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A Pursuit Strategy for Wheeled-vehicle Formations

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***Abstract*— Inspired by the so-called "bugs" problem from mathematics, we propose a cyclic pursuit strategy for multi-vehicle formations. A particular version of this pursuit problem is studied for a system of n wheeled vehicles, each subject to a single nonholonomic constraint, towards the achievement of certain trajectories in the plane. A full stability analysis is provided for the special case when $n = 2$ and it is revealed how the system's global behaviour can be shaped through appropriate controller gain assignments.**

I. INTRODUCTION

In this paper, we introduce a cyclic pursuit strategy for systems of n wheeled vehicles whereby each vehicle pursues the next modulo n. Multi-vehicle systems might find application in terrestrial, space, and oceanic exploration, military surveillance and rescue missions, or even automated highway systems. Hence, from an engineering standpoint, the question of how to prescribe desired *global* behaviours through the application of only simple and *local* interactions is of particular interest.

Indeed, patterns of this sort seem to appear in nature [3], although it is often argued that analysis of even the most simple cases can be an impractical task [2]. Much of multi-agent robotics research has focused on the use of reactive or behaviour-based techniques. However, the global outcome of these systems is often difficult to predict analytically. Thus, corresponding mathematical results are rare, as noted in [10], [12]. Yet, this has not deterred interest in analyzing "nearest-neighbour" strategies, where relatively simple navigational rules are employed locally to generate desired global formations [7], [8], [11], [13]. Others have studied aggregate behaviour in *swarms* of organisms, where operational models are analyzed for the purpose of potential engineering application (e.g., see [5] and its references). Still, many of these ideas have yet to be explored for agents subject to motion constraints, such as wheeled vehicles.

The so-called "bugs" problem refers to what is also variously known as the dogs, mice, ants, or beetles problem, and originally stems from the mathematics of *pursuit curves*, first studied by French scientist Pierre Bouguer (c. 1732). In 1877, Edouard Lucas asked, what trajectories would be generated if three dogs, initially placed at the vertices of an equilateral triangle, were to run one-after-the-other? Three years later, Henri Brocard replied with the answer that each dog would follow a logarithmic spiral and that the dogs would meet at a common point, known now as the *Brocard point* of a triangle. A modern variant of this problem has n ordered bugs that start at the vertices of a regular n-polygon. If each bug pursues the next modulo n (i.e., *cyclic* pursuit) at fixed speed, the bugs will trace out logarithmic spirals and eventually meet at the polygon's centre [1]. A similar result holds if each bug's speed is proportional to the distance between bugs. For a more complete historical review of cyclic pursuit, see [1] and references therein.

Consider a variation on this traditional pursuit problem where each "bug" is additionally subject to a single nonholonomic constraint, or equivalently, modelled as a kinematic unicycle. In this case, the unicycles will not generally be able to head towards their designated targets at each instant. Instead, depending on the allowed control energy, each vehicle will require some finite time to steer itself towards its preassigned leader. What trajectories can be generated?

In what follows, we introduce this concept of cyclic pursuit for multi-vehicle systems, and we study the case when each vehicle is subject to a single nonholonomic (rolling) constraint. A version of the described pursuit problem is explored, for a *homogeneous* system of unicycles, towards the achievement of certain geometric formations in the plane. The possible equilibria for a system of n interconnected vehicles are determined, and the special case when $n = 2$ is analyzed in full.

II. EQUATIONS OF PURSUIT

Consider the classical "bugs" problem, formalized as follows. Let there be n ordered agents $z_i = (x_i, y_i) \in \mathbb{R}^2$ with arbitrary initial conditions, where agent i pursues the next, $i+1$, modulo[1] n. Suppose the kinematics of each agent are described by an integrator $\dot{z}_i = u_i$, with control inputs

$$u_i = K_i(z_{i+1} - z_i) \tag{1}$$

for given constant matrices $K_i \in \mathbb{R}^{2\times 2}$. Thus, by adjusting the matrices $K_1, \ldots, K_n$, the group's behaviour can be assigned. As a simple example, let $n = 2$. Controls (1) then yield error $e = z_2 - z_1$ dynamics $\dot{e} = -(K_1 + K_2)e$. In particular, if $-(K_1 + K_2)$ is stable (resp. unstable) the agents will converge (resp. diverge). In the case of marginal stability, the agents in fact travel around a circle.

Now, suppose we extend the above *linear* pursuit scenario to one in which each agent is a kinematic unicycle with nonlinear state model

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{bmatrix} = \begin{bmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = G(\theta_i)u_i, \tag{2}$$

[1]Henceforth, all vehicle indices $i+1$ should be evaluated modulo n.

