Time Optimal Swing-Up of the Planar Pendulum

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Abstract

This paper presents qualitative results on the global structure of the time optimal trajectories of the planar pendulum on a cart. This mechanical system is a benchmark to test nonlinear control methods and various papers addressed the problem of computing time optimal open-loop controls. Relying on the theory of optimal synthesis, we provide a discontinuous feedback giving optimal solutions for any initial data. The approach is that of geometric control theory.

Keywords: inverted pendulum, time optimal control, stabilization

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1 Introduction

This paper concerns the global structure of the time optimal trajectories to swing up a planar pendulum on a cart. We consider only the dynamics of the pendulum and take the acceleration w of the cart as the control input. Let x_1 be the angle between the pendulum and the upright position, increasing in the clockwise direction. The equations of motion are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{mgl}{I}\sin x_1 - \frac{mgl}{I} u \cos x_1 \,, \end{aligned} \tag{1}$$

where m is the pendulum's mass, I its moment of inertia, l the distance from the pivot to centre of mass, $g \approx 9.81$ the gravitational field strength, and $u = \frac{w}{g}$ is the control input. The domain of the system is the cylinder $\mathbb{S} \times \mathbb{R}$.

Global stabilization of this model has been studied as a benchmark for nonlinear control by many researchers, for instance, [21], [3], [16], to name a few. Time optimal synthesis has been studied recently in [4] and [25]. These papers are focused on computing exact switching times for an open loop control starting from the down equilibrium. In contrast, we are interested in computing a globally defined feedback control. Notice that, in the recent

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paper [15], the same problem has been studied with the help of numerical simulations in the case of a pendulum with controlled torque.

Our approach is that of geometric control, whose birth, around the 1960's, was characterized by the systematic use of differential geometry tools. Since then, optimal control problems of increasing difficulty have been addressed, permitting solutions through a systematic method. Briefly, this method can be summarized in four steps:

(1) Study the properties of optimal trajectories using necessary conditions, for instance in the form of the well known Pontryagin Maximum Principle (PMP).

(2) Determine a finite dimensional family of trajectories, sufficiently large to contain an optimal trajectory for every initial datum. (By finite dimensional family it is meant a family of trajectories parametrized by a finite number of parameters. In this case one could choose as parameters the lengths of the bang and singular arcs arising from the PMP.)

(3) Select one trajectory for every initial datum to define a *synthesis.*

(4) Prove that the necessary conditions on each trajectory, plus mild regularity conditions on the whole synthesis, result in a synthesis that is optimal. Alternatively one can try to prove optimality directly by comparing the selected trajectories with all the other trajectories obtained at step (2) and corresponding to the same initial datum.

Some of the applications concern mechanical systems [17, 19] and quantum control [9, 10]. Still, success with this approach has been achieved only for special systems [22] or for systems in low dimension. Particular attention has been payed to minimum time problems on two dimensional manifolds. The first results are probably those of Baitman [5, 6]. Next, a series of works of Sussmann dealt with the analytic case [23, 24]. Finally, the generic $\mathcal{C}^\infty$ case was treated in [12, 13]. A general account of these results, together with a complete analysis of singularities and the minimum time function, can be found in [11].

For single input systems with bound on the control, e.g. $|u| \leq 1$, the general outcome is the following. Under generic assumptions, minimum time syntheses correspond to discontinuous controls which are equal to ± 1 on two dimensional regions, and can be singular (i.e. not equal to ± 1) only on special curves called *turnpikes.* The phase portrait of the optimal flow contains some special curves, called *switching curves*, where bang-bang optimal controls switch from $+1$ to -1 and viceversa. Moreover, some *overlap* curves appear, where points can follow more than one optimal trajectory to reach the target. Sometimes such curves are called *cut loci*, because two different optimal flows meet (going backwards in time) and lose optimality after such curves.

The system treated in this paper does not satisfy the genericity assumptions required in [11]. However, we can still use the same geometric control approach to determine the shape of the optimal synthesis. Some switching and overlap curves are determined numerically, by solving implicitly defined equations. However, even in the case of explicit equations for the optimal synthesis phase portrait, one has to deal with numerical evaluations for application to the real system. Thus, this is not a great limitation.

The paper is organized as follows. In Section 2 we give a brief description of the theory of time optimal syntheses for control affine systems in the plane. Then in Section 3 we apply these results to the pendulum problem. In particular we are able to determine the switching curves and overlap curves by solving numerically suitable implicit equations. In Section 4 we assemble all the results obtained in order to give a complete description of the optimal synthesis.

2 Background

We begin by reviewing some results from the geometric theory of optimal control [11] in the special case of control affine systems in $\mathbb{R}^2$. Consider a control system of the form

$$\dot{x} = F(x) + G(x)u \tag{2}$$

where F and G are smooth vector fields in $\mathbb{R}^2$ and u is a scalar input satisfying the constraint $|u| \leq 1$. Any trajectory of (2) joining two points x_0, x_1 in minimum time is called a *time optimal trajectory*. We then consider the problem of determining all the time optimal trajectories $x : [0,T] \to \mathbb{R}^2$ such that $x(T) = 0$; a solution to this problem is called a *time optimal synthesis.*

Let

$$X = F - G\,, \qquad Y = F + G\,.$$

A *bang-bang* trajectory of (2) is a concatenation of X and Y trajectories. We write $X * Y$ to mean a Y trajectory followed by an X trajectory.

If (γ, u) are a trajectory-control pair for the system (2) a *covector field* along (γ, u) is an absolutely continuous function $\lambda : [0,T] \to \mathbb{R}^2$ that satisfies the linear time-varying differential equation

$$\dot{\lambda}(t) = -\lambda(t) \cdot \left[\frac{\partial F}{\partial x}(\gamma(t)) + u(t)\frac{\partial G}{\partial x}(\gamma(t))\right] \qquad a.e. \;\; t \in [0,T]\,. \tag{3}$$

The *Hamiltonian* is defined as

$$H(x, \lambda, u) = \lambda \cdot (F(x) + uG(x))\;.$$

The Pontryagin Maximum Principle says that if (γ, u) is a time optimal trajectory-control pair, then there exists a nontrivial field of covectors λ along γ and a constant $\lambda_0 \leq 0$ such that for a.e. $t \in [0,T]$
(i) λ satisfies (3) ,
(ii) $H(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$,
(iii) $H(\gamma(t), \lambda(t), u(t)) = \max_{u' \in [-1,1]}\{H(\gamma(t), \lambda(t), u')\}$.
A trajectory that satisfies the PMP is called an *extremal trajectory* and (γ, λ) are called an *extremal pair.* If $\lambda_0 = 0$, then the trajectory is called an *abnormal extremal.*

The continuous function $\phi(t) = \lambda(t) \cdot G(\gamma(t))$ is called the *switching function.* If $\phi(t) \neq 0$ then the optimal control is $u(t) = \mathrm{sgn}(\phi(t))$. One would like to determine when $u(t)$ changes value, namely when $\phi(t) = 0$. First, it is easily shown that ϕ is differentiable and

$$\dot{\phi}(t) = \lambda(t) \cdot [F, G](\gamma(t))\,,$$

where $[F, G] = \nabla G \cdot F - \nabla F \cdot G$ is the Lie bracket of F and G. An extremal trajectory γ defined on an interval $[c, d]$ is said to be *singular* or a *Z-trajectory* if $\phi = 0$ on $[c, d]$. If γ

is singular on $[c,d]$ we have $\phi(t) = \dot{\phi}(t) = 0$ on $[c,d]$. That is, $\lambda(t)\cdot G = \lambda(t)\cdot[F,G] = 0$. Since $\lambda(t) \neq 0$ we have that G and $[F,G]$ are parallel.

Following [23] we define the following scalar functions on $\mathbb{R}^2$:

$$\begin{aligned} \Delta_A(x) &:= det(F(x), G(x)) \\ \Delta_B(x) &:= det(G(x), [F,G](x))\,, \end{aligned}$$

where the notation (F,G) means the 2×2 matrix formed by columns F and G. The first function $\Delta_A(x)$ is for locating abnormal extremals. Suppose that at some time $\bar{t}$ along an extremal trajectory γ, $\phi(\bar{t}) = 0$ and $G(\gamma(\bar{t})) \neq 0$. Then it can be shown ([11], p. 45) that γ is an abnormal extremal iff for every t with $\phi(t) = 0$ one has $\Delta_A(\gamma(t)) = 0$. Clearly, the second function $\Delta_B(x)$ is to locate singular trajectories. It can be shown that under certain generic conditions, singular trajectories are of a particular type called *regular turnpikes*. Since it will be shown that the pendulum does not have singular trajectories, we will not go further into characterizing properties of regular turnpikes. See [11] for details.

