Time Optimal Swing-Up of the Planar Pendulum

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***Abstract*— This paper presents results on the global structure of the time optimal trajectories of the planar pendulum on a cart. Relying on the geometric theory of time optimal synthesis, we provide a discontinuous feedback giving optimal solutions for any initial data.**

I. Introduction

This paper concerns the global structure of the time optimal trajectories to swing up a planar pendulum on a cart. We consider only the dynamics of the pendulum and take the acceleration of the cart as the control input. Global stabilization of the pendulum system has been studied as a benchmark for nonlinear control by many researchers, for instance, [10] and [2], to name a few. Time optimal synthesis has been studied recently in [3] and [14]. These papers are focused on computing exact switching times for an open loop control starting from the down equilibrium. In contrast, we are interested in computing a globally defined feedback control. In particular, some results concerning the problem of swinging up the pendulum via feedback in minimum time from any initial condition, have been recently obtained in [9].

Our approach is that of geometric time optimal control. For the general theory, the first results are probably those of Baitman [4], [5]. Next, a series of works of Sussmann dealt with the analytic case [12], [13]. Finally, the generic $\mathcal{C}^\infty$ case was treated in [7], [8]. A general account of these results, together with a complete analysis of singularities and the minimum time function, can be found in [6].

II. Analysis of the pendulum system

We consider the time optimal synthesis of the pendulum with equations of motion given by

$$\dot{x} = F(x) + G(x)u \tag{1}$$

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where $x \in \mathbb{S} \times \mathbb{R}$, $|u| \leq 1$, $F(x) = \begin{pmatrix} x_2 \\ \sin x_1 \end{pmatrix}$ and $G(x) = \begin{pmatrix} 0 \\ -\cos x_1 \end{pmatrix}$.

Problem. *For every* $(\bar{x}_1, \bar{x}_2) \in \mathbb{S} \times \mathbb{R}$ *find a trajectory-control pair* $(\gamma(\cdot), u(\cdot))$ *defined on* $[0, T]$ *and such that* γ *is time optimal between* $\gamma(0) = (\bar{x}_1, \bar{x}_2)$ *and* $\gamma(T) = 0$.

This problem is meaningful since the system (1) is globally stabilizable to the origin, as it can be proven by applying classical results on global controllability. Moreover, for every initial datum $(\bar{x}_1, \bar{x}_2)$, the existence of a time optimal trajectory reaching the origin can be derived, when $u(\cdot)$ belongs to the class of measurable functions with $|u| \leq 1$, from Filippov Theorem (see [1], Corollary 10.7, p. 143).

Now we apply the general theory of [6]. We define, for every *covector* $\lambda \in \mathbb{R}^2$, the *Hamiltonian* as

$$H(x,\lambda,u) = \lambda\cdot(F(x)+uG(x)) = \lambda_1 x_2 + \lambda_2(\sin x_1 - u\cos x_1)\,.$$

Then the Pontryagin Maximum Principle (PMP) says that if (γ, u) is a time optimal trajectory-control pair, then there exists a nontrivial field of covectors λ along γ and a constant $\lambda_0 \leq 0$ such that for a.e. $t \in [0, T]$
(i) λ satisfies $\dot{\lambda} = -\frac{\partial H}{\partial x}(\gamma(t), \lambda(t), u(t))$,
(ii) $H(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$,
(iii) $H(\gamma(t), \lambda(t), u(t)) = \max_{u' \in [-1,1]} H(\gamma(t), \lambda(t), u')$.
A trajectory that satisfies the PMP is called an *extremal trajectory*. In particular condition (i) translates into the following differential equation

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_2(\cos x_1 + u \sin x_1) \\ \dot{\lambda}_2 &= -\lambda_1\,, \end{aligned} \tag{2}$$

while from (iii) one easily derives the optimal control $u^*(t) = \mathrm{sgn}(\phi(t))$, where $\phi = -\lambda_2 \cos x_1$ is called the *switching function*. From this we deduce that switchings occur only if $\lambda_2 = 0$ or if $x_1 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

Note that pendulum system does not satisfy all the generic conditions in ([6], p. 48). Nevertheless, the analyticity of (1) allows to exclude pathological behaviours [13], such as the Fuller phenomenon, so that we will be able to construct an optimal synthesis.

A. Extremal trajectories

In this section we identify properties of the extremal trajectories. We assume w.l.o.g. that extremal trajectories reach the origin at $t = 0$; thus, time is negative and increasing. First, it can be verified that there are no optimal trajectories containing singular arcs. Next, we consider the bold, black curves in Figure 1 called γ^+ and γ^-.

