exist $\alpha_1, \ldots, \alpha_k$ with $1 \le k \le n+1$ such that

$$0 = \sum_{i=1}^{k} \alpha_i y_i\,, \qquad \alpha_i > 0\,, \quad \sum_i \alpha_i = 1\,.$$

Let $\overline{x} = \sum_{i=1}^{k} \alpha_i v_i \in \mathcal{P}$. Since $\{v_1, \ldots, v_k\}$ are in general position, one can apply the placing triangulation [13] to the ordered point set $V = \{v_1, \ldots, v_p\}$ such that $\mathcal{S} :=$ co $\{v_1, \ldots, v_k\}$ is a simplex of the resulting triangulation $\mathbb{T}$. Let $u(x)$ be the PWA feedback associated with $\mathbb{T}$ such that $u(v_i) = u_i$. Since $u(x)$ is affine on $\mathcal{S}$,

$$A\overline{x} + Bu(\overline{x}) + a = \sum_{i=1}^{k} \alpha_i \big(Av_i + Bu(v_i) + a\big) = \sum_{i=1}^{k} \alpha_i y_i = 0\,.$$

That is $\overline{x}$ is an equilibrium of the closed-loop system, so RCP is not solved using this triangulation, a contradiction. We conclude $0 \notin$ co $\{y_1, \ldots, y_p\}$. By the same argument as in the proof of Lemma 5.1, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y_i < 0$, $i \in I$. By the same argument as at the end of the proof of Theorem 6.3, we have $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by PWA feedback. ■

VIII. ALGORITHM FOR MRCP

In this section we present an algorithm for solving MRCP by PWA feedback for single-input systems. It is assumed that $\mathcal{O}_{\mathcal{P}} \neq \emptyset$, for if $\mathcal{O}_{\mathcal{P}} = \emptyset$, then Theorem 6.2 provides a solution. Also, if $\mathcal{O}_{\mathcal{P}}$ is a face of $\mathcal{P}$, then Theorem 6.3 provides a solution. The algorithm, inspired by

Theorem 6.4, is easily explained in words: for a single input system, there are only two control directions $b, -b \in \mathcal{B}$. Choose $b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ (step 1). At all those vertices v_i where $b \notin \mathcal{C}(v_i)$, we inject a maximal b component into the vector field by choice of control u_i (step 2). If MRCP is solvable, then Theorem 6.4 tells us that such an extremal solution exists. A flow condition must hold with extremal control values; that is, we can find a candidate $\xi \in \mathbb{R}^n$ for Problem 6.1 (step 3). Then we use ξ to select control values at the remaining vertices v_i where $b \in \mathcal{C}(v_i)$ (step 4). If ξ cannot be found, then the procedure is repeated with $-b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$(step 5). Theorem 8.1 shows that this procedure is sound and complete.

Algorithm 1:

1. Select $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$.
2. For each $i \notin I_b$, solve the LP for $\overline{u}_i \in \mathbb{R}$:

$$\begin{aligned} &\max_{u \in \mathbb{R}} \quad b \cdot (Av_i + a + Bu) \\ &\text{subject to:} \quad Av_i + a + Bu \in \mathcal{C}(v_i) \end{aligned} \tag{10}$$

3. Solve the LP for $\xi \in \mathbb{R}^n$:

$$\xi \cdot (Av_i + a + B\overline{u}_i) \;<\; 0\,, \qquad i \notin I_b \tag{11a}$$

$$\xi \cdot b \;<\; 0\,. \tag{11b}$$

4. If (11) is solvable, then for each $i \in I_b$, solve the LP for $\overline{u}_i \in \mathbb{R}$:

$$\xi \cdot (Av_i + a + B\overline{u}_i) \;<\; 0 \tag{12a}$$

$$Av_i + a + B\overline{u}_i \;\in\; \mathcal{C}(v_i)\,. \tag{12b}$$

5. If (11) is not solvable, select $-b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ and repeat steps $2-4$ after replacing b by $-b$.
6. Form a triangulation $\mathbb{T}$ of $\mathcal{P}$ using only vertices of $\mathcal{P}$. Construct an affine feedback $u(x) = K_j x + g_j$ for each $n-$dimensional simplex $\mathcal{S}_j \in \mathbb{T}$ such that $u(v_i) = \overline{u}_i,\ i = 1, \cdots, p$.