So far we have described a simple test to locate abnormal and singular trajectories. One would like to determine the behavior of optimal trajectories which are instead bang-bang. To this end, a point $x \in \mathbb{R}^2$ is called an *ordinary point* if $\Delta_A(x)\cdot\Delta_B(x) \neq 0$. If we define the sets $\Omega_A = \{x : \Delta_A(x) \neq 0\}$ and $\Omega_B = \{x : \Delta_B(x) \neq 0\}$ then the set of ordinary points is $\Omega = \Omega_A \cap \Omega_B$. Now we can study bang-bang trajectories on Ω. On Ω we define the scalar functions f and g which are the coefficients of the linear combination

$$[F,G](x) = f(x)F(x) + g(x)G(x)\,.$$

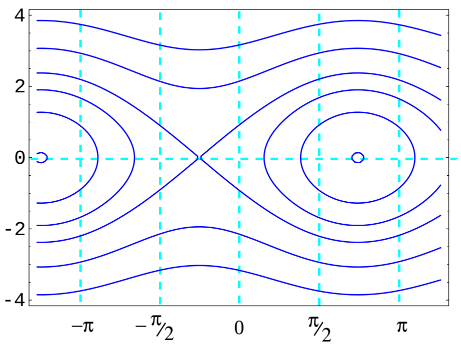
One can show that $f(x) = -\frac{\Delta_B}{\Delta_A}$ ([23], p. 447). Let $U \subset \Omega$ and suppose that $f > 0$ on U, the opposite case being analogous. Let (γ,λ) be an extremal pair such that γ is contained in U. Suppose that $\bar{t}$ is a switching time, i.e. $\lambda(\bar{t})\cdot G(\bar{t}) = 0$. Then

$$\begin{aligned} \dot{\phi}(\bar{t}) &= \lambda(\bar{t})\cdot[F,G](\gamma(\bar{t})) \\ &= \lambda(\bar{t})\cdot(fF + gG)(\gamma(\bar{t})) \\ &= f(\gamma(\bar{t}))\left(\lambda(\bar{t})\cdot F(\gamma(\bar{t}))\right)\,. \end{aligned}$$

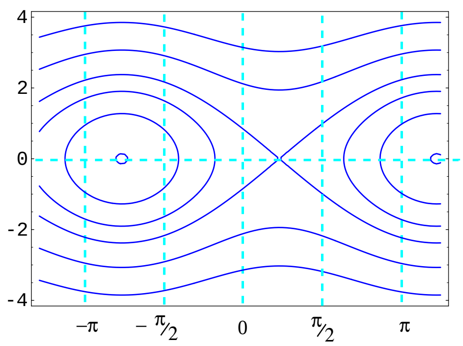
From the PMP we have that $H(\gamma(\bar{t}),\lambda(\bar{t})) = \lambda(\bar{t})\cdot F(\gamma(\bar{t})) \geq 0$. Hence, $\dot{\phi} \geq 0$. But we cannot have $\lambda(\bar{t})\cdot F(\gamma(\bar{t})) = 0$, otherwise $F(\gamma(\bar{t}))$ and $G(\gamma(\bar{t}))$ would be parallel and $\gamma(\bar{t})$ would not be a ordinary point. So we find $\dot{\phi}(\gamma(\bar{t})) > 0$. This shows that ϕ has at most one zero along γ with positive derivative at the switching time. We have just proved the following useful result due to Sussmann ([23], p. 443).

Theorem 1. *Let $U \subset \Omega$. Then all time optimal trajectories γ of the system* (2) *restricted to U are bang-bang with at most one switching. Moreover, if $f > 0$ throughout U then γ is an X, Y, or $Y * X$ trajectory; if $f < 0$ throughout U then γ is an X, Y, or $X * Y$ trajectory.*

Finally, we discuss a method to construct the optimal synthesis once the properties of singular and bang-bang trajectories have been exploited. Under certain generic conditions, one can prove that there exists a bound on the number of bang and singular arcs. See ([11], p. 48). One can then devise a finite, inductive, conceptual algorithm with the induction on the number of bang or singular arcs. Under the generic conditions, the qualitative features of the synthesis can be classified, in the spirit of the qualitative classification of generic flows



(a) $u = -1$



(b) $u = +1$

Figure 1: Phase portraits for $u = \pm 1$.

on 2D manifolds due to Peixoto [18]. The qualitative features are in the form of boundary curves and points called *frame curves* and *frame points*. Under the generic conditions, the algorithm generates five types of frame curves:

1. The trajectories $\hat{\gamma}^+$ and $\hat{\gamma}^-$ which are the X and Y trajectories that reach the origin.
2. Singular trajectories called S curves.
3. Switching curves, called C curves.
4. Overlap curves, formed by points where two distinct optimal trajectories can reach the origin; called K curves.
5. The topological frontier of the set of states that can reach the origin, called a B curve.

Topologically distinct frame points can similarly be classified. This information is then used to obtain a qualitative picture of the optimal synthesis. Unfortunately, the pendulum system does not satisfy the generic conditions of these results, but we can still use the inductive algorithm and see that it terminates.

3 Analysis of the pendulum system

We consider the time optimal synthesis of the pendulum with equations of motion corresponding to $\frac{mgl}{I} = 1$ and with the bound $|u| \leq 1$, so that (1) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u \cos x_1 \, . \end{aligned} \tag{4}$$

Therefore, we have $F(x) = \begin{pmatrix} x_2 \\ \sin x_1 \end{pmatrix}$ and $G(x) = \begin{pmatrix} 0 \\ -\cos x_1 \end{pmatrix}$. In the sequel, it will be useful to refer to the phase portraits for (4) with $u = \pm 1$, as shown in Figure 1. The optimal trajectories will be concatenations of trajectory segments of these two phase portraits. We consider the following problem.

Problem. *For every* $(\bar{x}_1, \bar{x}_2) \in \mathbb{S} \times \mathbb{R}$ *find a trajectory-control pair* $(\gamma(\cdot), u(\cdot))$ *defined on* $[0, T]$ *and such that* γ *is time optimal between* $\gamma(0) = (\bar{x}_1, \bar{x}_2)$ *and* $\gamma(T) = 0$.

In order to solve this problem one first needs to prove that the system is globally stabilizable at the origin. This can be easily done by using classical results in geometric control theory on global controllability. In particular, since all the trajectories of (4) corresponding to u constant, with the exception of the separatrices, are closed and $\{F, G\}$ defines a bracket-generating family of vector fields, we can apply the Rashevsky-Chow Theorem (see [1], Theorem 5.9, p. 67) to conclude that the attainable set from every point coincides with the whole cylinder. Finally, for every initial datum $(\bar{x}_1, \bar{x}_2)$, the existence of a time optimal trajectory reaching the origin can be derived, when $u(\cdot)$ belongs to the class of measurable functions with $|u| \leq 1$, from Filippov Theorem (see [1], Corollary 10.7, p. 143).

We can now apply the general theory recalled in the previous section. The Hamiltonian is

$$H = \lambda_1 x_2 + \lambda_2(\sin x_1 - u \cos x_1)\,,$$

and the adjoint variables satisfy the differential equation

$$\begin{aligned} \dot{\lambda}_1 &= -\lambda_2(\cos x_1 + u \sin x_1) \\ \dot{\lambda}_2 &= -\lambda_1\,. \end{aligned} \tag{5}$$

The switching function is $\phi = -\lambda_2 \cos x_1$ and the optimal control is $u^*(t) = \mathrm{sgn}(\phi(t))$. From this we deduce that switchings occur only if $\lambda_2 = 0$ or if $x_1 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$.