Definition 1: Let γ^+ (resp. γ^-) be the trajectory of (1) defined on $(-\infty, 0]$ that reaches the origin with $u = 1$ (resp. $u = -1$) at time $t = 0$ and such that the control switches occur exactly at $x_1 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$. The controls corresponding to γ^+ and γ^- are a.e. $u(t) = \operatorname{sgn}(\cos x_1(t))$ and $u(t) = -\operatorname{sgn}(\cos x_1(t))$, respectively.

The following result exhibits the main properties of the extremal trajectories for the pendulum system, and it provides our main guide for determining the switching curves.

Proposition 1: Consider a bang-bang trajectory of (1), $x(\cdot) = (x_1(\cdot), x_2(\cdot)) : [t_1, t_2] \to \mathbb{R}$, with $x_2(t) \neq 0$ on (t_1, t_2) and let S be the set of switching times of $x(\cdot)$ and $K = \{t \in (t_1, t_2) : \ x_1(t) = \frac{\pi}{2} + k\pi\,, \ \exists k \in \mathbb{Z}\}$. Then $x(\cdot)$ is extremal if and only if one of the following three possibilities is satisfied:

(i) $K = S$,

(ii) there exists $\bar{t} \in (t_1, t_2) \setminus K$ such that $S = K \cup \{\bar{t}\}$ and $u(t) = -\operatorname{sgn}(x_2)\operatorname{sgn}(\cos x_1(t))$ a.e. on $(t_1, \bar{t})$, while $u(t) = \operatorname{sgn}(x_2)\operatorname{sgn}(\cos x_1(t))$ a.e. on $(\bar{t}, t_2)$

(iii) there exists $\bar{t} \in K$ such that $S = K \setminus \{\bar{t}\}$ and $u(t) = -\operatorname{sgn}(x_2)\operatorname{sgn}(\cos x_1(t))$ a.e. on $(t_1, \bar{t})$, while $u(t) = \operatorname{sgn}(x_2)\operatorname{sgn}(\cos x_1(t))$ a.e. on $(\bar{t}, t_2)$.

Proof: Let $x(\cdot)$ be an extremal trajectory which does not intersect the x_1 axis. Then we have $u(t) = \operatorname{sgn}(\phi(t)) = -\operatorname{sgn}(\lambda_2(t))\operatorname{sgn}(\cos x_1(t))$. Therefore (i) is equivalent to $\operatorname{sgn}(\lambda_2(t)) = \varepsilon \quad a.e$ (where $\varepsilon \in \{-1, 1\}$), while one among the cases (ii) and (iii) holds if and only if $\operatorname{sgn}(\lambda_2(t)) = \operatorname{sgn}(x_2(t))$ a.e on $(t_1, \bar{t})$ and $\operatorname{sgn}(\lambda_2(t)) = -\operatorname{sgn}(x_2(t))$ a.e on $(\bar{t}, t_2)$. Assume that $x(\cdot)$ does not satisfy (i), then in particular there exists $\bar{t}$ such that $\lambda_2(\bar{t}) = 0$. Then, using the fact that $H = \lambda_1(\bar{t})x_2(\bar{t}) > 0$, we obtain $\operatorname{sgn}(\lambda_1(\bar{t})) = \operatorname{sgn}(x_2(\bar{t}))$. From this equality and since $\dot{\lambda}_2 = -\lambda_1$, we find that $\operatorname{sgn}(\lambda_2(t)) = \operatorname{sgn}(x_2(\bar{t}))$ on $(\bar{t} - \epsilon, \bar{t})$ and $\operatorname{sgn}(\lambda_2(t)) = -\operatorname{sgn}(x_2(\bar{t}))$ on $(\bar{t}, \bar{t} + \epsilon)$. To prove that $x(\cdot)$ satisfies either (ii) or (iii) it is enough to see that, if the sign of x_2 is fixed, then there is only one time $\bar{t}$ with $\lambda_2(\bar{t}) = 0$. Assume by contradiction that $\bar{t}_1 < \bar{t}_2$ are such that $\lambda_2(\bar{t}_1) = \lambda_2(\bar{t}_2) = 0$ and $\lambda_2(\cdot) \neq 0$ on $(\bar{t}_1, \bar{t}_2)$. Then, since $\operatorname{sgn}(\lambda_2(t)) = -\operatorname{sgn}(x_2)$ on $(\bar{t}_1, \bar{t}_1 + \epsilon)$ and $\operatorname{sgn}(\lambda_2(t)) = \operatorname{sgn}(x_2)$ on $(\bar{t}_2 - \epsilon, \bar{t}_2)$ the continuous function $\lambda_2(\cdot)$ must be zero somewhere on $(\bar{t}_1, \bar{t}_2)$ and we find a contradiction. We have therefore proved that every extremal trajectory with $x_2(\cdot) \neq 0$ satisfies one among (i), (ii) and (iii). Conversely, it is clear from the previous arguments that every bang-bang trajectory $x(\cdot)$ satisfying (ii) or (iii) satisfies the PMP with $\lambda(\bar{t}) = (\operatorname{sgn}(x_2), 0)$. When (i) holds the PMP is satisfied with $\lambda(t_1) = (\operatorname{sgn}(x_2), 0)$ if $u(\cdot) = \operatorname{sgn}(x_2)\operatorname{sgn}(\cos x_1(\cdot))$ or $\lambda(t_2) = (\operatorname{sgn}(x_2), 0)$ if $u(\cdot) = -\operatorname{sgn}(x_2)\operatorname{sgn}(\cos x_1(\cdot))$. ■