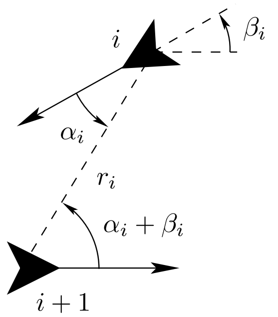

Fig. 1. New coordinates, with vehicle i in pursuit of $i+1$.

where $(x_i, y_i) \in \mathbb{R}^2$ denotes the i-th vehicle's Cartesian position, $\theta_i \in \mathbb{R}$ is the vehicle's orientation, and $u_i = (v_i, \omega_i) \in \mathbb{R}^2$ are control inputs. In this paper, we allow angles to take values in $\mathbb{R}$ to avoid a discontinuity in our feedback law, which depends on angles.

Let r_i denote the distance between vehicles i and $i+1$, and let α_i be the difference between the i-th vehicle's heading and the heading that would take it directly towards vehicle $i+1$ (see Fig. 1). In analogy with the linear controls (1), an intuitive control law for (2) is to assign vehicle i's linear speed v_i in proportion to r_i, while assigning its angular speed ω_i in proportion to α_i. In the sections that follow, we set out to study multi-vehicle systems of this sort.

A. Coordinate Transformation

Before beginning our analysis, it is useful to consider a transformation of coordinates into ones that involve the variables r_i and α_i. Firstly, for $q_i = (x_i, y_i, \theta_i)$, let

$$\tilde{q}_i = R(\theta_{i+1})(q_i - q_{i+1}),$$

where $R(\theta)$ is the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which yields dynamics

$$\dot{\tilde{q}}_i = G(\tilde{\theta}_i)u_i - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u_{i+1} + \begin{bmatrix} \omega_{i+1}\tilde{y}_i \\ -\omega_{i+1}\tilde{x}_i \\ 0 \end{bmatrix}.$$

In these coordinates, vehicle i views itself in a coordinate frame centred at vehicle $i+1$ and aligned with vehicle $i+1$'s heading. Define the variables (see Fig. 1)

$$\begin{aligned} r_i &= \sqrt{\tilde{x}_i^2 + \tilde{y}_i^2} \\ \alpha_i &= \arctan\left(\frac{\tilde{y}_i}{\tilde{x}_i}\right) + \pi - \tilde{\theta}_i \\ \beta_i &= \tilde{\theta}_i - \pi, \end{aligned}$$

with $r_i \in \mathbb{R}^+$ and $\alpha_i, \beta_i \in \mathbb{R}$. When $\tilde{x}_i = 0$, it is assumed that $\arctan\left(\frac{\tilde{y}_i}{0}\right)$ evaluates to $\pm\frac{\pi}{2}$, depending on the sign of $\tilde{y}_i$. After some (rather tedious) algebraic manipulation, the kinematic equations become

$$\begin{aligned} \dot{r}_i &= -v_i \cos\alpha_i - v_{i+1}\cos(\alpha_i + \beta_i) \\ \dot{\alpha}_i &= \frac{1}{r_i}\left[v_i \sin\alpha_i + v_{i+1}\sin(\alpha_i + \beta_i)\right] - \omega_i \\ \dot{\beta}_i &= \omega_i - \omega_{i+1}\,. \end{aligned} \tag{3}$$

This system describes the relationship between vehicle i and the one that it is pursuing, $i+1$, where r_i and α_i are as previously described. Note that, in these coordinates, it is assumed that $r_i > 0$.

B. Formation Control and Sample Simulations

As previously suggested, we investigate the case when

$$v_i = k_r r_i \quad \text{and} \quad \omega_i = k_\alpha \alpha_i \tag{4}$$

where $k_r, k_\alpha > 0$ are constant gains. Substituting these controls into (3) gives a system of n cyclically interconnected and identical subsystems

$$\begin{aligned} \dot{r}_i &= -k_r\left[r_i \cos\alpha_i + r_{i+1}\cos(\alpha_i + \beta_i)\right] \\ \dot{\alpha}_i &= k_r\left[\sin\alpha_i + \frac{r_{i+1}}{r_i}\sin(\alpha_i + \beta_i)\right] - k_\alpha \alpha_i \\ \dot{\beta}_i &= k_\alpha(\alpha_i - \alpha_{i+1})\,. \end{aligned} \tag{5}$$

At each instant in time, the multi-vehicle system's geometric formation in the plane can be described by a *pursuit graph*, defined as follows.

Definition 1 (Pursuit Graph): A *pursuit graph* G consists of a pair (V, E) such that

(i) V is a finite set of vertices, $|V| = n$, where each vertex $z_i = (x_i, y_i) \in \mathbb{R}^2$, $i \in \{1, \ldots, n\}$, represents the position of vehicle i in the plane, and;

(ii) E is a finite set of directed edges, $|E| = n$, where each edge $e_i : V \times V \to \mathbb{R}^2$, $i \in \{1, \ldots, n\}$, is the vector from z_i to z_{i+1}.