We compute the Lie bracket

$$[F, G](x) = \cos x_1 \frac{\partial}{\partial x_1} + x_2 \sin x_1 \frac{\partial}{\partial x_2}$$

and the functions

$$\begin{aligned} \Delta_A(x) &= -x_2 \cos x_1 \\ \Delta_B(x) &= \cos^2 x_1\,. \end{aligned}$$

The set of ordinary points is $\Omega = \{x : x_2 \neq 0, x_1 \neq \pm\frac{\pi}{2}\}$. Also

$$f(x) = -\frac{\Delta_B}{\Delta_A} = \frac{\cos x_1}{x_2}\,.$$

Ω is split into four regions where f has a constant sign. In the region

$$\left\{x \mid x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}), x_2 > 0\right\} \ \cup\ \left\{x \mid x_1 \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi), x_2 < 0\right\} \tag{6}$$

$f(x) > 0$ so by Theorem 1, the optimal control can switch at most once from $u = -1$ to $u = +1$. In the region

$$\left\{x \mid x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}), x_2 < 0\right\} \ \cup\ \left\{x \mid x_1 \in (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi), x_2 > 0\right\} \tag{7}$$

$f(x) < 0$, and the optimal control can switch at most once from $u = +1$ to $u = -1$. See Figure 2. Note that pendulum system does not satisfy all the generic conditions in ([11], p. 48), since both the sets $\Delta_A^{-1}(0)$ and $\Delta_B^{-1}(0)$ contain the line $x_2 = 0$ and therefore

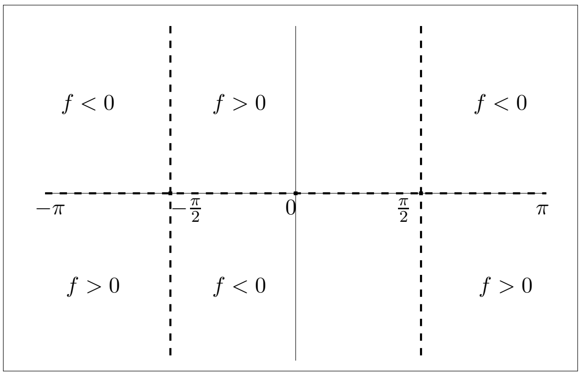


Figure 2: Regions where $f > 0$ and $f < 0$ for the pendulum system.

$\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)$ is not locally finite. Nevertheless, the analyticity of (4) allows to exclude pathological behaviours [24], such as the Fuller phenomenon, so that we will be able to construct an optimal synthesis.

We have established some basic features of the optimal synthesis using ideas from [23]. In the next two subsections, these results are refined. In Section 3.1, we identify properties of extremal trajectories. First, we eliminate the possibility of singular extremals. Then we examine the trajectories γ^+ and γ^- which form a skeleton of the synthesis, and we examine those extremal trajectories that switch onto γ^+ and γ^-. Second, in Section 3.2, we identify and analytically characterize candidate switching and overlap curves.

3.1 Extremal trajectories

In this section we identify properties of the extremal trajectories. We assume w.l.o.g. that extremal trajectories reach the origin at $t = 0$; thus, time is negative and increasing.

First, we observe that there are no optimal trajectories containing singular arcs. Indeed a singular arc must be contained inside $\Delta_B^{-1}(0)$ i.e. it must be a vertical segment, which is not allowed by equation (4).

Now we consider γ^+ and γ^-, those trajectories that reach the origin and whose final bang arcs before reaching the origin, $\hat{\gamma}^+$ and $\hat{\gamma}^-$, form the first step of the algorithm described in Section 2.

Definition 1. *Let γ^+ (resp. γ^-) be the trajectory of (4) defined on $(-\infty, 0]$ that reaches the origin with $u = 1$ (resp. $u = -1$) at time $t = 0$ and such that the control switches occur exactly at $x_1 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$.*

The bold, black curve in Figure 3 depicts γ^+ and γ^-.

Remark 1. *The controls corresponding to γ^+ and γ^- are (almost everywhere) $u(t) = \operatorname{sgn}(\cos x_1(t))$ and $u(t) = -\operatorname{sgn}(\cos x_1(t))$, respectively.*

The trajectories γ^+ and γ^- play an important role in the construction of the synthesis, as will be clear after the following proposition and corollary.

The following result exhibits the main properties of the extremal trajectories for the pendulum system, and it is our main guide for determining the switching curves.

Proposition 1. *Consider a bang-bang trajectory of* (4), $x(\cdot) = (x_1(\cdot), x_2(\cdot)) : [t_1, t_2] \to \mathbb{R}$, *with* $x_2(t) \neq 0$ *on* (t_1, t_2) *and let* S *be the set of switching times of* $x(\cdot)$ *and* $K = \{t \in (t_1, t_2) : \; x_1(t) = \frac{\pi}{2} + k\pi \,, \; \exists k \in \mathbb{Z}\}$. *Then* $x(\cdot)$ *is extremal if and only if one of the following three possibilities is satisfied:*

(i) $K = S$,

(ii) there exists $\bar{t} \in (t_1, t_2) \setminus K$ *such that* $S = K \cup \{\bar{t}\}$ *and* $u(t) = -\operatorname{sgn}(x_2) \operatorname{sgn}(\cos x_1(t))$ *a.e. on* $(t_1, \bar{t})$, *while* $u(t) = \operatorname{sgn}(x_2) \operatorname{sgn}(\cos x_1(t))$ *a.e. on* $(\bar{t}, t_2)$

(iii) there exists $\bar{t} \in K$ *such that* $S = K \setminus \{\bar{t}\}$ *and* $u(t) = -\operatorname{sgn}(x_2) \operatorname{sgn}(\cos x_1(t))$ *a.e. on* $(t_1, \bar{t})$, *while* $u(t) = \operatorname{sgn}(x_2) \operatorname{sgn}(\cos x_1(t))$ *a.e. on* $(\bar{t}, t_2)$.

Proof. Let $x(\cdot)$ be an extremal trajectory which does not intersect the x_1 axis. Then we have $u(t) = \operatorname{sgn}(\phi(t)) = -\operatorname{sgn}(\lambda_2(t)) \operatorname{sgn}(\cos x_1(t))$. Therefore (i) is equivalent to $\operatorname{sgn}(\lambda_2(t)) = \kappa \quad a.e$ (where $\kappa \in \{-1, 1\}$), while one among the cases (ii) and (iii) holds if and only if $\operatorname{sgn}(\lambda_2(t)) = \operatorname{sgn}(x_2(t))$ a.e on $(t_1, \bar{t})$ and $\operatorname{sgn}(\lambda_2(t)) = -\operatorname{sgn}(x_2(t))$ a.e on $(\bar{t}, t_2)$. Assume that $x(\cdot)$ does not satisfy (i), then in particular there exists $\bar{t} \in (t_1, t_2)$ such that $\lambda_2(\bar{t}) = 0$. Then, using the fact that $H = \lambda_1(\bar{t}) x_2(\bar{t}) > 0$, we obtain $\operatorname{sgn}(\lambda_1(\bar{t})) = \operatorname{sgn}(x_2(\bar{t}))$. From this equality and since $\dot{\lambda}_2 = -\lambda_1$, we find that $\operatorname{sgn}(\lambda_2(t)) = \operatorname{sgn}(x_2(\bar{t}))$ on $(\bar{t} - \epsilon, \bar{t})$ and $\operatorname{sgn}(\lambda_2(t)) = -\operatorname{sgn}(x_2(\bar{t}))$ on $(\bar{t}, \bar{t} + \epsilon)$. To prove that $x(\cdot)$ satisfies either (ii) or (iii) it is enough to see that, if the sign of x_2 is fixed, then there is only one time $\bar{t}$ with $\lambda_2(\bar{t}) = 0$. Assume by contradiction that $\bar{t}_1 < \bar{t}_2$ are such that $\lambda_2(\bar{t}_1) = \lambda_2(\bar{t}_2) = 0$ and $\lambda_2(\cdot) \neq 0$ on $(\bar{t}_1, \bar{t}_2)$. Then, since $\operatorname{sgn}(\lambda_2(t)) = -\operatorname{sgn}(x_2)$ on $(\bar{t}_1, \bar{t}_1 + \epsilon)$ and $\operatorname{sgn}(\lambda_2(t)) = \operatorname{sgn}(x_2)$ on $(\bar{t}_2 - \epsilon, \bar{t}_2)$ the continuous function $\lambda_2(\cdot)$ must be zero somewhere on $(\bar{t}_1, \bar{t}_2)$ and we find a contradiction. We have therefore proved that every extremal trajectory with $x_2(\cdot) \neq 0$ satisfies one among (i), (ii) and (iii). Conversely, it is clear from the previous arguments that every bang-bang trajectory $x(\cdot)$ satisfying (ii) or (iii) satisfies the PMP with $\lambda(\bar{t}) = (\operatorname{sgn}(x_2), 0)$. When (i) holds the PMP is satisfied with $\lambda(t_1) = (\operatorname{sgn}(x_2), 0)$ if $u(\cdot) = \operatorname{sgn}(x_2) \operatorname{sgn}(\cos x_1(\cdot))$ or $\lambda(t_2) = (\operatorname{sgn}(x_2), 0)$ if $u(\cdot) = -\operatorname{sgn}(x_2) \operatorname{sgn}(\cos x_1(\cdot))$. □

Remark 2. *Note that Proposition 1 agrees with the conclusions of Theorem 1. The extra information provided in Proposition 1 concerns the possibility of switching when* $cos(x_1) = 0$, *i.e. when* $f(x)$ *changes sign.*

The previous statement can be applied to the trajectories γ^+ and γ^- so that we obtain the following important result.