The following is a straightforward consequence of Proposition 1.

Corollary 1: The trajectory γ^+ (resp. γ^-) is extremal on any interval $[\bar{t}, 0]$, $\bar{t} < 0$, and for every point p of γ^+ (resp. γ^-), there exists an extremal trajectory that reaches γ^+ (resp. γ^-) for the first time at p and then follows γ^+ (resp. γ^-) until it touches the origin.

The results so far on extremal trajectories permit a partition of the cylinder into regions in which the behaviour of extremal trajectories has common qualitative properties.

Referring to Figure 1, we describe boundaries of these regions by identifying particular trajectories. Notice in the following descriptions that we make no distinction between a trajectory and its support. Let $\xi^+ : [0, +\infty) \to \mathbb{S} \times \mathbb{R}$ be the trajectory of (1) corresponding to $u = \operatorname{sgn}(\cos x_1(t))$, such that $\lim_{t \to +\infty} \xi^+(t) = (\frac{\pi}{4}, 0)$, $\xi_2^+(0) = 0$ and $\xi_2^+(t) < 0\,, \ \forall t > 0$. Let $\chi^+ : \mathbb{R} \to \mathbb{S} \times \mathbb{R}$ be the trajectory of (1) corresponding to $u = \operatorname{sgn}(\cos x_1(t))$, such that $\lim_{t \to +\infty} \chi^+(t) = (\frac{\pi}{4}, 0)$, $\chi_1^+(0) = 0$ and $\chi_1^+(t) \neq 0\,, \ \forall t > 0$. Let $\eta^+ : (-\infty, 0] \to \mathbb{S} \times \mathbb{R}$ be the trajectory of (1) corresponding to $u = \operatorname{sgn}(\cos x_1(t))$, such that $\lim_{t \to -\infty} \eta^+(t) = (\frac{\pi}{4}, 0)$, and $\eta^+ \cap \gamma^- = \eta^+(0)$. Let $t^+ < 0$ be the largest negative time such that the x_1 component of $\gamma^+(t^+)$ vanishes.

Regions **A**, **B**, **C**, **D**, **A′**, **B′**, **C′**, **D′** are then defined by means of the previous trajectories as in Figure 1.

Now we give a brief description of the trajectories of (1) satisfying the conditions given by Proposition 1 inside the regions **A**, **B**, **C**, and **D**, until they reach their corresponding boundaries.

- If the initial condition for the minimization problem is inside **A** then it is easy to see that the trajectories corresponding to $u = -\operatorname{sgn}(\cos x_1(t))$ must reach the boundary of **A** at some point of γ^+. On the other hand all the trajectories corresponding to $u = \operatorname{sgn}(\cos x_1(t))$ reach the boundary of **D**, **D′** or **C′**.

