Chattering in the Reach Control Problem

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Abstract

The Reach Control Problem (RCP) is a fundamental problem in hybrid control theory. The goal of the RCP is to find a feedback control that drives the state trajectories of an affine system to leave a polytope through a predetermined exit facet. In the current literature, the notion of leaving a polytope through a facet has an ambiguous definition. There are two different notions. In one, at the last time instance when the trajectory is inside the polytope, it must also be inside the exit facet. In the other, the trajectory is required to cross from the polytope into the outer open half-space bounded by the exit facet. In this paper, we provide a counterexample showing that these definitions are not equivalent for general continuous or smooth state feedback. On the other hand, we prove that analyticity of the feedback control is a sufficient condition for equivalence of these definitions. We generalize this result to several other classes of feedback control previously investigated in the RCP literature, most notably piecewise affine feedback. Additionally, we clarify or complete a number of previous results on the exit behavior of trajectories in the RCP.

Key words: Reach Control Problem, Piecewise affine feedback, Switched control, Zeno behaviour

1 Introduction

In the past period there has been a significant effort to formalize the mathematical foundations of switched and hybrid control systems. Due to the discontinuous nature of such systems, fundamental results guaranteeing the existence and uniqueness of solutions of classical ODEs with continuous vector fields no longer hold automatically. A new theory of existence and uniqueness of solutions of switched and hybrid systems has been formulated by, among others, Imura and van der Schaft (2000); Heemels et al. (2002); Lygeros et al. (2003).

An additional property specific to switched and hybrid systems is *Zeno* behaviour, in which a trajectory, even if guaranteed to exist and be unique, undergoes an infinite number of switches, i.e., discontinuous changes in the governing vector field, in a finite time interval. This property has been the subject of intense recent research, e.g., by Ames and Sastry (2005); Heymann et al. (2005); Çamlibel (2008); Goebel and Teel (2008).

Additionally, a number of classical control notions such as controllability, observability, and Lyapunov stability do not apply to systems governed by discontinuous vector fields. There has been significant work to extend these concepts to switched and hybrid systems (see, e.g., Ezzine and Haddad (1988); Petterson and Lennartson (1996); Branicky (1998)). For a comprehensive treatment of discontinuous dynamical systems see the work by Liberzon (2003); Cortés (2008).

This paper follows the above line of research on deepening the mathematical foundations of hybrid control system, but here we focus on reach control theory. The central problem in this theory is the Reach Control Problem (RCP) (Habets et al., 2006; Roszak and Broucke, 2006). Further work appeared in Belta et al. (2002); Habets and van Schuppen (2004); Broucke (2010); Ashford and Broucke (2013); Helwa and Broucke (2013, 2014); Broucke and Ganness (2014); Semsar-Kazerooni and Broucke (2014); Helwa and Broucke (2015); Helwa et al. (2016); Moarref et al. (2016); Wu and Shen (2016); Ornik and Broucke (2017). The goal of the RCP is to find a feedback control u such that, for any initial state x_0 inside a polytope $\mathcal{P}$, the trajectory $\phi(\cdot, x_0)$ of an affine control system $\dot{x} = Ax + Bu + a$ leaves $\mathcal{P}$ through a predetermined exit facet $\mathcal{F}_0$ in finite time, without first leaving $\mathcal{P}$ through any other facets. While there is an extensive literature on reach control theory, this is the first paper that focuses solely on a formal and complete discussion of existence, uniqueness, and behaviour of solutions.

The intention is for the RCP to serve as a building block in a hybrid control strategy that rests upon triangulating the state space to achieve some control objective. For example, if the system state is desired to go from one area of the state space to another, this can be achieved by partitioning the entire

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state space into simplices or polytopes, and constructing a sequence of polytopes such that the state trajectories move through the polytopes in the desired order, until they finally reach the last polytope (see Figure 1).

Numerous applications of the RCP have already been identified. These include biomolecular networks (Belta et al., 2002), control of aircraft (Belta and Habets, 2006), process control (Haugwitz and Hagander, 2007), aggressive maneuvers of mechanical systems (Vukosavljev and Broucke, 2014), quadcopter motion (Vukosavljev et al., 2016), and automatic parallel parking of vehicles (Ornik et al., 2017).