Theorem 8.1: Consider the system (1) defined on a polytope $\mathcal{P}$. Suppose $m = 1$ and $\mathcal{O}_{\mathcal{P}} \neq \emptyset$. MRCP is solvable by PWA feedback if and only if Algorithm 1 terminates successfully.

Proof: ($\Longleftarrow$) Suppose the algorithm terminates successfully. It is required to show that the

PWA feedback $u(x)$ calculated in step 6 solves MRCP on $\mathcal{P}$. From (10) and (12b), $u(x)$ satisfies the invariance conditions (2). From (11a) and (12a), a flow condition holds at the vertices of $\mathcal{P}$. By the same argument as at the end of the proof of Theorem 6.3 (with β replaced by ξ), $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically by the PWA feedback $u(x)$.

($\Longrightarrow$) Suppose that MRCP is solvable by PWA feedback. By way of contradiction, we show that if Algorithm 1 does not terminate successfully, then MRCP is not solvable by PWA feedback. Let's consider all the cases where the algorithm does not terminate successfully. Let $\overline{y}_i := Av_i + B\overline{u}_i + a$, $i = 1, \dots, p$.

1. The algorithm terminates in step 1 if $\mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}}) = \mathbf{0}$. By Theorem 4.3, MRCP is not solvable by continuous state feedback.
2. The algorithm terminates in step 2 if either (10) is not solvable, but then by Proposition 3.1 in [11], MRCP is not solvable by continuous state feedback. Alternatively, for some $i \notin I_b$, the maximum does not exist under the constraint (10). Because $i \notin I_b$, $b \notin \mathcal{C}(v_i)$ so there exists $j \in J(v_i)$ such that $h_j \cdot b > 0$. Then we have that the b-component in Bu can be made arbitrary large, while also $h_j \cdot (Av_i + Bu + a) \leq 0$. This is clearly impossible.
3. The algorithm terminates in step 4 if the LP is not feasible. As above, if (12b) is not solvable, then MRCP is not solvable by continuous state feedback. Instead, suppose (12a) is not achievable simultaneously with (12b). This can't happen because $\xi \cdot b < 0$ and $b \in \mathcal{C}(v_i)$, $i \in I_b$, so any sufficiently large b-component added to a velocity vector already satisfying (12b) solves the LP.
4. The algorithm terminates in step 5 if either for every $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$ the LP problem in step 3 is not solvable. Equivalently, by a result analogous to Lemma 5.1, for every $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$, $0 \in \text{co}\ \{b,\ \overline{y}_i \mid i \notin I_b\}$. Then by Theorem 6.4(ii), MRCP is not solvable by PWA feedback. Alternatively, the LP problem in step 2, or 4 is not solvable (using $-b$). By a similar argument to the previous two points, MRCP is not solvable by continuous state feedback.

$\blacksquare$

Remark 8.1: Indeed if (12b) is satisfied using u_i', then it can be easily verified that a control that satisfies (12) is calculated as follows. For $i \in I_b$, let $c_i > \max(0, -\frac{\xi \cdot (Av_i + a + Bu_i')}{\xi \cdot b})$. Select $\overline{u}_i = u_i' + w(v_i)$, where $w(v_i)$ is determined by $Bw(v_i) = c_i b$.

Remark 8.2: One advantage of Algorithm 1 is that it can be used for general polytopes. Also, in Algorithm 1 the number of LP problems does not exceed $2p + 2$.