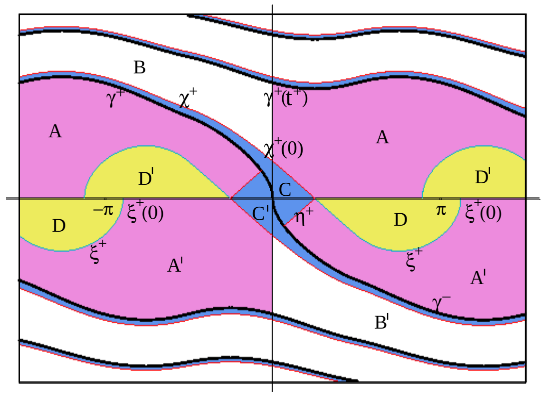

Fig. 1. Partition of $\mathbb{S} \times \mathbb{R}$

• If the initial condition is inside **B** then all the trajectories reach the boundary in a point of γ^+ or of the segment connecting $\chi^+(0)$ and $\gamma^+(t^+)$.
• If the initial condition is inside **C** then the trajectories corresponding to $u = \operatorname{sgn}(\cos x_1(t))$ stay in **C** until they reach γ^-, while every trajectory corresponding to $u = -\operatorname{sgn}(\cos x_1(t))$ must cross χ^+.
• All the trajectories that start inside **D** must cross the x_1 axis.
The descriptions of the trajectories starting from the regions **A′**, **B′**, **C′** and **D′** are analogous.

B. Switching curves

Collecting the results of the previous section, we find that every extremal trajectory reaching the origin and belonging to the upper or lower half plane can switch only on $\gamma^\pm$ or if $x_1 = \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$. Therefore, in order to detect other nontrivial switching curves, we need to look for extremal trajectories reaching the origin and crossing the x_1 axis. These switching curves are identified with the regular submanifold of points such that $\lambda_2 = 0$.

The following observations will facilitate our analysis both of switching curves and of overlap curves. For a fixed value of u, the system (1) admits a first integral

$$h(x) = \frac{1}{2}x_2^2 + \cos x_1 + u \sin x_1 \,. \tag{3}$$

Moreover we know that along an extremal trajectory the value of the Hamiltonian is a constant, which we call H. Using these two facts, we would like to obtain an explicit formula for λ_2 in order to find the switching curves. We have $-\dot{\lambda}_2 x_2 + \lambda_2(\sin x_1 - u\cos x_1) = H$ so $\dot{\lambda}_2 = \frac{\lambda_2}{x_2}(\sin x_1 - u\cos x_1) - \frac{H}{x_2}$. If $x_2 \neq 0$ and $\dot{x}_2 \neq 0$ we can locally view λ_2 as a function of x_2 and we obtain

$$\frac{\dot{\lambda}_2}{\dot{x}_2} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\lambda_2}{x_2} - \frac{H}{x_2(\sin x_1 - u\cos x_1)} \,. \tag{4}$$

The right-hand side of this equation can be written in terms of x_2 only by using $h(x)$. If $u = \pm 1$ we have $\sin x_1 - u\cos x_1 = \sqrt{2}\sin(x_1 - u\frac{\pi}{4})$. while $h(x) = \frac{1}{2}x_2^2 + \sqrt{2}\cos(x_1 - u\frac{\pi}{4})$. Combining this information we obtain

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\lambda_2}{x_2} - \varepsilon\frac{H}{x_2\sqrt{2 - (h - \frac{1}{2}x_2^2)^2}} \tag{5}$$

where $\varepsilon = \operatorname{sgn}(\sin x_1 - u\cos x_1)$ is the only term depending on x_1. The explicit solution to this equation, with initial condition $\lambda_2(x_2^0) = \lambda_2^0$, is:

$$\lambda_2(x_2) = x_2\left(\frac{\lambda_2^0}{x_2^0} - \varepsilon H \int_{x_2^0}^{x_2} \frac{\mathrm{d}y}{y^2\sqrt{2 - (h - \frac{1}{2}y^2)^2}}\right). \tag{6}$$

This integral cannot be solved exactly, but it can be written in terms of elliptic integrals, and therefore it can be easily computed numerically.

The formula (6) for λ_2 is applied on intervals where ε and u are constant. At switching points, the value of h must be updated by using the previous value of h and the value of x at the switching point. In this manner the formula (6) can be used to derive an equation for a particular switching curve $\lambda_2(x_2) = 0$.