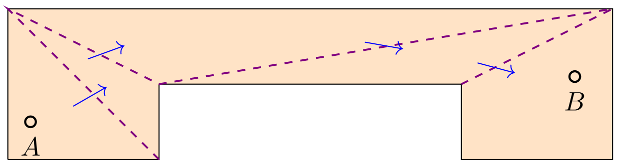

Fig. 1. An example of a reach control approach to solving a control problem. The state space is given in red, and the control objective is to guide the system state from point A on the left to point B on the right. The state space is cut into polytopes, and the goal is to define a controller on each polytope such that the desired sequences of polytopes (denoted by blue arrows) is followed. The exit facets of all polytopes in sequence are marked in purple.

This paper focuses on what happens when trajectories transition from one polytope to the next. In order to make the transitions between polytopes work, it is not only necessary for a trajectory to exit a polytope $\mathcal{P}$ with its last point in $\mathcal{P}$ lying on the desired exit facet $\mathcal{F}_0$. We must ensure that this exit will simultaneously result in the trajectory entering the next polytope in the desired sequence. This paper investigates a fundamental question in reach control theory which has not been addressed by previous work on general hybrid or switched systems: *what is the appropriate notion of leaving a polytope or a simplex through a facet?*

In Habets and van Schuppen (2001, 2004) it was required that velocity vectors must point strictly outside the polytope at points in the exit facet. This condition implies that a trajectory arriving at the exit facet will immediately enter the open half-space outside $\mathcal{P}$ and bounded by the exit facet. Sufficient conditions were given in Habets and van Schuppen (2004) for a Lipschitz continuous feedback to solve this problem. The proof assumes strict inward or outward conditions on velocity vectors along the facets of the polytope. When these conditions are not strict, certain pathologies can arise, as this paper will show, and arguments about whether trajectories lie in certain half-spaces with respect to facets are considerably more delicate. Lemma 3 of Roszak and Broucke (2006) regards trajectories exiting a polytope via a facet but without necessarily crossing into the outer open half-space. In Section 3.4 we provide a complete proof of a stronger version of this result.

One goal of this paper is to explore the relationship between the two notions for exiting a polytope. We consider the following question: *Is it possible for a trajectory to leave a polytope $\mathcal{P}$ but without crossing into an outer half-space?* When a trajectory exits $\mathcal{P}$ but does not cross into an outer half-space, we say it *chatters*. A second goal of the paper is to identify appropriate classes of feedback controls that do not allow chattering.

The paper is organized as follows. In Section 2 we define the Reach Control Problem and discuss the nuanced notions of *exiting through* a facet, *crossing* a facet, and *chattering*. In Section 3 we explore conditions on the vector field to disallow chattering. Section 3.1 discusses chattering under various feedback classes previously studied for the RCP. Section 3.2 focuses on the important class of continuous piecewise affine feedbacks. In Section 3.3 we apply these results to the Output Reach Control Problem (ORCP), first studied in Kroeze and Broucke (2016). Section 3.4 further discusses affine feedback control. Finally, Section 4 explores discontinuous piecewise affine feedback, as developed in Broucke and Ganness (2014).

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of $\mathcal{K}$ is $\mathcal{K}^c := \mathbb{R}^n \setminus \mathcal{K}$, and the set difference of two sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^n$ is denoted by $\mathcal{K}_1 \backslash \mathcal{K}_2$. The closure of set $\mathcal{K}$ is $\overline{\mathcal{K}}$. For two vectors $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of the two vectors. The notation co$\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$, and aff$(\mathcal{K})$ is the affine hull of set $\mathcal{K}$.