In other words, $e_i = z_{i+1} - z_i$ and consequently $\sum_i^n e_i = 0$ for vehicles in cyclic pursuit. Also, note that our coordinate $r_i = \|e_i\|_2$. In the next section, we employ this definition in characterizing the possible equilibrium formations of our multi-vehicle system.

Preliminary computer simulations suggest the possibility of achieving circular pursuit trajectories in the plane. Fig. 2, Fig. 3, and Fig. 4 show results for a system of $n = 5$ vehicles, initially positioned at random, where $k_\alpha = 1$ is fixed and k_r is different in each case. Note that, when $k_r = k^* := \frac{\pi}{10}\csc\left(\frac{\pi}{5}\right)$ in Fig. 2, the vehicles converge to evenly spaced motion around a circle with a pursuit graph that is similar to a regular pentagon. In Fig. 3 and Fig. 4, the vehicles converge to a point and diverge, respectively.

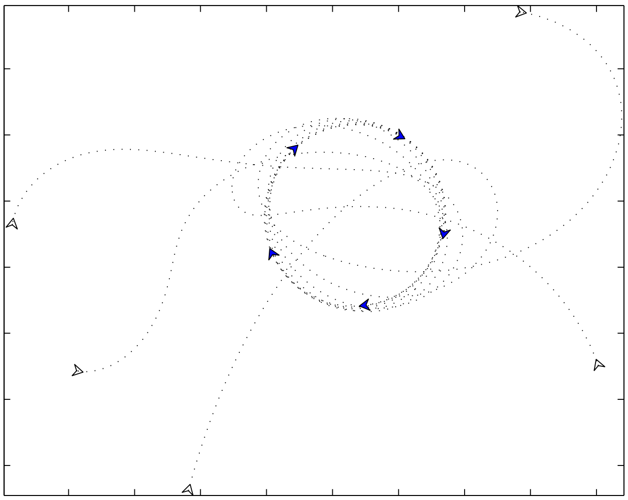

Fig. 2. Five vehicles, $k_\alpha = 1$, $k_r = k^*$.

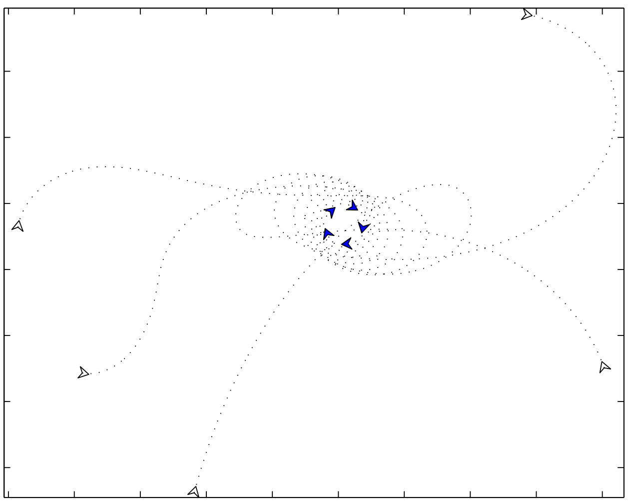

Fig. 3. Five vehicles, $k_\alpha = 1$, $k_r < k^*$.

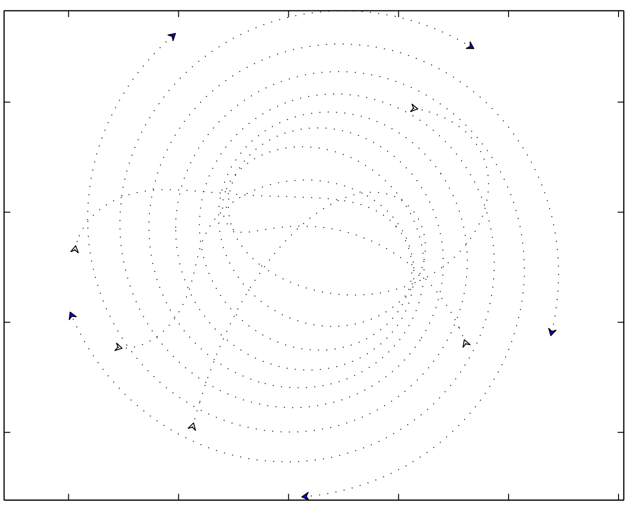

Fig. 4. Five vehicles, $k_\alpha = 1$, $k_r > k^*$.

III. GENERAL EQUILIBRIA

In this section, we analyze the system of interconnected vehicles (5) to determine the possible equilibrium formations under control law (4). Towards achieving this goal, we need to adequately describe the state of our system's pursuit graph at equilibrium. The following definition for a plane polygon has been adapted from [4] to allow for vertices that are not necessarily distinct and for directed edges.