The above observations are also useful for determining overlap curves, so we briefly outline the technique here, though it will be applied later in Section II-C. Overlap curves occur where two or more extremal trajectories reach the origin in minimum time. In order to compute them, we would like to derive an explicit formula for the time elapsed for an extremal trajectory to reach the origin. From above we know that

$$\dot{x}_1 = x_2 = \operatorname{sgn}(x_2)\sqrt{2\big(h(x) - \sqrt{2}\cos(x_1 - u\frac{\pi}{4})\big)} \tag{7}$$

and, similarly

$$\dot{x}_2 = \operatorname{sgn}\left(\sin\left(x_1 - u\frac{\pi}{4}\right)\right)\sqrt{2 - \left(h(x) - \frac{x_2^2}{2}\right)^2} \,. \tag{8}$$

Using equation (8) we obtain a formula for $T(x^0, x^1)$, the time elapsed along a extremal trajectory starting from x^0 and ending at x^1, as a function of x_2

$$T(x^0, x^1) = \int_{x_2^0}^{x_2^1} \frac{\mathrm{d}y}{\sqrt{2 - \left(h(x) - \frac{y^2}{2}\right)^2}} \,. \tag{9}$$

This formula must be used on segments of an extremal

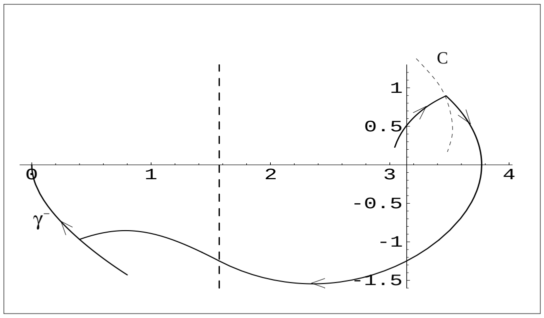

Fig. 2. Switching curve C.

trajectory where the control does not switch and where $\operatorname{sgn}\left(\sin\left(x_1 - u\frac{\pi}{4}\right)\right)$ is constant.

1) Switching curve C: In this section we consider the family of extremal trajectories that cross, with $u = -1$, the segment of the x_1 axis between $\mathbf{D}'$ and $\mathbf{A}'$ with $x_1 < \frac{3\pi}{2}$, then switch from $u = -1$ to $u = 1$ on the line $x_1 = \frac{\pi}{2}$, and finally, switch at a point x^0 on γ^-, and follow γ^- until reaching the origin. See Figure 2. Let C be (if it exists) the set of points belonging to these trajectories and corresponding to $\lambda_2 = 0$. We want to determine an equation that describes C. To do this we follow backwards the extremal trajectories that reach the origin, integrating by quadratures the equation (6).

Take an extremal trajectory $\tilde{\gamma}$ as described above, and assume that the last switching before reaching the origin is at $x^0 = (x_1^0, x_2^0) \in \gamma^-$, with $x_1^0 < \pi/4$ and with $\lambda_2^0 = 0$. (We will comment on the case $x_1^0 \geq \pi/4$ below). Also, let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be a point of $\tilde{\gamma}$ with $\bar{x}_1 > 3\pi/4$ and $\bar{x}_2 < 0$. Recall that H is a fixed positive real number along $\tilde{\gamma}$, and we can rescale $\lambda_1(x_2^0)$ such that $H = 1$. We must determine the value of $h(\cdot)$ along each bang arc of $\tilde{\gamma}$. Using (3) we have that $h(\cdot) = 1$ along γ^-; $h(\cdot) = h_1 := 1 + 2\sin x_1^0$ for the bang arc of $\tilde{\gamma}$ when $x_1 \in [x_1^0, \frac{\pi}{2}]$; and $h(\cdot) = h_2 := -1 + 2\sin x_1^0$ for the bang arc of $\tilde{\gamma}$ when $x_1 \in [\frac{\pi}{2}, \bar{x}_1]$. Also, referring to Figure 2, the segments on which $\tilde{\gamma}$ is monotone in x_2 are $x_1 \in [x_1^0, \frac{\pi}{4}]$, $x_1 \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, and $x_1 \in [\frac{3\pi}{4}, \bar{x}_1]$. Also, $\varepsilon = -1$ when $x_1 \in [x_1^0, \frac{\pi}{4}]$, $\varepsilon = +1$ when $x_1 \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, and $\varepsilon = -1$ when $x_1 \in [\frac{3\pi}{4}, \bar{x}_1]$, where $\varepsilon = \operatorname{sgn}(\sin x_1 - u\cos x_1)$. Combining this information with (6) we obtain

$$\begin{aligned}\lambda_2(\bar{x}_2) = H\bar{x}_2\Big(&\int_{x_2^0}^{x_2(x_1=\frac{\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} \\ &- \int_{x_2(x_1=\frac{\pi}{4})}^{x_2(x_1=\frac{\pi}{2})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \int_{x_2(x_1=\frac{\pi}{2})}^{x_2(x_1=\frac{3\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} \\ &+ \int_{x_2(x_1=\frac{3\pi}{4})}^{\bar{x}_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}\Big),\end{aligned} \tag{10}$$