2 Problem Statement

Consider an n-dimensional polytope $\mathcal{P} := \text{co}\{v_0, \ldots, v_p\}$ with vertex set $V := \{v_0, \ldots, v_p\}$. A *facet* of $\mathcal{P}$ is an $(n-1)$-dimensional face of $\mathcal{P}$. Let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$ denote the facets of $\mathcal{P}$. The facet $\mathcal{F}_0$ is referred to as the *exit facet*, while $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are called *restricted facets*. Let $J = \{1, \ldots, r\}$ and let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. We note that each point on the boundary of $\mathcal{P}$ can belong to one or more facets. An example is given in Figure 2, where vertices of $\mathcal{P}$ belong to two facets, and other points on the boundary of $\mathcal{P}$ to one.

We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and rank$(B) = m$. Let $\mathcal{B} = \text{Im}(B)$, the image of B. Let $\phi(\cdot, x_0)$ denote the trajectory of (1) under some control law u starting from $x_0 \in \mathcal{P}$. The standard formulation of the RCP is as follows (Habets et al., 2006; Roszak and Broucke, 2006).

Problem 1 (Reach Control Problem (RCP)) *Consider system (1) defined on $\mathcal{P}$. Find a map $u : \mathcal{P} \to \mathbb{R}^m$ such that for every $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\varepsilon > 0$ such that*

(i) $\phi(t, x_0) \in \mathcal{P}$ *for all* $t \in [0, T]$,
(ii) $\phi(T, x_0) \in \mathcal{F}_0$, *and*

(iii) $\phi(t, x_0) \notin \mathcal{P}$ *for all* $t \in (T, T + \varepsilon)$.

We emphasize that the current setting of the RCP as given in Problem 1 does not stipulate that a system trajectory should leave $\mathcal{P}$ immediately after first entering the exit facet $\mathcal{F}_0$. Indeed, it is allowed for a system trajectory to touch the exit facet $\mathcal{F}_0$ and then go back into $\mathcal{P}$ before leaving through $\mathcal{F}_0$ at some later point.

Notice that in order for the RCP to make sense it is assumed that the dynamics (1) are extended to a small neighborhood of $\mathcal{P}$. This feature will not be recalled in the remainder of the paper except in Theorem 30 where we extend the dynamics in a non-obvious way. In line with the previous work, in the remainder of this paper we use the shorthand notation $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ to denote that Problem 1 is solved by some map u. In the remainder of the text, a map $u : \mathcal{P} \to \mathbb{R}^m$ will be referred to as *state feedback*.

We also note that the statement of Problem 1 implicitly assumes that the trajectory $\phi(\cdot, x_0)$ exists and is unique. This is a known result in the case where u is continuous. In the case of discontinuous feedback u and open-loop controls, both discussed in Section 4, this is true in the sense of Carathéodory: see, e.g., Hale (1980), Ch. I.5 for a longer discussion. In other words, there exists a unique absolutely continuous function $\phi(\cdot, x_0)$ such that

$$\phi(t, x_0) = x_0 + \int_0^t \left[A\phi(\tau, x_0) + Bu\big(\phi(\tau, x_0)\big) + a\right] d\tau$$

for all $t \geq 0$.

It is well-known that solvability of so-called *invariance conditions*, which ensure that trajectories do not exit the polytope through the restricted facets, is necessary to solve the RCP by various classes of feedbacks (Habets and van Schuppen, 2004; Habets et al., 2006; Roszak and Broucke, 2006). Let $J(x) = \{j \in J \mid x \in \mathcal{F}_j\}$. That is, $J(x)$ is the set of indices of the restricted facets that contain x. For each $x \in \mathcal{P}$, we define the closed, convex cone

$$\mathcal{C}(x) := \left\{ \, y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in J(x) \, \right\}. \qquad (2)$$

Notice that for any $x \in \mathcal{P} \setminus \mathcal{F}_0$, $\mathcal{C}(x)$ equals

$$T_{\mathcal{P}}(x) = \left\{ v \in \mathbb{R}^n \mid \liminf_{t \to 0+} \frac{\min_{y \in \mathcal{P}} \|x + tv - y\|}{t} = 0 \right\},$$

the Bouligand tangent cone to $\mathcal{P}$ at x. For further details on the Bouligand tangent cone, see Clarke et al. (1998). At points $x \in \mathcal{F}_0$, $\mathcal{C}(x)$ and $T_{\mathcal{P}}(x)$ are different since $\mathcal{C}(x)$ includes directions pointing out of $\mathcal{P}$ through $\mathcal{F}_0$. An illustration of the above notions, adapted from Moarref et al. (2016), is given in Figure 2.