IX. EXAMPLES

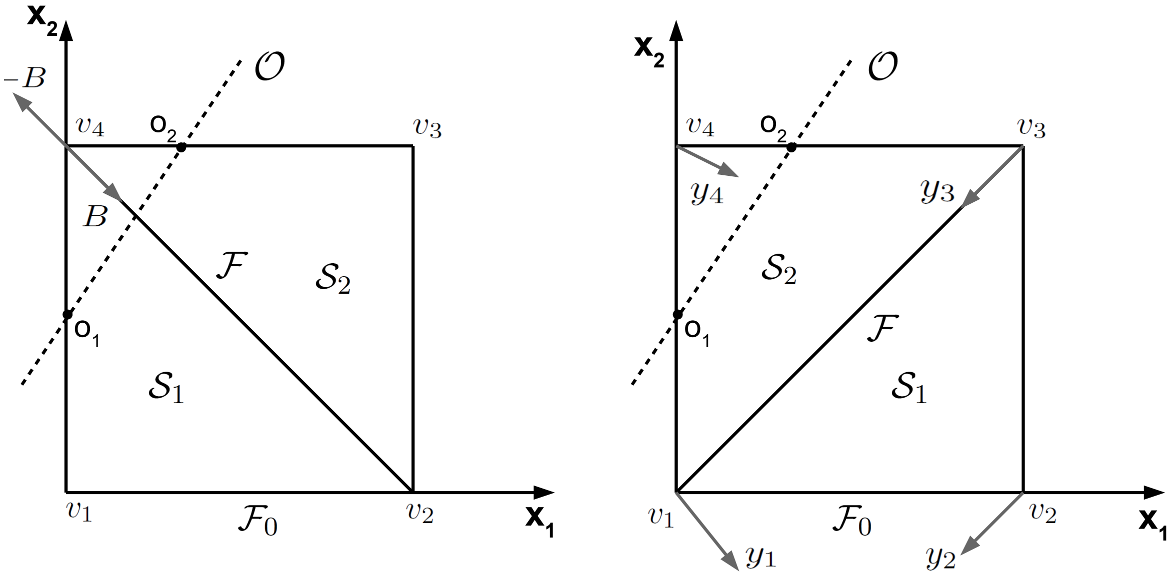
We give several examples motivating why new research is needed on reach control for polytopes and illustrating some of the findings of the paper.

Example 9.1: In the first example we show that using simplex-based methods for reach control, RCP is solvable for one triangulation but not for another. However, MRCP is solvable using any triangulation, thereby illustrating Theorem 7.1. Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u\,.$$

The polytope is shown in Figure 4. The vertices of $\mathcal{P}$ are: $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,1)$, and $v_4 = (0,1)$. We find by direct computation that $\mathcal{O} = \{x \mid -3x_1 + 2x_2 = 1\}$, depicted as a dashed line in Figure 4. It can be easily verified that $\mathcal{O}_{\mathcal{P}} = \text{co}\ \{o_1, o_2\}$, where $o_1 = (0, 1/2)$ and $o_2 = (1/3, 1)$. The control objective is to achieve $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by PWA feedback using existing simplex methods in the literature. Suppose we triangulate $\mathcal{P}$ as in Figure 4(a). Then the control objective splits as $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback. We study the invariance conditions of $\mathcal{S}_1$ at v_4. We have $Av_4 + a = (1,0)$, thus for any choice of control the invariance conditions of $\mathcal{S}_1$ are always violated at v_4. Consequently, $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ is not achievable. Instead, suppose we triangulate $\mathcal{P}$ as in Figure 4(b). The control objective is again $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F}$ by affine feedback. We choose the control values at the vertices to be $u_1 = 4$, $u_2 = 0$, $u_3 = 0$, $u_4 = 2$. The corresponding velocity vector, y_i, at each vertex v_i is shown in Figure 4(b). Based on these selected control values at the vertices, one can construct a PWA feedback such that the control objective based on existing simplex methods is achieved [11].