Corollary 1. *The trajectory* γ^+ *(resp.* γ^-*) is extremal on any interval* $[\bar{t}, 0]$, $\bar{t} < 0$, *and for every point* p *of* γ^+ *(resp.* γ^-*), there exists an extremal trajectory that reaches* γ^+ *(resp.* γ^-*) for the first time at* p *and then follows* γ^+ *(resp.* γ^-*) until it touches the origin.*

Remark 3. *We comment on the existence of abnormal extremals. If we choose initial conditions* $x(0) = 0$ *and* $\lambda(0) = (1, 0)$, *then* $H = 0$. *This implies* $\lambda_2 \neq 0$ *for every point of the extremal trajectory such that* $x_2 \neq 0$ *(otherwise* $\lambda_2 = 0$ *and* $H = 0$ *imply* $\lambda_1 = 0$, *which is impossible from PMP). Thus, for* $x(0) = 0$ *and* $\lambda(0) = (1, 0)$, γ^- *is the corresponding* *extremal trajectory and it is an abnormal extremal. Analogously, taking $\lambda(0) = (-1, 0)$, we find that γ^+ is an abnormal extremal. Clearly there are no other abnormal extremals reaching the origin. Otherwise, there would exist a point in $\gamma^\pm$ with $\lambda_2 = 0$ and so $H = \lambda_1 x_2 = 0$ would imply $\lambda_1 = 0$, which is impossible.*

The results so far on extremal trajectories permit a partition of the cylinder into regions in which the behaviour of extremal trajectories has common qualitative properties. Referring to Figure 3, we describe boundaries of these regions by identifying particular trajectories. Notice in the following descriptions that we often make no distinction between a trajectory and its graph.

- Let $\xi^+ : [0, +\infty) \to \mathbb{S} \times \mathbb{R}$ be the trajectory of (4) corresponding to $u = \operatorname{sgn}(\cos x_1(t))$, such that $\lim_{t \to +\infty} \xi^+(t) = (\frac{\pi}{4}, 0)$, $\xi_2^+(0) = 0$ and $\xi_2^+(t) < 0$, $\forall t > 0$.
- Let $\chi^+ : \mathbb{R} \to \mathbb{S} \times \mathbb{R}$ be the trajectory of (4) corresponding to $u = \operatorname{sgn}(\cos x_1(t))$, such that $\lim_{t \to +\infty} \chi^+(t) = (\frac{\pi}{4}, 0)$, $\chi_1^+(0) = 0$ and $\chi_1^+(t) \neq 0$, $\forall t > 0$.
- Let $\eta^+ : (-\infty, 0] \to \mathbb{S} \times \mathbb{R}$ be the trajectory of (4) corresponding to $u = \operatorname{sgn}(\cos x_1(t))$, such that $\lim_{t \to -\infty} \eta^+(t) = (\frac{\pi}{4}, 0)$, and $\eta^+ \cap \gamma^- = \eta^+(0)$.
- Let $t^+ < 0$ be the largest negative time such that the x_1 component of $\gamma^+(t^+)$ vanishes.

Remark 4. *Although χ^+ is an extremal trajectory for the minimization problem, an arbitrary arc of it can never belong to an optimal trajectory reaching the origin. Indeed, from Proposition 1, this trajectory should follow χ^+ until it reaches the point $(\frac{\pi}{4}, 0)$, which actually never happens, since $(\frac{\pi}{4}, 0)$ is an equilibrium point.*

Referring to Figure 3, we call **D** the region enclosed by ξ^+ and by the segment joining $\xi^+(0)$ with $(\frac{\pi}{4}, 0)$. The region **C** is defined as the strip whose boundary is the union of γ^+, χ^+, η^+ and the arc of γ^- between $\eta^+(0)$ and the origin. The region **B** is the strip whose boundary is the union of the restriction of χ^+ to $(-\infty, 0)$, the restriction of γ^+ to $(-\infty, t^+)$, and the segment joining $\chi^+(0)$ with $\gamma^+(t^+)$. The regions **B′**, **C′**, **D′** are obtained from **B**, **C**, **D** by symmetry with respect to the origin. Finally **A** (resp. **A′**) is the complement of **B**∪**C**∪**D**∪**B′**∪**C′**∪**D′** in the upper (resp. lower) half-plane.

Now we give a brief description of the trajectories of (4) satisfying the conditions given by Proposition 1 inside the regions **A**, **B**, **C**, and **D**, until they reach their corresponding boundaries.

- If the initial condition for the minimization problem is inside **A** then it is easy to see that the trajectories corresponding to $u = -\operatorname{sgn}(\cos x_1(t))$ must reach the boundary of **A** at some point of γ^+. On the other hand all the trajectories corresponding to $u = \operatorname{sgn}(\cos x_1(t))$ reach the boundary of **D**, **D′** or **C′**.
- If the initial condition is inside **B** then all the trajectories reach the boundary in a point of γ^+ or of the segment connecting $\chi^+(0)$ and $\gamma^+(t^+)$.
- If the initial condition is inside **C** then the trajectories corresponding to $u = \operatorname{sgn}(\cos x_1(t))$ stay in **C** until they reach γ^-, while every trajectory corresponding to $u = -\operatorname{sgn}(\cos x_1(t))$ must cross χ^+.

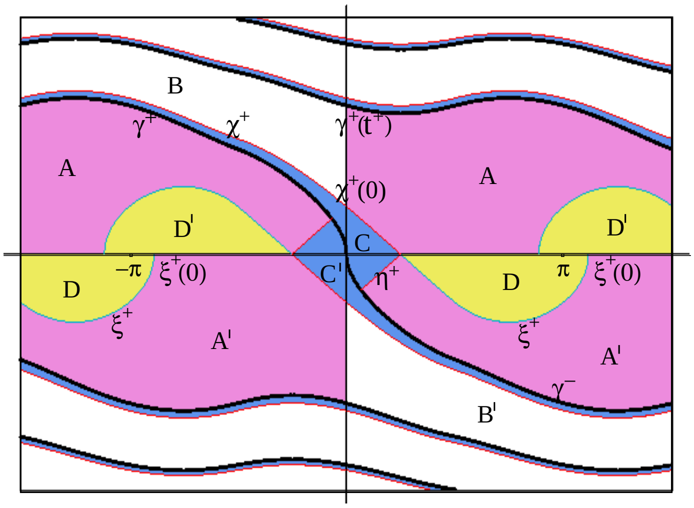


Figure 3: Partition of $\mathbb{S} \times \mathbb{R}$

- All the trajectories that start inside **D** must cross the x_1 axis.

The descriptions of the trajectories starting from the regions $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$ and $\mathbf{D}'$ are analogous.

3.2 Switching curves

Collecting the results of the previous section, we find that every extremal trajectory reaching the origin and belonging to the upper or lower half plane can switch only on $\gamma^{\pm}$ or if $x_1 = \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$. Therefore, in order to detect other nontrivial switching curves, we need to look for extremal trajectories reaching the origin and crossing the x_1 axis. The switching curves are identified with the regular submanifold of points such that $\lambda_2 = 0$.