where $x_2^0 = -\sqrt{2(1-\cos x_1^0 + \sin x_1^0)}$, $x_2(x_1 = \frac{\pi}{4}) = -\sqrt{2(1 + 2\sin x_1^0 - \sqrt{2})}$, $x_2(x_1 = \frac{\pi}{2}) = -2\sqrt{\sin x_1^0}$ and $x_2(x_1 = \frac{3\pi}{4}) = -\sqrt{2(-1 + 2\sin x_1^0 + \sqrt{2})}$

Note that the integrals in (10) are generalized integrals; that is, the integrands are not well-defined at the extremes. The formula (10) applies to $\bar{x}_2 < 0$. Now this must be connected with the segment of $\tilde{\gamma}$ for which $x_2 > 0$, by using a continuity argument. In particular, we use the continuity of λ at the intersection of $\tilde{\gamma}$ and the x_1 axis. The value of λ at the intersection point can be derived from (10) and from the equation (2) by passing to the limit as $\bar{x}_2$ goes to 0. The result is a function of switching point x^0 and the choice of H. However, we observe that $H = \lambda_2(\sin x_1 - u\cos x_1)$ if $x_2 = 0$, so the value of λ_2 at a point of the x_1 axis is the same for any extremal trajectory with Hamiltonian H and passing through that point with $u = -1$. So the fact that λ_2 must be continuous at the intersection between $\tilde{\gamma}$ and the x_1 axis is not sufficient to determine the corresponding point of C. We must use the additional information that $\lambda_1 = -\dot{\lambda}_2$ is also continuous. We have that

$$\lim_{x_2\to 0^-} \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \lim_{x_2\to 0^+} \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\dot{\lambda}_2}{\dot{x}_2}\Big|_{x_2=0}. \tag{11}$$

Assume that there exists a switching point $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in \tilde{\gamma}$ with $\tilde{x}_2 > 0$ and $\tilde{x}_1 > \frac{3\pi}{4}$, so that we have $u = -1$ and $\varepsilon = -1$ between $\tilde{x}$ and the axis x_1. We know that $\lambda_2 = 0$ at the point $\tilde{x}$. Then we can find an expression for $\lambda_2(x_2),\ x_2 > 0$ similar to (10)

$$\lambda_2(\bar{x}_2) = H\bar{x}_2 \int_{\tilde{x}_2}^{\bar{x}_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}. \tag{12}$$

Now we differentiate (10) and (12) with respect to $\bar{x}_2$, and after some manipulation pass to the limit as $\bar{x}_2$ tends to 0. The resulting expressions must be equal thanks to (11). Finally, we arrive at the following equation

$$\begin{aligned}&\int_{x_2^0}^{x_2(x_1=\frac{\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \int_{x_2(x_1=\frac{\pi}{4})}^{x_2(x_1=\frac{\pi}{2})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} \\ &- \int_{x_2(x_1=\frac{\pi}{2})}^{x_2(x_1=\frac{3\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} \\ &- \frac{1}{4(2-h_2^2)} \int_{x_2(x_1=\frac{3\pi}{4})}^{0} \frac{y^2\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} \\ &= \frac{\sqrt{2-(h_2-\frac{1}{2}\tilde{x}_2^2)^2}}{\tilde{x}_2(2-h_2^2)} - \frac{1}{4(2-h_2^2)} \int_{\tilde{x}_2}^{0} \frac{y^2\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}.\end{aligned} \tag{13}$$

This gives $\tilde{x}_2$ (and therefore also $\tilde{x}_1$, since $h(\tilde{x}_1, \tilde{x}_2) = h_2$) in terms of $x^0 \in \gamma^-$. Therefore the switching curve C can be determined by solving numerically the previous equation.

We conclude this section with a few remarks. First, there exists a switching curve C' symmetric to C ($x \in C$ if and only if $-x \in C'$). Second, for the same family of extremal trajectories, any switching curve that precedes C (in time) cannot be optimal. Indeed, if we assume that the x_1 coordinate for the new switching point is less than $3\pi/2$ (otherwise there would be a self-intersection of the corresponding extremal trajectory that would imply a loss of optimality) and we write the equality obtained from (11), it turns out that the right-hand side and the left-hand side of such an equation has different signs. For the case when $x_1^0 \geq \frac{\pi}{4}$, one simply removes the first integral in (13) and modifies the lower limit of the second integral to be x_2^0. By solving numerically the resulting equation, one obtains an expression for a switching curve which starts at the final point of the switching curve obtained by (13). Finally we mention that it is possible to prove that the only switching curves inside regions **C**, **C**$'$ are those corresponding to $x_1 = \frac{\pi}{2} + k\pi$.