Now we show MRCP is solvable. We choose the same control values at the vertices as above. Let $\xi = (0,1)$. It can be verified that $\xi \cdot (Av_i + Bu_i + a) < 0$, for $i = 1, \ldots, 4$. Triangulate $\mathcal{P}$ using any triangulation $\mathbb{T}$ and construct the associated PWA feedback $u(x)$ based on the control values at the vertices [11]. For instance, if the second triangulation in Figure 4(b) is selected,



(a) (b)

Fig. 4. Two triangulations of $\mathcal{P}$ for Example 9.1.

then we obtain the following PWA control law:

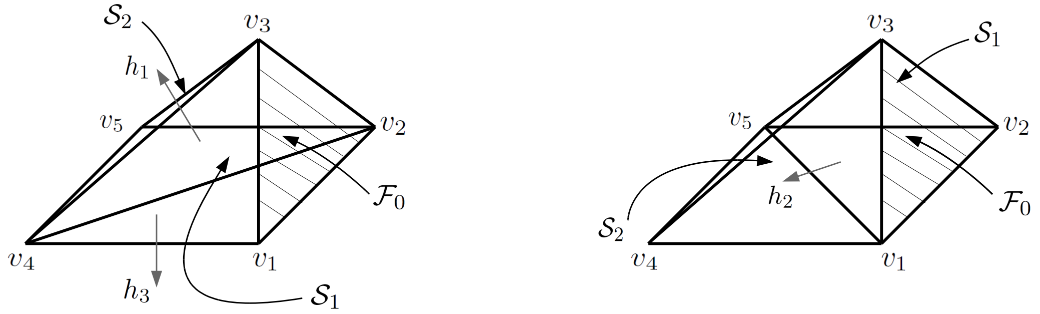
$$u(x) = \begin{cases} \begin{bmatrix} -4 & 0 \end{bmatrix} x + 4\,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} -2 & -2 \end{bmatrix} x + 4\,, & x \in \mathcal{S}_2\,. \end{cases}$$

Since the invariance conditions of $\mathcal{P}$ are satisfied, we get $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically. Notice the result holds even if the first triangulation were selected.

Example 9.2: In the previous example simplex methods could be used to solve RCP, although there was an advantage to the solution via MRCP, since it was valid for any triangulation. Now we consider an example where simplex methods fail for any choice of triangulation, but MRCP is solvable. Consider the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -3 & -2 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u\,.$$

The polytope is shown in Figure 9.2. The vertices of $\mathcal{P}$ are: $v_1 = (1,0,0)$, $v_2 = (1,1,0)$, $v_3 = (1,0,1)$, $v_4 = (0,0,0)$, and $v_5 = (0,1,0)$.



(a)

(b)

Fig. 5. Two triangulations of $\mathcal{P}$ for Example 9.2.

First, we check if the problem is solvable using simplex methods. There are two possible triangulations of $\mathcal{P}$, shown in Figures 9.2 and 5.

For the first triangulation the control objective is $\mathcal{S}_1 \xrightarrow{\mathcal{S}_1} \mathcal{F}_0$ by affine feedback and $\mathcal{S}_2 \xrightarrow{\mathcal{S}_2} \mathcal{F} = \mathcal{S}_1 \cap \mathcal{S}_2$ by affine feedback. We examine the invariance conditions of $\mathcal{S}_1$ at v_4. We have $Av_4 + Bu_4 + a = (0, 1, u_4)$. The normal vectors to facets $\mathcal{F}_3$ and $\mathcal{F}_1$ in $\mathcal{S}_1$ are $h_3 = (0, 0, -1)$, and $h_1 = (-0.5774, 0.5774, 0.5774)$ respectively. The invariance conditions of $\mathcal{S}_1$ at v_4 yield $h_3 \cdot (Av_4 + Bu_4 + a) \leq 0$ and $h_1 \cdot (Av_4 + Bu_4 + a) \leq 0$. That is, $u_4 \geq 0$ and $u_4 \leq -1$. Thus, RCP is not solvable by simplex methods using this triangulation. Now we try the second triangulation in Figure 5. We examine the invariance conditions of $\mathcal{S}_1$ at v_3. In this case, we have $Av_3 + Bu_3 + a = (-1, 0, -1 + u_3)$, $h_2 = (-0.7071, -0.7071, 0)$, and $h_2 \cdot (Av_3 + Bu_3 + a) > 0$ for all $u_3 \in \mathbb{R}$. Again, RCP is not solvable by simplex methods using this triangulation.