The following observations will facilitate our analysis both of switching curves and of overlap curves. For a fixed value of u, the system (4) admits a first integral

$$h(x) = \frac{1}{2}x_2^2 + \cos x_1 + u \sin x_1 \,. \tag{8}$$

Moreover we know that along an extremal trajectory the value of the Hamiltonian is a constant, which we call H. Notice that, since we can rescale λ by an arbitrary positive factor, we can assume w.l.o.g. that $H = 1$. Using these facts, we would like to obtain an explicit formula for λ_2 in order to find the switching curves. We have

$$-\dot{\lambda}_2 x_2 + \lambda_2(\sin x_1 - u \cos x_1) = H \quad \Longrightarrow \quad \dot{\lambda}_2 = \frac{\lambda_2}{x_2}(\sin x_1 - u \cos x_1) - \frac{H}{x_2}\,.$$

If $x_2 \neq 0$ and $\dot{x}_2 \neq 0$ we can locally view λ_2 as a function of x_2 and we obtain

$$\frac{\dot{\lambda}_2}{\dot{x}_2} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\lambda_2}{x_2} - \frac{H}{x_2(\sin x_1 - u \cos x_1)}\,.$$

The right-hand side of this equation can be written in terms of x_2 only by using $h(x)$. If $u = \pm 1$ we have

$$\sin x_1 - u \cos x_1 = \sqrt{2} \sin(x_1 - u\frac{\pi}{4}) ,$$

while

$$h(x) = \frac{1}{2}x_2^2 + \cos x_1 + u \sin x_1 = \frac{1}{2}x_2^2 + \sqrt{2} \cos(x_1 - u\frac{\pi}{4}) .$$

Combining this information we obtain

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\lambda_2}{x_2} - \kappa \frac{H}{x_2 \sqrt{2 - (h - \frac{1}{2}x_2^2)^2}}$$

where $\kappa = \operatorname{sgn}(\sin x_1 - u \cos x_1)$ is the only term depending on x_1. The explicit solution to this equation, with initial condition $\lambda_2(x_2^0) = \lambda_2^0$, can be obtained via the method of variation of parameters:

$$\lambda_2(x_2) = x_2 \left(\frac{\lambda_2^0}{x_2^0} - \kappa H \int_{x_2^0}^{x_2} \frac{\mathrm{d}y}{y^2 \sqrt{2 - (h - \frac{1}{2}y^2)^2}} \right) \tag{9}$$

This integral cannot be solved exactly, but it can be written in terms of elliptic integrals, and therefore it can be easily computed numerically.

The formula (9) for λ_2 is applied on intervals where κ and u are constant. At switching points, the value of h must be updated by using the previous value of h and the value of x_1. In this manner the formula (9) can be used to integrate through all the switchings and segments with constant κ of an extremal trajectory to obtain an analytical expression for a particular switching curve $\lambda_2(x_2) = 0$.

The above observations are also useful for determining overlap curves, so we briefly outline the technique here, though it will be applied later in Section 3.3. Overlap curves occur where two or more extremal trajectories reach the origin in minimum time. In order to compute overlap curves, we would like to derive an explicit formula for the time elapsed for an extremal trajectory to reach the origin. From above we know that

$$\dot{x}_1 = x_2 = \operatorname{sgn}(x_2) \sqrt{2\big(h(x) - \sqrt{2} \cos(x_1 - u\frac{\pi}{4})\big)} \tag{10}$$

and, similarly

$$\dot{x}_2 = \sqrt{2} \sin\left(x_1 - u\frac{\pi}{4}\right) = \operatorname{sgn}\left(\sin\left(x_1 - u\frac{\pi}{4}\right)\right) \sqrt{2 - \left(h(x) - \frac{x_2^2}{2}\right)^2} . \tag{11}$$

Using equation (11) we obtain a formula for $T(x^0, x^1)$, the time elapsed along a extremal trajectory starting from x^0 and ending at x^1, as a function of x_2

$$T(x^0, x^1) = \int_{x_2^0}^{x_2^1} \frac{\mathrm{d}y}{\sqrt{2 - \left(h(x) - \frac{y^2}{2}\right)^2}} . \tag{12}$$

This formula must be used on segments of an extremal trajectory where the control does not switch and where $\operatorname{sgn}\left(\sin\left(x_1 - u\frac{\pi}{4}\right)\right)$ is constant.

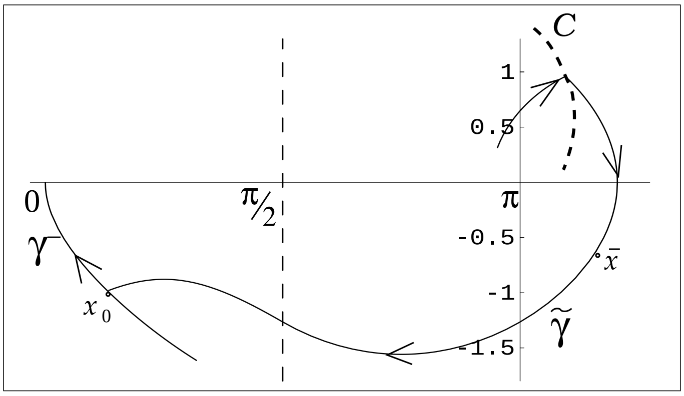


Figure 4: Switching curve C.

3.2.1 Switching curve C

In this section we consider the family of extremal trajectories that cross, with $u = -1$, the segment of the x_1 axis between $\mathbf{D}'$ and $\mathbf{A}'$ with $x_1 < \frac{3\pi}{2}$, then switch from $u = -1$ to $u = 1$ on the line $x_1 = \frac{\pi}{2}$, and finally, switch at a point x^0 on γ^-, and follow γ^- until reaching the origin. See Figure 4. Let C be (if it exists) the set of points belonging to these trajectories and corresponding to $\lambda_2 = 0$. We want to determine an equation that describes C. To do this we follow backwards the extremal trajectories that reach the origin, integrating by quadratures the equation (9).

Take an extremal trajectory $\tilde{\gamma}$ as described above, and assume that the last switching before reaching the origin is at $x^0 = (x_1^0, x_2^0) \in \gamma^-$, with $x_1^0 < \pi/4$ and with $\lambda_2^0 = 0$. (We will comment on the case $x_1^0 \geq \pi/4$ below). Also, let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be a point of $\tilde{\gamma}$ with $\bar{x}_1 > 3\pi/4$ and $\bar{x}_2 < 0$. Recall that H is a fixed positive real number along $\tilde{\gamma}$, and we can rescale $\lambda_1(x_2^0)$ such that $H = 1$. We must determine the value of $h(\cdot)$ along each bang arc of $\tilde{\gamma}$. Using (8) we have that $h(\cdot) = 1$ along γ^-; $h(\cdot) = h_1 := 1 + 2\sin x_1^0$ for the bang arc of $\tilde{\gamma}$ when $x_1 \in [x_1^0, \frac{\pi}{2}]$; and $h(\cdot) = h_2 := -1 + 2\sin x_1^0$ for the bang arc of $\tilde{\gamma}$ when $x_1 \in [\frac{\pi}{2}, \bar{x}_1]$. Also, $\kappa = -1$ when $x_1 \in [x_1^0, \frac{\pi}{4}]$, $\kappa = +1$ when $x_1 \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, and $\kappa = -1$ when $x_1 \in [\frac{3\pi}{4}, \bar{x}_1]$, where $\kappa = \operatorname{sgn}(\sin x_1 - u\cos x_1)$. Referring to Figure 4, we notice that in such segments $\tilde{\gamma}$ is monotone with respect to x_2. Combining these informations with (9) we obtain

$$\lambda_2(\bar{x}_2) = H\bar{x}_2\Bigg(\int_{x_2^0}^{x_2(x_1=\frac{\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \int_{x_2(x_1=\frac{\pi}{4})}^{x_2(x_1=\frac{\pi}{2})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}}$$

$$- \int_{x_2(x_1=\frac{\pi}{2})}^{x_2(x_1=\frac{3\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} + \int_{x_2(x_1=\frac{3\pi}{4})}^{\bar{x}_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}\Bigg), \tag{13}$$

where

$$x_2^0 = -\sqrt{2(1-\cos x_1^0+\sin x_1^0)}, \quad x_2(x_1=\frac{\pi}{4}) = -\sqrt{2(1+2\sin x_1^0-\sqrt{2})},$$
$$x_2(x_1=\frac{\pi}{2}) = -2\sqrt{\sin x_1^0}, \quad x_2(x_1=\frac{3\pi}{4}) = -\sqrt{2(-1+2\sin x_1^0+\sqrt{2})}.$$

Note that the integrals in (13) are generalized integrals; that is, the integrands are not well-defined at the extremes.

The formula (13) applies to $\bar{x}_2 < 0$. Now this must be connected with the segment of $\tilde{\gamma}$ for which $x_2 > 0$, by using a continuity argument. In particular, we use the continuity of λ at the intersection of $\tilde{\gamma}$ and the x_1 axis. The value of λ at the intersection point can be derived from (13) and from the equation (5) by passing to the limit as $\bar{x}_2$ goes to 0. The result is a function of switching point x^0 and the choice of H. However, we observe that $H = \lambda_2(\sin x_1 - u\cos x_1)$ if $x_2 = 0$, so the value of λ_2 at a point of the x_1 axis is the same for any extremal trajectory with Hamiltonian H and passing through that point with $u = -1$. So the fact that λ_2 must be continuous at the intersection between $\tilde{\gamma}$ and the x_1 axis is not sufficient to determine the corresponding point of C.