Definition 2 (after [4], p. 93): Let n and $d < n$ be positive integers so that $p := n/d > 1$ is a rational number. Let T be the positive rotation in the plane, about the origin, through angle $2\pi/p$ and let $z_1 \neq 0$ be a point in the plane. Then, the points $z_{i+1} = Tz_i$, $i = 1, \ldots, n-1$ and edges $e_i = z_{i+1} - z_i$, $i = 1, \ldots, n$, define a *generalized regular polygon*, which is denoted $\{p\}$.

By this definition, $\{p\}$ can be interpreted as a directed graph with vertices z_i (not necessarily distinct) connected by edges e_i as determined by the ordering of points.

Since p is rational, the period of T is finite and, when n and d are coprime, this definition is equivalent to the well-known definition of a regular polygon as a polygon that is both *equilateral* and *equiangular*. Moreover, when $d = 1$, $\{p = n\}$ is an *ordinary* regular polygon (i.e., its edges do not cross one another). However, when $d > 1$ is coprime to n, $\{p\}$ is a *star* polygon since its sides intersect at certain extraneous points, which are not included among the vertices [4, pp. 93–94]. If n and d have a common factor $m > 1$, then $\{p\}$ has $\tilde{n} = n/m$ distinct vertices and $\tilde{n}$ edges traversed m times. Note that the trivial case when $d = n$ has not been included since this corresponds to the geometrically uninteresting situation where the vertices are all coincident (i.e., $r_i = 0$ for all i). In the next section we do consider the stability of such a point.

Fig. 5 illustrates some example possibilities for $\{p\}$ when $n = 9$. In the first instance, $\{9/1\}$ is an ordinary polygon. In the second instance, $\{9/2\}$ is a star polygon since 9 and 2 are coprime. In the third instance, the edges of $\{9/3\}$ traverse a $\{3/1\}$ polygon 3 times, because $m = 3$ is a common factor of both 9 and 3.

Lemma 1 (after [4], p. 94): The *internal angle* at every vertex of $\{p\}$ is given by $\psi = \pi(1 - 2d/n)$.

We are now ready to discuss the possible equilibrium formations for our system of n vehicles in cyclic pursuit.

Theorem 1: At equilibrium, the n-vehicle pursuit graph corresponding to (5) is a generalized regular polygon $\{p\}$, where $p = n/d$ and $d \in \{1, \ldots, n-1\}$. Consequently, the equilibrium angles in the range $[-\pi, \pi)$ are

$$\alpha_i = \pm\frac{\pi d}{n} \text{ and } \beta_i = \pm\left(\pi - \frac{2\pi d}{n}\right) \tag{6}$$

for all $i \in \{1, \ldots, n\}$.

Proof. When $\dot{\beta}_i = 0$, (5) yields $\alpha_i = \alpha_{i+1}$. When $\dot{r}_i = 0$,

$$\frac{-\cos\alpha_i}{\cos(\alpha_i + \beta_i)} = \frac{r_{i+1}}{r_i} > 0 \tag{7}$$

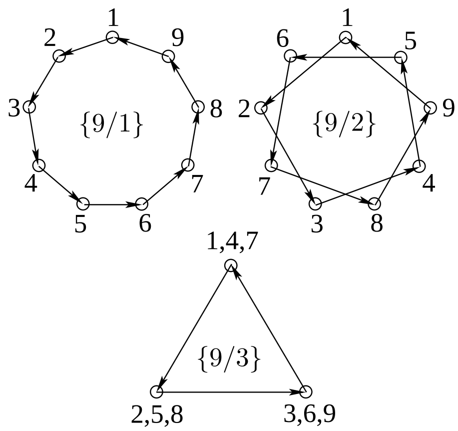

Fig. 5. Example generalized regular polygons $\{9/d\}$, $d \in \{1,2,3\}$.

and when $\dot{\alpha}_i = 0$,

$$k_\alpha \alpha_i = k_r \left[\sin\alpha_i - \cos\alpha_i \tan(\alpha_i + \beta_i)\right] \tag{8}$$

by substituting (7). The left-hand side of (8) is constant over i, thus β_i should satisfy $\beta_i = \beta_{i+1} + \pi a$, with $a \in \mathbb{Z}$. But since, by assumption, the right-hand side of (7) is strictly positive, its left-hand side cannot change sign, which implies a is even. For $\beta_i \in [-\pi, \pi)$, $\beta_i = \beta_{i+1}$. Consequently, $r_i = r_{i+1}$ since $r_i = r_{i+n}$.