Now we study solvability of MRCP. Select control values $u_1 = 0$, $u_2 = 0$, $u_3 = -20$, $u_4 = 0$, and $u_5 = 0$. It can be verified they satisfy the invariance conditions of $\mathcal{P}$. Let $\xi = (-1, -0.01, 1)$. One can then verify that $\xi \cdot (Av_i + Bu_i + a) < 0$ for all $i = 1, \ldots, 5$. Triangulate $\mathcal{P}$ using any triangulation $\mathbb{T}$ and construct the associated PWA feedback $u(x)$ based on the control values at the vertices [11]. For instance, if the first triangulation in Figure 9.2 is selected, then we obtain the following control law:

$$u(x) = \begin{cases} \begin{bmatrix} 0 & 0 & -20 \end{bmatrix} x\,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} 0 & 0 & -20 \end{bmatrix} x\,, & x \in \mathcal{S}_2\,. \end{cases}$$

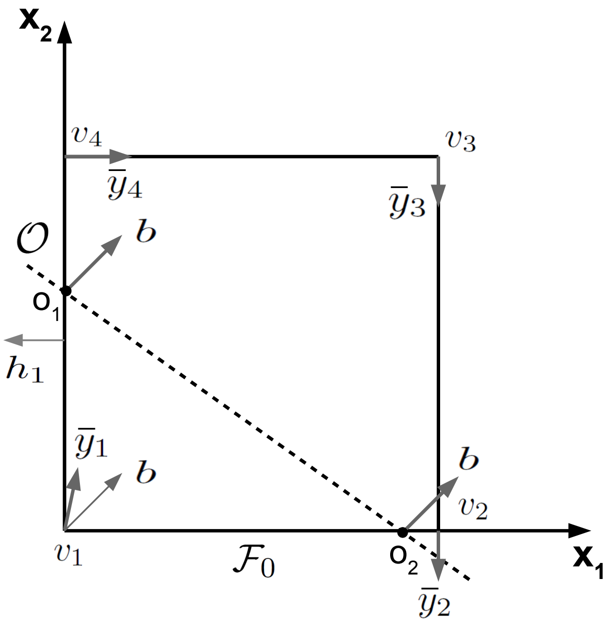


Fig. 6. Polytope for Example 9.3

Since the invariance conditions of $\mathcal{P}$ are satisfied, we get $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ monotonically.

Example 9.3: In this example we study a case where $\mathcal{O}_{\mathcal{P}} \cap \mathcal{P}^{\circ} \neq \emptyset$, and we use Algorithm 1 to solve MRCP. Consider the system

$$\dot{x} = \begin{bmatrix} 2.1 & 2.5 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

The polytope is shown in Figure 6. The vertices of $\mathcal{P}$ are: $v_1 = (0,0)$, $v_2 = (1,0)$, $v_3 = (1,1)$, and $v_4 = (0,1)$. We find by direct computation that $\mathcal{O} = \{x \mid 1.1x_1 + 1.5x_2 = 1\}$, depicted as a dashed line in Figure 6. Clearly, $\mathcal{P}^{\circ} \cap \mathcal{O} \neq \emptyset$. It can be easily verified that $\mathcal{O}_{\mathcal{P}} = \text{co}\ \{o_1, o_2\}$, where $o_1 = (0, 2/3)$ and $o_2 = (0.90909, 0)$. By definition, cone$(\mathcal{O}_{\mathcal{P}})$ is the cone of directions that simultaneously satisfy the union of the invariance conditions at all the points $o_i \in \mathcal{O}_{\mathcal{P}}$. So, in this example cone$(\mathcal{O}_{\mathcal{P}}) = \{y \in \mathbb{R}^2 \mid h_1 \cdot y \leq 0\}$, where $h_1 = (-1, 0)$ as shown in Figure 6. Now we follow the steps of Algorithm 1 to solve MRCP by PWA feedback.