We must use the additional information that $\lambda_1 = -\dot{\lambda}_2$ is also continuous. We have that

$$\lim_{x_2\to 0^-} \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \lim_{x_2\to 0^+} \frac{\mathrm{d}\lambda_2}{\mathrm{d}x_2} = \frac{\dot{\lambda}_2}{\dot{x}_2}\Big|_{x_2=0}. \tag{14}$$

Assume that there exists a switching point $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in \tilde{\gamma}$ with $\tilde{x}_2 > 0$ and $\tilde{x}_1 > \frac{3\pi}{4}$, so that we have $u = -1$ and $\kappa = -1$ between $\tilde{x}$ and the axis x_1. We know that $\lambda_2 = 0$ at the point $\tilde{x}$. Then we can find an expression for $\lambda_2(x_2),\ x_2 > 0$ similar to (13)

$$\lambda_2(\bar{x}_2) = H\bar{x}_2 \int_{\tilde{x}_2}^{\bar{x}_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}. \tag{15}$$

Now we differentiate (13) and (15) with respect to $\bar{x}_2$, then apply the identity

$$\int_{z_1}^{z_2} \frac{\mathrm{d}y}{y^2\sqrt{2-(h-\frac{1}{2}y^2)^2}} = -\frac{\sqrt{2-(h-\frac{1}{2}y^2)^2}}{y(2-h^2)}\Bigg|_{z_1}^{z_2} - \frac{1}{4(2-h^2)}\int_{z_1}^{z_2} \frac{y^2\mathrm{d}y}{\sqrt{2-(h-\frac{1}{2}y^2)^2}} \tag{16}$$

to the last integral of (13) and to the one of (15), and finally pass to the limit as $\bar{x}_2$ tends to 0. The resulting expressions must be equal thanks to (14). Thus, we arrive at the following equation

$$\begin{aligned}
&\int_{x_2^0}^{x_2(x_1=\frac{\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \int_{x_2(x_1=\frac{\pi}{4})}^{x_2(x_1=\frac{\pi}{2})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_1-\frac{1}{2}y^2)^2}} - \\
&\quad - \int_{x_2(x_1=\frac{\pi}{2})}^{x_2(x_1=\frac{3\pi}{4})} \frac{\mathrm{d}y}{y^2\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} - \frac{1}{4(2-h_2^2)}\int_{x_2(x_1=\frac{3\pi}{4})}^{0} \frac{y^2\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} = \\
&\quad = \frac{\sqrt{2-(h_2-\frac{1}{2}\tilde{x}_2^2)^2}}{\tilde{x}_2(2-h_2^2)} - \frac{1}{4(2-h_2^2)}\int_{\tilde{x}_2}^{0} \frac{y^2\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}.
\end{aligned} \tag{17}$$

This gives $\tilde{x}_2$ (and therefore also $\tilde{x}_1$, since $h(\tilde{x}_1, \tilde{x}_2) = h_2$) in terms of $x^0 \in \gamma^-$. Therefore the switching curve C can be determined by solving numerically the previous equation.

We conclude this section with a few remarks. First, there exists a switching curve C' symmetric to C ($x \in C$ if and only if $-x \in C'$). Second, for the same family of extremal trajectories, any switching curve that precedes C (in time) cannot be optimal. Indeed, if we assume that the x_1 coordinate for the new switching point is less than $3\pi/2$ (otherwise there would be a self-intersection of the corresponding extremal trajectory that would imply a loss of optimality) and we write the equality obtained from (14), it turns out that the right-hand side and the left-hand side of such an equation have different signs. Finally, for the case when $x_1^0 \geq \frac{\pi}{4}$, one simply removes the first integral in (17) and modifies the lower limit of the second integral to be x_2^0. By solving numerically the resulting equation, one obtains an expression for a switching curve which starts at the final point of the switching curve obtained by (17). However, as will become more clear in Section 3.3, this switching curve is not optimal due to the presence of an overlap curve.

3.2.2 Switching curves in region $\mathbf{C}$

In this section we analyze the extremal trajectories that reach γ^- with a coordinate $x_1^0 < \arcsin((\sqrt{2}-1)/2)$, i.e. the trajectories that are contained in region $\mathbf{C}$ before the last switching on γ^-. In particular, these trajectories form a front that crosses the x_1 axis with $x_1 \in [0, \frac{\pi}{4})$. See the left side of Figure 5. We will show in the following proposition that this front cannot generate a switching curve (different from $x_1 = \frac{\pi}{2} + k\pi$) on the half-plane $x_2 > 0$. This possibility cannot be excluded apriori using the qualitative results of Section 3.1.

Proposition 2. *An optimal trajectory that switches for the last time at $x^0 \in \gamma^-$ after having crossed the x_1 axis must switch before x^0 only at the points such that $x_1 = \frac{\pi}{2} + k\pi$.*

Let Λ denote the strip of trajectories that reach γ^- after crossing the axis x_1 and correspond to $u(t) = \text{sgn}(\cos x_1(t))$. This strip corresponds exactly to the region $\mathbf{C}$. The proof of the proposition is based on the following lemma.

Lemma 1. *Let $T(x)$ be the time needed to reach the origin starting from $x \in \mathbf{C}$ and following the corresponding trajectory in Λ. Then $\frac{\partial T}{\partial x_2}(x) > 0$.*

Proof. Consider a trajectory of Λ that switches if $x_1 = \pm\pi/2$ and at a point $x^0 \in \gamma^-$. For every initial condition $x \in \mathbf{C}$ one can associate a value x_1^0 where the corresponding trajectory reaches γ^-. See the left of Figure 5. If we restrict initial conditions in $\mathbf{C}$ to a line $x_1 = \bar{x}_1$, one can locally consider the inverse map $x_2(x_1^0)$, that takes values on a neighborhood of $\bar{x}_2$ with $(\bar{x}_1, \bar{x}_2) \in \mathbf{C}$ and which is increasing. Then it is enough to show that $\frac{\mathrm{d}T}{\mathrm{d}x_1^0}\big((\bar{x}_1, x_2(x_1^0))\big) > 0$.

Using (12) it is possible to write $T\big((-\frac{\pi}{2}, x_2(x_1^0))\big)$ as a sum of two integrals:

$$
\begin{aligned}
T\big((-\frac{\pi}{2}, x_2(x_1^0))\big) &= \int_0^{\sqrt{2-2\cos x_1^0+2\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-\left(1-\frac{y^2}{2}\right)^2}} \\
&+ \int_{-\sqrt{2-2\cos x_1^0+2\sin x_1^0}}^{2\sqrt{1+\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-\left(1+2\sin x_1^0-\frac{y^2}{2}\right)^2}}. \qquad (18)
\end{aligned}
$$

Therefore it is possible to compute formally the derivative of this quantity and one can verify numerically that it is larger than 9.9 for every value of x_1^0 corresponding to trajectories of Λ.

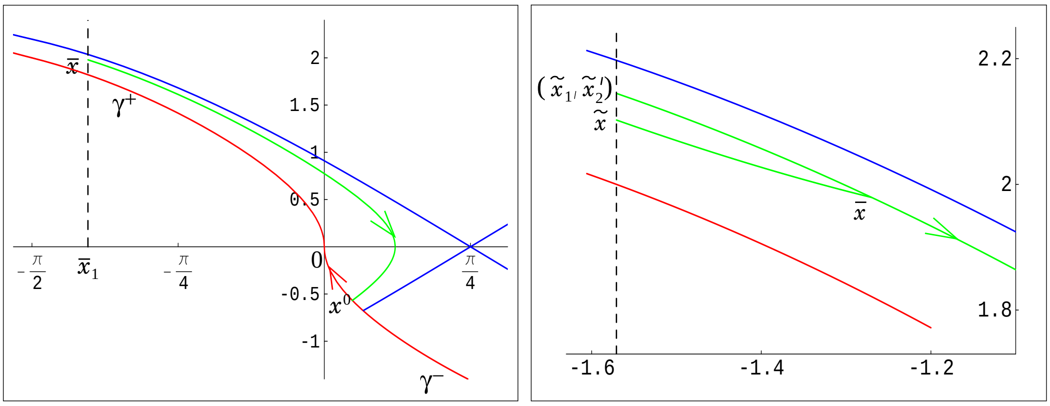


Figure 5: Region **C**

Similarly, it is possible to compute the time between two consecutive switchings at $x_1 = -\frac{\pi}{2} - n\pi$ and $x_1 = \frac{\pi}{2} - n\pi$, $n \geq 1$, up to the point $(-\frac{\pi}{2}, x_2(x_1^0))$, using equation (10), and the corresponding derivative with respect to x_1^0 has the following form:

$$-2\cos x_1^0 \int_{-\frac{\pi}{2}-n\pi}^{\frac{\pi}{2}-n\pi} \frac{\mathrm{d}z}{\big(2\,(h - \cos z - u \sin z)\big)^{3/2}} \tag{19}$$

where $h = 2n + 1 + 2\sin x_1^0$ and $u = \cos n\pi\,,\ n \geq 1$. The derivative $\frac{\mathrm{d}T}{\mathrm{d}x_1^0}\big((\bar{x}_1, x_2(x_1^0))\big)$ is obtained as the sum of the derivative of (18) and the series of the terms (19) for $n \geq 1$. The latter in modulus is bounded by $\sum_{n=1}^{\infty} n^{-\frac{3}{2}} < 3$. The lemma follows immediately from these estimates. □