Let $\bar{\alpha} \equiv \alpha_i$ and $\bar{\beta} \equiv \beta_i$ at equilibrium. Since $r_i = r_{i+1}$, the system's pursuit graph G is equilateral (i.e., $\|e_i\|_2 = \|e_{i+1}\|_2$). Let ψ_i be the internal angle at vertex i of the pursuit graph. The pursuit graph is equiangular (i.e., $\psi_i = \psi_{i+1}$) since it can be checked using the geometry of Fig. 1 that the internal angle at each vertex is given by

$$\bar{\psi} \equiv \psi_i = \begin{cases} \bar{\beta} & \text{for } \bar{\alpha} > 0 \\ -\bar{\beta} & \text{for } \bar{\alpha} < 0 \end{cases}$$

at equilibrium. Therefore, by Definition 2, the pursuit graph must correspond to a generalized regular polygon $\{p\}$.

Equation (7) simplifies to $\cos\bar{\alpha} = -\cos(\bar{\alpha}+\bar{\beta})$. For fixed $\bar{\alpha}$, it can be checked that $\bar{\beta} = \{\pi, \pi - 2\bar{\alpha}\}$. However, by Lemma 1, the internal angles of $\{p\}$ must sum to

$$n\bar{\psi} = n\pi\left(1 - \frac{2d}{n}\right) < n\pi,$$

since $d > 0$. Thus, $\bar{\beta} = \pi$ is not feasible for vehicles in cyclic pursuit, and so $\bar{\beta} = \pi - 2\bar{\alpha}$. Again, using Lemma 1, the internal angle $\bar{\psi} = \bar{\beta}$ at each vertex gives

$$\bar{\beta} = \pi\left(1 - \frac{2d}{n}\right)$$

which, together with $\bar{\beta} = \pi - 2\bar{\alpha}$, implies that

$$\bar{\alpha} = \frac{\pi d}{n}\,.$$

However, when $\bar{\alpha} < 0$, $\bar{\psi} = -\bar{\beta}$ implies that

$$\bar{\alpha} = \frac{-\pi d}{n}, \text{ and } \ \bar{\beta} = -\pi\left(1 + \frac{2d}{n}\right)$$

at equilibrium. ∎

The case when n and d of Theorem 1 are not coprime is physically undesirable (e.g., as in $\{9/3\}$ of Fig. 5) since it requires that multiple vehicles occupy the same point in space. From geometry, it is clear that, for each possible $\{n/d\}$ formation, $\bar{\alpha} = \pm\frac{\pi d}{n}$ corresponds exactly to a relative heading for each vehicle that points it in a direction that is *tangent* to the circle circumscribed by the vertices of the corresponding polygon.

At equilibrium, (8) simplifies to

$$\begin{aligned} k_r/k_\alpha &= \bar{\alpha}\left[\sin\bar{\alpha} + \sin(\bar{\alpha}+\bar{\beta})\right]^{-1} \\ &= \pm\frac{\pi d}{n}\left[\sin\left(\pm\frac{\pi d}{n}\right) + \sin\left(\pm\frac{\pi d}{n}\right)\right]^{-1} \\ &= \frac{\pi d}{2n}\csc\left(\frac{\pi d}{n}\right) =: k^*(n)\,. \end{aligned} \tag{9}$$

In other words, the ratio $k^*(n)$ must be as defined in order that an equilibrium (with $\bar{r} > 0$) exists. Thus, without loss of generality, we can choose $k_\alpha = 1$ and $k_r = k^*(n)$ to ensure the existence of regular polygon equilibria. For example, a polygon $\{5/1\}$ has $k^* = \frac{\pi}{10}\csc\left(\frac{\pi}{5}\right)$, which corresponds to the critical gain used to generate the results of Fig. 2.

IV. Stability Analysis for $n = 2$

In general, a full stability analysis of the multi-vehicle system (5) is not a trivial task. However, when $n = 2$ the analysis is simplified in that $r_1 = r_2$, $\alpha_2 = \alpha_1 + \beta_1$, and $\alpha_1 = \alpha_2 + \beta_2$ (see Fig. 6). Consequently, by choosing $k_\alpha = 1$ and $k_r = k \in \mathbb{R}^+$, system (5) reduces to

$$\dot{r}_1 = -kr_1\left[\cos\alpha_1 + \cos(\alpha_1+\beta_1)\right] \tag{10a}$$
$$\dot{\alpha}_1 = k\left[\sin\alpha_1 + \sin(\alpha_1+\beta_1)\right] - \alpha_1 \tag{10b}$$
$$\dot{\beta}_1 = -\beta_1 \tag{10c}$$
$$\begin{aligned} \dot{r}_2 &= -kr_2\left[\cos\alpha_2 + \cos(\alpha_2+\beta_2)\right] \\ \dot{\alpha}_2 &= k\left[\sin\alpha_2 + \sin(\alpha_2+\beta_2)\right] - \alpha_2 \\ \dot{\beta}_2 &= -\beta_2\,. \end{aligned}$$

Since the vehicle equations are decoupled, we drop the indices to simplify notation and proceed by analyzing (10).