C. Overlap curves and switching curves around $(\pi, 0)$

In Section II-B.1 we examined a candidate switching curve C and we analyzed an extremal trajectory that switches at $x^0 \in \gamma^-$. We can determine by direct computation that if x_1^0 is large enough, then the corresponding switching point of C, obtained by solving (10), belongs to region **A**. Therefore there exists a second extremal trajectory starting at $\bar{x}$ with control $u = 1$, then switching at $x_1 = \frac{3\pi}{2}$ and reaching the origin with an arc of γ^+. It is also possible to determine numerically that if x_1^0 is larger than a value ≈ 0.53, the extremal trajectory starting at $\bar{x} \in C$ with $u = 1$ is time optimal.

We deduce that the switching curve C is not completely optimal and there exists an overlap curve K_1 which starts at a point of C. On the other hand, it is possible to see, still numerically, that there is a point of C corresponding to $x_1^0 \approx 0.3$, at which the tangent vector to C is parallel to the vector field corresponding to $u = 1$. At this point a second overlap curve K_2 starts. See Figure 3. Associated to each point of this curve there are two optimal trajectories: the first one starts with control $u = -1$ and reaches γ^- at a point corresponding to a small value of x_1^0, while the second one starts with control $u = 1$, then switches on C and ends again on γ^-. The curve K_2 ends at a point in which a further overlap curve K_3 is generated. This curve contains the point $(\pi, 0)$ and the two time optimal

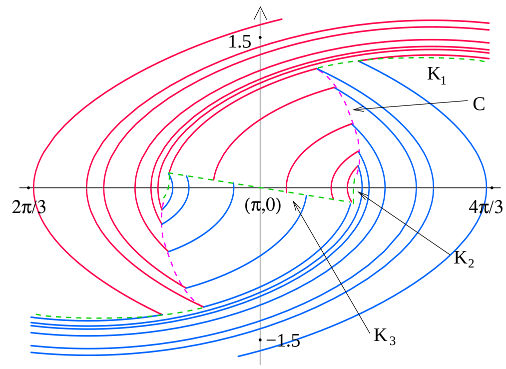

Fig. 3. Optimal synthesis around $(\pi, 0)$

trajectories starting at a point of K_3 have a symmetric behaviour, in the sense that they switch for the first time, respectively, on C and C' and for the last time on γ^- and γ^+. The complete synthesis around the point $(\pi, 0)$ can be completed using symmetry with respect to the point $(\pi, 0)$. Figure 3 provides a sketch of the optimal synthesis around the point $(\pi, 0)$.

In order to show how to determine explicitly the overlap curves previously defined we derive an equation for K_1. As in Section II-B.1, we consider an extremal trajectory that starts at a point $\bar{x} \in K_1$ and whose last switching before reaching the origin with $u = -1$ occurs at $x^0 \in \gamma^-$. Using equations (7) and (8) on each bang arc where the extremal trajectory is either strictly monotone in x_1 or in x_2, it is possible to obtain a formula for the elapsed time, as in (9). We assume that $x_1^0 < \pi/2$, but if not, one can proceed in the same way to find a similar equation. This procedure can now be repeated for the other trajectory that reaches the origin by switching at a point $x^1 \in \gamma^+$. By propagating values of $h(\cdot)$ on each bang arc it can be determined that $x_1^1 := 2\pi - \arcsin\left(\sin x_1^0 + \sin \bar{x}_1\right)$. The equation for the overlap curve is obtained by setting the times to reach the origin along the two possible extremal trajectories equal, to yield:

$$\int_0^{x_2^0} \frac{\mathrm{d}y}{\sqrt{2-(1-\frac{1}{2}y^2)^2}} + \int_{x_1^0-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_1 - 2\sqrt{2}\cos y}}$$
$$+ \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_2 - 2\sqrt{2}\cos y}} + \int_0^{2\sqrt{\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}$$
$$+ \int_0^{\bar{x}_2} \frac{\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} = \int_{\bar{x}_1-\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_3 - 2\sqrt{2}\cos y}}$$
$$+ \int_{\frac{7\pi}{4}}^{x_1^1+\frac{\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_4 - 2\sqrt{2}\cos y}} + \int_0^{x_2^1} \frac{\mathrm{d}y}{\sqrt{2-(1-\frac{1}{2}y^2)^2}}$$