Proof of Proposition 2. Assume by contradiction that there exists an optimal trajectory γ starting from a point $\tilde{x} \in \mathbf{C}$ with $u(t) = -\operatorname{sgn}(\cos x_1(t))$ and switching to $u(t) = \operatorname{sgn}(\cos x_1(t))$ at a point $\bar{x} \in \mathbf{C}$, after which it follows the corresponding trajectory $\bar{\gamma}$ of Λ until the origin. Before reaching $\bar{x}$, $\bar{\gamma}$ crosses the line $x_1 = \tilde{x}_1$ at a point $(\tilde{x}_1, \tilde{x}_2')$ with $\tilde{x}_2' > \tilde{x}_2$. See the right of Figure 5. Let T be the time to reach the origin along γ starting from $\tilde{x}$. Then from the equation $\dot{x}_1 = x_2$, one immediately sees that $T\big((\tilde{x}_1, \tilde{x}_2')\big) < T$. Moreover by the previous lemma it must be $T(\tilde{x}) < T\big((\tilde{x}_1, \tilde{x}_2')\big)$. Therefore, $T(\tilde{x}) < T$, a contradiction. □

3.3 Overlap curves and switching curves around $(\pi, 0)$

In Section 3.2.1 we examined a candidate switching curve C and we analyzed an extremal trajectory that switches at $x^0 \in \gamma^-$. We can determine by direct computation that if x_1^0 is large enough, then the corresponding switching point $\tilde{x}$ of C, obtained by solving (17), belongs to region **A**. Therefore there exists a second extremal trajectory starting at $\tilde{x}$ with control $u = 1$, then switching at $x_1 = \frac{3\pi}{2}$ and reaching the origin with an arc of γ^+. It is also possible to determine numerically that if x_1^0 is larger than a value ≈ 0.53, the extremal trajectory starting at $\tilde{x} \in C$ with $u = 1$ is time optimal.

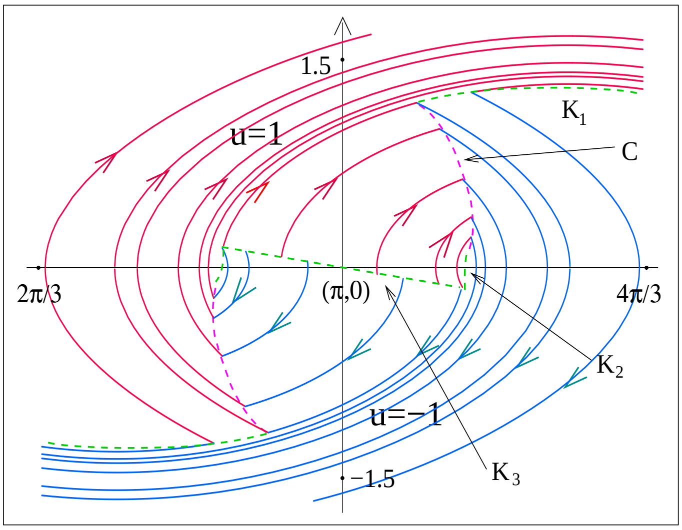


Figure 6: Optimal synthesis around $(\pi, 0)$

We deduce that the switching curve C is not completely optimal and there exists an overlap curve K_1 which starts at a point of C. On the other hand, it is possible to see, still numerically, that there is a point of C corresponding to $x_1^0 \approx 0.3$, at which the tangent vector to C is parallel to the vector field corresponding to $u = 1$. At this point a second overlap curve K_2 starts. See Figure 6. Associated to each point of this curve there are two optimal trajectories: the first one starts with control $u = -1$ and reaches γ^- at a point corresponding to a small value of x_1^0, while the second one starts with control $u = 1$, then switches on C and ends again on γ^-. The curve K_2 ends at a point in which a further overlap curve K_3 is generated. This curve contains the point $(\pi, 0)$ and the two time optimal trajectories starting at a point of K_3 have a symmetric behaviour, in the sense that they switch for the first time, respectively, on C and C' and for the last time on γ^- and γ^+. The complete synthesis around the point $(\pi, 0)$ can be completed using symmetry with respect to the point $(\pi, 0)$. Figure 6 provides a sketch of the optimal synthesis around the point $(\pi, 0)$.

In order to show how to determine explicitly the overlap curves previously defined we derive an equation for K_1. As in Section 3.2.1, we consider an extremal trajectory that starts at a point $\bar{x} \in K_1$ and whose last switching before reaching the origin with $u = -1$ occurs at $x^0 \in \gamma^-$. Using equations (10) and (11) on each bang arc where the extremal trajectory is either strictly monotone in x_1 or in x_2, it is possible to obtain a formula for the elapsed time, as in (12). We assume that $x_1^0 < \pi/2$, but if not, one can proceed in the same way to find a similar equation. This procedure can now be repeated for the other trajectory that reaches the origin by switching at a point $x^1 \in \gamma^+$. By propagating values of $h(\cdot)$ on each bang arc it can be determined that $x_1^1 := 2\pi - \arcsin\left(\sin x_1^0 + \sin \bar{x}_1\right)$. The equation for the overlap curve is obtained by setting the times to reach the origin along the two possible extremal trajectories equal, to yield:

$$\int_0^{x_2^0} \frac{\mathrm{d}y}{\sqrt{2 - (1 - \frac{1}{2}y^2)^2}} + \int_{x_1^0 - \frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_1 - 2\sqrt{2}\cos y}} + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_2 - 2\sqrt{2}\cos y}}$$

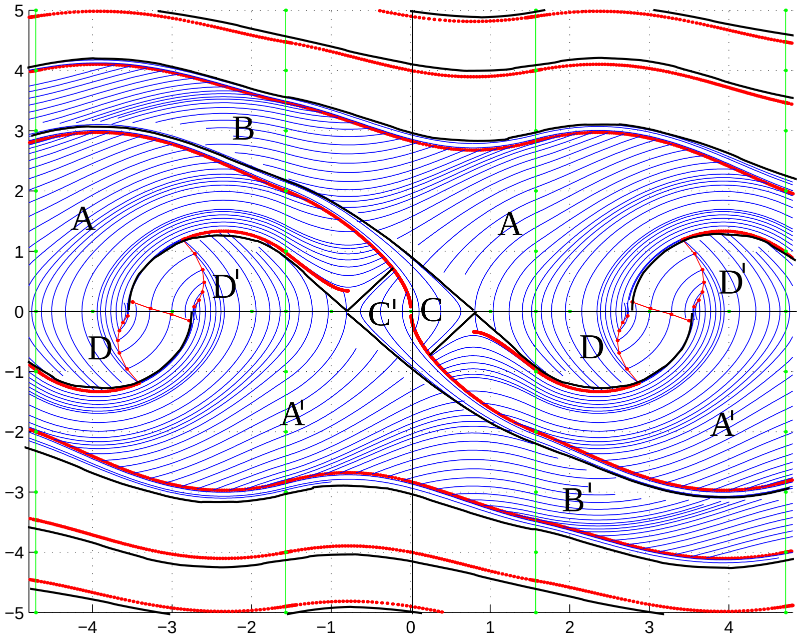


Figure 7: The complete time optimal synthesis

$$+\int_0^{2\sqrt{\sin x_1^0}} \frac{\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}} + \int_0^{\bar{x}_2} \frac{\mathrm{d}y}{\sqrt{2-(h_2-\frac{1}{2}y^2)^2}}$$

$$= \int_{\bar{x}_1-\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_3-2\sqrt{2}\cos y}} + \int_{\frac{7\pi}{4}}^{x_1^1+\frac{\pi}{4}} \frac{\mathrm{d}y}{\sqrt{2h_4-2\sqrt{2}\cos y}} + \int_0^{x_2^1} \frac{\mathrm{d}y}{\sqrt{2-(1-\frac{1}{2}y^2)^2}}$$

where

$$x_2^0 = -\sqrt{2(1-\cos x_1^0+\sin x_1^0)}, \quad \bar{x}_2 = \sqrt{2(h_2-\cos\bar{x}_1+\sin\bar{x}_1)},$$

$$x_2^1 = \sqrt{2(1-\cos x_1^1-\sin x_1^1)}, \quad h_1 = 1+2\sin x_1^0, \quad h_2 = -1+2\sin x_1^0,$$

$$h_3 = -1+2\sin x_1^0+2\sin\bar{x}_1, \quad h_4 = 1+2\sin x_1^0+2\sin\bar{x}_1.$$

It is also possible to find equations for K_2 and K_3, using (10) and (11) and with the help of the expressions of the switching curves C and C' found above. In order to avoid long and not very interesting computations (similar to those made above), we will not present these derivations.