The behaviour of this two-vehicle system depends on the choice of gain k. However, observe that when $\beta(0) = -2\alpha(0)$, subsystems (10b) and (10c) respectively reduce to $\dot{\alpha} = -\alpha$ and $\dot{\beta} = -\beta$ for all $t \geq 0$, independent of any particular choice for k.

Theorem 2: Consider $n = 2$ vehicles in cyclic pursuit, each with kinematics (10). Let $M = \{\xi = (\alpha,\beta) : \beta = -2\alpha\}$ and $k^* = \frac{\pi}{4}$ after (9). Then, (i) if $0 < k < k^*$ or if $\xi(0) \in M$ and $0 < k < \frac{5\pi}{4}$, the vehicles converge to a common point; (ii) if $k^* < k < \frac{5\pi}{4}$ and $\xi(0) \notin M$, the vehicles diverge, or; (iii) if $k = k^*$ and $\xi(0) \notin M$, the vehicles converge to equally spaced motion around a circle.

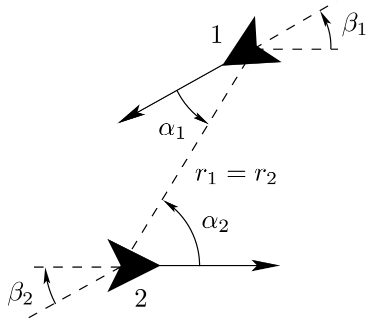

Fig. 6. Coordinates for $n = 2$ vehicles.

When $k \geq \frac{5\pi}{4}$ the analysis is further complicated by more equilibria, as will become clear in the proof.

In proving Theorem 2, we employ Theorem 10.3.1 of [6, p. 15], which is not reproduced here due to space restrictions.

Proof of Theorem 2. Since (10b) and (10c) do not depend on r, they can be viewed as an autonomous system in $\xi = (\alpha, \beta)$. Let $(\bar{\alpha}, \bar{\beta} = 0)$ denote an equilibrium point of (10b,c). From (10b), $\bar{\alpha}$ must satisfy

$$2k \sin \bar{\alpha} - \bar{\alpha} = 0 \tag{11}$$

at equilibrium. If $k \leq \frac{1}{2}$, (10b,c) has only one equilibrium point, namely $(0, 0)$, since $\bar{\alpha} = 0$ is the only solution to (11). However, when the gain k is increased to $\frac{1}{2} < k < \frac{5\pi}{4}$, a bifurcation occurs so that the system acquires two equilibrium points (locations dependent on k) in addition to the one at the origin. In general, the following cases exist.

Case I ($0 < k \leq \frac{1}{2}$)*:* In this case, $(0, 0)$ is the sole equilibrium point. System (10b,c) can be viewed as a pair of cascade connected subsystems (cf. Theorem 10.3.1 of [6])

$$\begin{aligned} \dot{\alpha} &= f_\alpha(\alpha, \beta) \\ \dot{\beta} &= f_\beta(\beta), \end{aligned}$$

where β is an input to (10b). We show that the origin of

$$\dot{\alpha} = f_\alpha(\alpha, 0) \tag{12}$$

is globally asymptotically stable (GAS). Let $V : \mathbb{R} \to \mathbb{R}$ be the continuously differentiable function $V(\alpha) = \frac{1}{2}\alpha^2$ which has the derivative along (12) given by $\dot{V}(\alpha) = -\alpha(\alpha - 2k \sin \alpha)$. But $\alpha > 0 \Rightarrow \alpha > 2k \sin \alpha \Rightarrow \dot{V} < 0$ and $\alpha < 0 \Rightarrow \alpha < 2k \sin \alpha \Rightarrow \dot{V} < 0$. Since $V(0) = 0, V(\alpha) > 0$ in $\mathbb{R} - \{0\}$, $V(\alpha)$ is radially unbounded, and $\dot{V}(\alpha) < 0$ in $\mathbb{R} - \{0\}$, the origin of (12) must be GAS by the Barbashin-Krasovskii theorem (cf. Theorem 4.2 of [9]). Choose $S = \mathbb{R}$.

It is clear that the origin of $\dot{\beta} = -\beta$ is GAS.

Next, we show that trajectories of $\dot{\alpha} = f_\alpha(\alpha, \beta(t))$ are bounded for all $t \geq 0$ and for every $\alpha(0) \in S$ by showing that trajectories of the full system (10b,c) are bounded for all trajectories starting at $\xi(0) \in \mathbb{R}^2$. Consider the positive definite function $V_\Omega : \mathbb{R}^2 \to \mathbb{R}$, $V_\Omega(\xi) = \frac{1}{2}\alpha^2 + \frac{1}{4}\beta^2$, which has the derivative along (10b,c) given by

$$\begin{aligned} \dot{V}_\Omega &= \alpha g(\xi) - \alpha^2 - \frac{\beta^2}{2} \\ &\leq -\frac{1}{2}\left(\alpha^2 + \beta^2\right) + \frac{1}{2} g^2(\xi) \\ &\leq -\frac{1}{2}\|\xi\|_2^2 + k^2 < 0 \end{aligned}$$