Step 1: Let $b := (1,1) \in \mathcal{B}$. It can be easily verified that $h_1 \cdot b < 0$. So, $0 \neq b \in \mathcal{B} \cap \text{cone}(\mathcal{O}_{\mathcal{P}})$. Note: It can be verified that $I_b = \{1\}$. As seen in Figure 6, b dips into $\mathcal{C}(v_1)$.

Step 2: For the vertices v_2, v_3, and v_4: we solve at each vertex a LP problem to obtain $\overline{u}_2$, $\overline{u}_3$, and $\overline{u}_4$ respectively. We get $\overline{u}_2 = -2.1$, $\overline{u}_3 = -4.6$, and $\overline{u}_4 = -2$. As seen in the figure, certain invariance conditions evaluate to zero at these vertices.

Step 3: We solve the LP problem. The problem is feasible, and we get $\xi = (-2.01, 0.01)$.

Step 4: For v_1, we solve the LP. We obtain $\overline{u}_1 = 0.0075$.

Note: The control value $\overline{u}_1$ can also be calculated as shown in Remark 8.1. Let $u_1' = 0$. It can be easily verified that u_1' satisfies the invariance conditions at v_1. We must select $c_1 > \max(0, -\frac{\xi\cdot(Av_1+a+Bu_1')}{\xi\cdot b}) = 0.005$. Let $c_1 = 0.0075$. Then, we find $\overline{u}_1 = 0.0075$.

Step 6: We form a triangulation $\mathbb{T}$ of $\mathcal{P}$ consisting of two simplices $\mathcal{S}_1 =$ co $\{v_1, v_2, v_3\}$ and $\mathcal{S}_2 =$ co $\{v_1, v_3, v_4\}$. The corresponding piecewise affine feedback is:

$$u(x) = \begin{cases} \begin{bmatrix} -2.1075 & -2.5 \end{bmatrix} x + 0.0075\,, & x \in \mathcal{S}_1 \\ \begin{bmatrix} -2.6 & -2.0075 \end{bmatrix} x + 0.0075\,, & x \in \mathcal{S}_2\,. \end{cases}$$

The example shows that the method of pushing b works even if b does not point to the exit facet $\mathcal{F}_0$. Also, we did not need to push a large amount of b at v_1. It turns out that a small push ($c_1 b,\ c_1 > 0.005$) is enough to construct a flow condition on $\mathcal{P}$. This small push is important. If we select $c_1 = 0$, it can be verified that $0.090909(Av_1 + B\overline{u}(v_1) + a) + 0.90909(Av_2 + B\overline{u}(v_2) + a) = 0$, so a flow condition cannot be achieved on $\mathcal{P}$.

X. CONCLUSION

In this paper we have extended the results in [5] for reach control on simplices to general polytopes. First, we have translated the geometric sufficient conditions for constructing a flow condition on simplices in [5] to analogous conditions for polytopes. Then, we have formulated the monotonic reach control problem (MRCP) in which we incorporate the requirement of a flow condition into the problem statement. Finally, we have obtained intrinsic necessary and sufficient conditions for solvability of MRCP by PWA feedback analogous to the conditions in [5]. One advantage of MRCP is that it allows a solution that can be implemented using any triangulation. Indeed, we have shown that RCP by arbitrary triangulations and MRCP are equivalent for generic polytopes. Our future research will explore solvability of RCP on polytopes when neither simplex-based methods nor a monotonic solution is available.

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