4 Time optimal synthesis

In this section we assemble all of the previous results to obtain the time optimal synthesis for the inverted pendulum on the whole cylinder $\mathbb{S}\times\mathbb{R}$.

Following the approach of geometric control, it would be natural to apply the known sufficiency results that guarantee the optimality of an extremal synthesis (see for instance [8] and [20]). To do this, one first needs to determine a candidate extremal synthesis, i.e. select a single extremal trajectory for each initial point, and then verify the regularity conditions required by the theorems in the above references. Roughly speaking, such regularity conditions, for instance according to [8], amount to requiring piecewise regularity w.r.t. time of trajectories and w.r.t. to space of the corresponding cost function. The verification of such conditions involves tedious but straightforward computations.

On the other hand, a careful selection of trajectories, obtained by combining the conditions established in Section 3.2 with the results of Section 3.3, is enough to isolate a unique extremal trajectory, which is also a candidate for optimality, for each initial point. The obtained synthesis is clearly optimal. We begin with the following lemma.

Lemma 2. *The trajectories γ^+ and γ^- are optimal on $(-\infty, 0]$.*

Proof. We know from Proposition 1 that the only extremal trajectory starting from $\gamma^+(t_0)$ (for some $t_0 < 0$) with control $u = \operatorname{sgn}(\cos x_1(t_0))$ is $\gamma^+|_{[t_0,0]}$. On the other hand, from Section 3.2, the optimal control corresponding to a trajectory starting from $\gamma^+(t_0)$ with $u = -\operatorname{sgn}(\cos x_1)$ may switch to $u = \operatorname{sgn}(\cos x_1)$ only at a point belonging to the switching curve C or to γ^+ itself. For the first case, it is clear that the trajectories starting from γ^+ with control $-\operatorname{sgn}(\cos x_1)$ never reach C. This is because C loses optimality close to the boundary of the region $\mathbf{D}'$ but the proposed trajectories do not reach $\mathbf{D}'$. For the second case of switching on γ^+ itself, the control passes from $-\operatorname{sgn}(\cos x_1)$ to $\operatorname{sgn}(\cos x_1)$ at some $\gamma^+(t_1)$, with $t_1 < t_0$, so that this trajectory can not be optimal. Therefore the only optimal trajectory starting from $\gamma^+(t_0)$ must be $\gamma^+|_{[t_0,0]}$. The same reasoning shows that γ^- is optimal on $(-\infty, 0]$. □

The optimal synthesis is described by the following result, and is depicted in Figure 7.

Theorem 2. *Consider an optimal trajectory starting from $x_0 \in S \times \mathbb{R}$. Then:*

- *If $x_0 \in \mathbf{B}$ the corresponding optimal trajectory starts with control $u = -\operatorname{sgn}(\cos x_1)$ and switches to $\operatorname{sgn}(\cos x_1)$ when it reaches γ^+.*
- *If $x_0 \in \mathbf{A}$ and if it is far enough from the boundary with $\mathbf{D}'$, the optimal trajectory corresponds to $u = -\operatorname{sgn}(\cos x_1)$ and switches to $\operatorname{sgn}(\cos x_1)$ when it reaches γ^+.*
- *If $x_0 \in \mathbf{D}'$ or $x_0 \in \mathbf{A}$ is close to the boundary with $\mathbf{D}'$, then, according to Section 3.3, there are three possibilities. If x_0 is "below" the overlap curve K_3 the optimal trajectory starts with $u = -1$, switches when it reaches C' and continues with $u = -\operatorname{sgn}(\cos x_1)$ until it reaches γ^+. If x_0 is between K_3 and $C \cup K_2$ the optimal trajectory starts with $u = 1$, switches when it reaches C and continues with $u = \operatorname{sgn}(\cos x_1)$ until it reaches γ^-. Otherwise the optimal trajectory corresponds to $u = \operatorname{sgn}(\cos x_1)$ until it reaches γ^-.*
- *If $x_0 \in \mathbf{C}$ and it is "close" to γ^+, then the optimal trajectory corresponds to $u = \operatorname{sgn}(\cos x_1)$ until γ^-. Otherwise $u = -\operatorname{sgn}(\cos x_1)$ until γ^+. More precisely there is an overlap curve K winding around the cylinder, between γ^+ and χ^+, dividing C in two parts with different optimal strategies.*

If x_0 belongs to the regions $\mathbf{B}'$, $\mathbf{A}'$, $\mathbf{D}$, $\mathbf{C}'$ the optimal strategy is obtained by symmetry with respect to the origin.

Proof of Theorem 2. Consider an extremal trajectory starting from the region **B** with control $u = \operatorname{sgn}(\cos x_1)$. Then from Proposition 1 the control remains $\operatorname{sgn}(\cos x_1)$ until the trajectory reaches the x_1 axis. We deduce that this trajectory is not optimal, since there are no optimal trajectories crossing the x_1 axis and starting from **B**, as can be deduced from Sections 3.2, 3.3. On the other hand, still from 3.2, we know that there are no optimal trajectories switching on a point of the switching curve C, so that the optimal trajectory must be the one starting with control $-\operatorname{sgn}(\cos x_1)$ that switches to $u = \operatorname{sgn}(\cos x_1)$ when it reaches γ^+.

Consider now the region **C**. Let $\gamma^* : [t^*, 0] \to S \times \mathbb{R}$ be the trajectory starting from $\gamma^+(t_0)$ (for some $t_0 < 0$) with control $u = -\operatorname{sgn}(\cos x_1)$ and switching to $u = \operatorname{sgn}(\cos x_1)$ when it reaches again γ^+. As we proved above, this trajectory is optimal on every subinterval $[t, 0] \subset [t^*, 0]$ if $\gamma^*(t) \in \mathbf{B}$ but it is not optimal on the whole interval $[t^*, 0]$. In particular, from the continuity of the minimum time function (see for instance [11]) it is not optimal on $[t, 0]$ if $t > t^*$ is close enough to t^*. We deduce that there is a strip of optimal trajectories corresponding to the control $u = \operatorname{sgn}(\cos x_1)$ "just above" γ^+. This strip is delimited by an overlap curve K that extends the overlap curve K_1 found in Section 3.3 (K can be computed in an analogous way). Above K, the optimal control is $u = -\operatorname{sgn}(\cos x_1)$. This concludes the description of the optimal synthesis inside **C**.

The most delicate regions are the regions **A** and **D**. However we know that every trajectory that does not cross the x_1 axis and does not cross C, C' must switch from $u = -\operatorname{sgn}(x_2 \cos x_1)$ to $u = \operatorname{sgn}(x_2 \cos x_1)$ at a point of $\gamma^\pm$. Therefore, the synthesis inside these regions is completely determined in Sections 3.2 and 3.3 (see Figure 6). □

The qualitative shape of the optimal synthesis is now completely clarified. After solving the equations given in the previous sections that describe the switching curves and the overlap curve, and the analogous equations that determine the curve K defined above, one can easily obtain the global shape of the synthesis, as depicted in Figure 7.

5 Conclusion

In this paper, by using tools from geometric control theory, we have obtained a complete description of the time optimal synthesis for the problem of swinging up a pendulum on a cart, with the acceleration of the cart as control input. In particular, in order to determine the synthesis, we combined general results on the structure of the optimal trajectories with the computation of the special curves of the synthesis, overlap curves and switching curves.

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