for all $\|\xi\|_2 > \sqrt{2}k$, where $g(\xi) = k[\sin \alpha + \sin(\alpha + \beta)]$. Let $\Omega = \{\xi \in \mathbb{R}^2 : V_\Omega \leq c\}$ with $c > k^2$, which corresponds to a ball of radius $\rho > \sqrt{2c}$ so that Ω defines a compact, positively invariant set with respect to (10b,c). Since we can take $\rho \to \infty$, it follows that solutions to $\dot{\alpha} = f_\alpha(\alpha, \beta(t))$ are bounded for all $t \geq 0$ and for all $\alpha(0) \in S$.

Having satisfied the conditions of Theorem 10.3.1 from [6], we conclude that $\lim_{t\to\infty} \alpha(t) = 0$ for all $\alpha(0) \in \mathbb{R}$, which implies that the origin of the full system (10b,c) is GAS[2] when $0 < k \leq \frac{1}{2}$. In a neighbourhood of the origin, $[\cos \alpha + \cos(\alpha + \beta)] > 0$, which by (10a) implies that, after some finite time $t^* > 0$, $r \to 0$ as $t \to \infty$ (i.e., the vehicles converge to a common point).

Case II ($\frac{1}{2} < k < k^*$)*:* In the cases that remain, the origin of (10b,c) is a saddle point and two equilibrium solutions to (11) exist, namely $\pm|\bar{\alpha}|$. It can be checked that $M = \{\xi : \beta = -2\alpha\}$ is invariant, making it a stable manifold of the origin. Thus, following the conclusion of Case I, for every $\xi(0) \in M$, $r \to 0$ as $t \to \infty$ for all k.

Consider a change of coordinates from (α, β) to (χ, β), where $\chi = 2\alpha + \beta$ and $\dot{\chi} = f_\chi(\chi, \beta)$ with

$$f_\chi(\chi, \beta) = 2k \sin\left(\frac{\chi}{2}\right) \cos\left(\frac{\beta}{2}\right) - \frac{\chi - \beta}{2}.$$

Let $S = \{\tilde{\chi} : \tilde{\chi} > 0\} \subset \mathbb{R}$. Define the function $V : \mathbb{R} \to \mathbb{R}$ by $V(\tilde{\chi}) = \frac{1}{2}\left(\tilde{\chi} - 2|\bar{\alpha}|\right)^2$ which has a derivative along the solutions of $\dot{\tilde{\chi}} = f(\tilde{\chi}, 0)$ given by

$$\dot{V}(\tilde{\chi}) = \underbrace{(\tilde{\chi} - 2|\bar{\alpha}|)}_{(*)} \underbrace{\left(2k \sin\left(\frac{\tilde{\chi}}{2}\right) - \frac{\tilde{\chi}}{2}\right)}_{(**)}.$$

Note that $(*) < 0$ for all $\tilde{\chi} < 2|\bar{\alpha}|$ and $(*) > 0$ for all $\tilde{\chi} > 2|\bar{\alpha}|$, where $\tilde{\chi} \in S$. Moreover, for $\tilde{\chi} \in S$

$$\begin{aligned} (**) < 0 &\Longleftrightarrow 2k \sin\left(\frac{\tilde{\chi}}{2}\right) < \frac{\tilde{\chi}}{2} \\ &\overset{(a)}{\Longleftrightarrow} \frac{\sin(\tilde{\chi}/2)}{\tilde{\chi}/2} < \frac{\sin|\bar{\alpha}|}{|\bar{\alpha}|} \\ &\overset{(b)}{\Longleftrightarrow} \tilde{\chi} > 2|\bar{\alpha}|. \end{aligned}$$

The equivalence (a) comes from (11) and the equivalence (b) follows from the fact that $|\bar{\alpha}| < \pi$ for $k < \frac{5\pi}{4}$ and the function $\sin x / x$ is strictly positive and monotone decreasing

[2] Interestingly, when $k = \frac{1}{2}$ the linearization of system (10b,c) cannot determine the stability of $(0, 0)$.

on $[0, \pi)$. It follows that $(**) > 0$ for $\tilde{\chi} < 2|\bar{\alpha}|$ when $\tilde{\chi} \in S$. Since $V(0) = 0$, $V(\tilde{\chi}) > 0$ in $S - \{2|\bar{\alpha}|\}$, and $\dot{V}(\tilde{\chi}) < 0$ in $S - \{2|\bar{\alpha}|\}$, the equilibrium point $\tilde{\chi} = 2|\bar{\alpha}|$ of $\dot{\tilde{\chi}} = f_\chi(\tilde{\chi}, 0)$ is asymptotically stable (AS) by Lyapunov's stability theorem (cf. Theorem 4.1 of [9]). Moreover, it can be checked (using the same argument given for $(**)$ above) that the set S is invariant with respect to $\dot{\tilde{\chi}} = f_\chi(\tilde{\chi}, 0)$, which implies that $\lim_{t\to\infty} \tilde{\chi}(t) = 2|\bar{\alpha}|$ for every trajectory starting in S.