where
$x_2^0 = -\sqrt{2(1-\cos x_1^0+\sin x_1^0)}$, $\bar{x}_2 = \sqrt{2(h_2-\cos\bar{x}_1+\sin\bar{x}_1)}$, $x_2^1 = \sqrt{2(1-\cos x_1^1-\sin x_1^1)}$, $h_1 = 1+2\sin x_1^0$, $h_2 =$

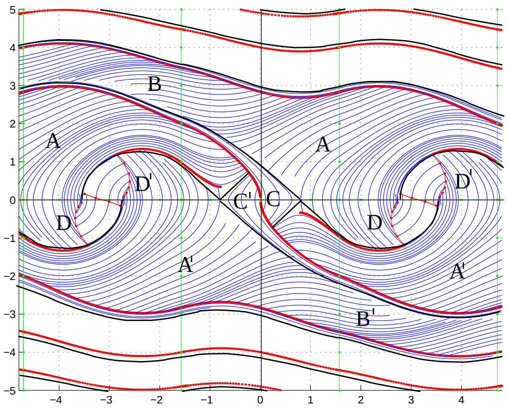

Fig. 4. The complete time optimal synthesis

$-1 + 2\sin x_1^0$, $h_3 = -1 + 2\sin x_1^0 + 2\sin \bar{x}_1$, and $h_4 = 1+2\sin x_1^0+2\sin \bar{x}_1$. It is also possible to find equations for K_2 and K_3, using (7) and (8) and with the help of the expressions of the switching curves C and C' found above. In order to avoid long and not very interesting computations (similar to those made above), we will not present these derivations.

III. Time optimal synthesis

The following result describes the time optimal synthesis for the inverted pendulum on the whole cylinder $\mathbb{S} \times \mathbb{R}$. Its proof, that we will skip, is based on the results obtained in the previous sections.

Theorem 1: Consider an optimal trajectory starting from $x_0 \in S \times \mathbb{R}$. Then:

- If $x_0 \in \mathbf{B}$ the corresponding optimal trajectory starts with control $u = -\operatorname{sgn}(\cos x_1)$ and switches to $\operatorname{sgn}(\cos x_1)$ when it reaches γ^+.
- If $x_0 \in \mathbf{A}$ and if it is far enough from the boundary with $\mathbf{D}'$, the optimal trajectory corresponds to $u = -\operatorname{sgn}(\cos x_1)$ and switches to $\operatorname{sgn}(\cos x_1)$ when it reaches γ^+.
- If $x_0 \in \mathbf{D}'$ or $x_0 \in \mathbf{A}$ is close to the boundary with $\mathbf{D}'$, then, according to Section II-C, there are three possibilities. If x_0 is "below" the overlap curve K_3 the optimal trajectory starts with $u = -1$, switches when it reaches C' and continues with $u = -\operatorname{sgn}(\cos x_1)$ until it reaches γ^+. If x_0 is between K_3 and $C \cup K_2$ the optimal trajectory starts with $u = 1$, switches when it reaches C and continues with $u = \operatorname{sgn}(\cos x_1)$ until it reaches γ^-. Otherwise the optimal trajectory corresponds to $u = \operatorname{sgn}(\cos x_1)$ until it reaches γ^-.
- If $x_0 \in \mathbf{C}$ and it is "close" to γ^+, then the optimal trajectory corresponds to $u = \operatorname{sgn}(\cos x_1)$ until γ^-. Otherwise $u = -\operatorname{sgn}(\cos x_1)$ until γ^+. More precisely there is an overlap curve K winding around the cylinder, between γ^+ and χ^+, dividing C in two parts with different optimal strategies.
- If x_0 belongs to the regions $\mathbf{B}'$, $\mathbf{A}'$, $\mathbf{D}$, $\mathbf{C}'$ the optimal strategy is obtained by symmetry with respect to the origin.

The qualitative shape of the optimal synthesis is now completely clarified. After solving the equations given in the previous sections that describe the switching curves and the overlap curves, and the analogous equations that determine the curve K defined above, one can easily obtain, with the help of matlab simulations, the global shape of the synthesis, as depicted in Figure 4.

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