It remains to show that trajectories of $\dot{\chi} = f_\chi(\chi, \beta(t))$ that start in S, remain in S for all $t \geq 0$. Suppose the converse is true and that for some $\chi(0) \in S$ it happens that $\chi(t_1) = 0$ at some time $t_1 > 0$. Then $\xi(t_1) \in M$. Since it has already been established that M is itself an invariant set, it must have been that $\chi(t) = 0$ for all future and past times. But this is a contradiction. Hence, trajectories of $\dot{\chi} = f_\chi(\chi, \beta(t))$ starting in S must remain in S for all $t \geq 0$.

Therefore, the conditions of Theorem 10.3.1 from [6] have been satisfied and so $\lim_{t\to\infty} \chi(t) = 2|\bar{\alpha}|$ for every $\alpha(0) \in S$, where $\chi(t)$ is the solution of $\dot{\chi} = f_\chi(\chi, \beta(t))$. In the original (α, β) coordinates, $\chi > 0$ corresponds to the condition that $\beta > -2\alpha$. Thus, together with the GAS of $\beta = 0$, this implies that all solutions starting in the set $S_+ = \{\xi : \beta > -2\alpha\}$ converge to $(|\bar{\alpha}|, 0)$. An identical argument can be used to show that all solutions starting in the set $S_- = \{\xi : \beta < -2\alpha\}$ converge to $(-|\bar{\alpha}|, 0)$. For $\frac{1}{2} < k < \frac{\pi}{4}$ this corresponds to $\bar{\alpha} \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, wherein (10a) yields $r \to 0$ as $t \to \infty$.

Case III ($k = \frac{\pi}{4}$)*:* In this case, the nonzero equilibria correspond to a $\{2/1\}$ polygon and are $(\pm\frac{\pi}{2}, 0)$ since $k^* = \frac{\pi}{4}$ according to (9). Indeed, these equilibria are AS following the technique of Case II. However, as $t \to \infty$, $\dot{r} \to 0$, thus $r \to \bar{r}$, where $\bar{r} > 0$ is some diameter of encirclement. Still, as noted in Case II, if $\xi(0) \in M$, $r \to 0$ as $t \to \infty$ for all k.

Case IV ($\frac{\pi}{4} < k < \frac{5\pi}{4}$)*:* When $k \geq \frac{5\pi}{4}$ more than three equilibria exist, further complicating the analysis. Therefore, we do not study gains equal to or exceeding $\frac{5\pi}{4}$. Again, following the technique of Case II, for every $\xi(0) \notin M$, the two equilibria $\bar{\alpha} \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ are AS, which by (10a) yields $r \to \infty$ as $t \to \infty$ (i.e., the vehicles diverge). ■

Whether the vehicles travel in the counterclockwise or clockwise direction depends on whether they start in S_+ or S_-, respectively. Also, the set of initial states $\xi(0) \in M$, for which changes in k have no effect, corresponds to $\alpha_1(0) = \alpha_2(0) + \beta_2(0) = -\alpha_2(0)$ (see Fig. 7a). Fig. 7b shows the special case when $\alpha_1(0) = \alpha_2(0) = 0$. Fig. 7c illustrates the case when $\alpha_1(0) = \pi$ and $\alpha_2(0) = -\pi$. Note that $\alpha_1(0) = \alpha_2(0) = \pi$ describes a geometric arrangement similar to that of Fig. 7c. However, in this case the system's behaviour does depend on k.

V. CONCLUSION

Over the last century, several pursuit problems have appeared in the mathematical literature. In this paper, we have introduced a pursuit strategy for multi-vehicle systems that is in essence a nonlinear version of the so-called "bugs" problem from mathematics. A particular version of this pursuit problem has been studied for a system of unicycles, towards the achievement of certain circular trajectories in the plane. A full stability analysis has been provided for the special case when $n = 2$ and it was shown that changes can be made to the system's global behaviour through appropriate controller gain assignments.

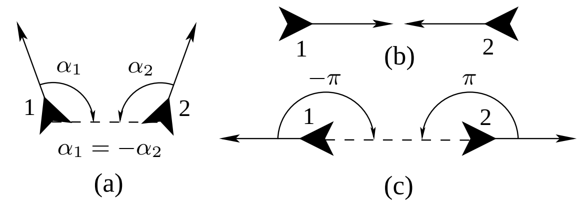

Fig. 7. Possible configurations for $\xi(0) \in M$.

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