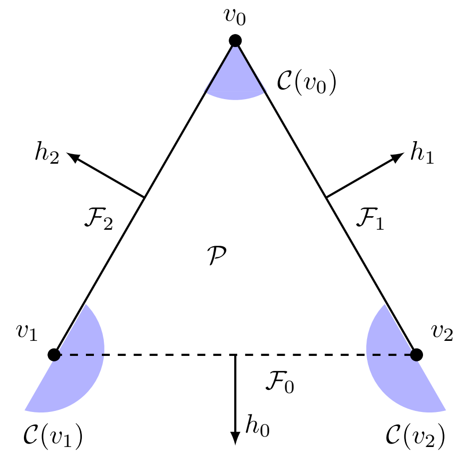

Fig. 2. An illustration of notation used in the paper. The polytope $\mathcal{P} = \mathrm{co}\{v_0, v_1, v_2\}$ is given by vertices $V_{\mathcal{S}} = \{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$, with each facet indexed by the vertex it does not contain. h_i is the unit normal vector pointing out of $\mathcal{S}$. $\mathcal{F}_0$ is designated as the exit facet. Because of their previously discussed geometric meaning, the cones $\mathcal{C}(v_i)$ are illustrated attached at each v_i. However, by (2), each cone $\mathcal{C}(x)$ has its apex at 0.

Definition 2 *We say that* the invariance conditions are solvable *if for each* $x \in \mathcal{P}$ *there exists* $u \in \mathbb{R}^m$ *such that*

$$Ax + Bu + a \in \mathcal{C}(x)\,. \qquad (3)$$

We now come to the central issue studied in this paper: how trajectories exit $\mathcal{F}_0$. First, it can be seen that a notion of trajectories exiting a facet of $\mathcal{P}$ appears implicitly in the statement of the RCP. This notion is formalized as follows.

Definition 3 *We say* $\phi(\cdot, x_0)$ exits $\mathcal{P}$ through facet $\mathcal{F}_0$ *if there exist* $T \geq 0$ *and* $\varepsilon > 0$ *such that*

(i) $\phi(t, x_0) \in \mathcal{P}$ *for all* $t \in [0, T]$*;*
(ii) $\phi(T, x_0) \in \mathcal{F}_0$*;*
(iii) $\phi(t, x_0) \notin \mathcal{P}$ *for all* $t \in (T, T + \varepsilon)$.

We say $\phi(\cdot, x_0)$ exits $\mathcal{P}$ *if it exits* $\mathcal{P}$ *through some facet of* $\mathcal{P}$.

Despite the fact that the statement of the RCP in Problem 1 is clear on the meaning of exiting a facet, other notions are used in the literature, as discussed in Section 1. We now define a stronger notion of exiting a facet compared to Definition 3. To that end, let $\mathcal{H}_i$ be the closed half-space bounded by aff$(\mathcal{F}_i)$ which contains $\mathcal{P}$; that is

$$\mathcal{H}_i = \{x \in \mathbb{R}^n \mid h_i \cdot (x - y) \leq 0\}, \qquad y \in \mathcal{F}_i. \qquad (4)$$

Since $h_i \perp (y_1 - y_2)$ for all $y_1, y_2 \in \mathcal{F}_i$, definition of $\mathcal{H}_i$ in (4) does not depend on the choice of y.

Definition 4 *We say* $\phi(\cdot, x_0)$ crosses *facet* $\mathcal{F}_i$ *if there exist* $T \geq 0$ *and* $\varepsilon > 0$ *such that*

(i) $\phi(t, x_0) \in \mathcal{P}$ *for all* $t \in [0, T]$*;*
(ii) $\phi(T, x_0) \in \mathcal{F}_i$*;*
(iii) $\phi(t, x_0) \notin \mathcal{H}_i$ *for all* $t \in (T, T + \varepsilon)$*.*

Both notions of exiting the polytope — either *exiting through* a facet or by *crossing* a facet — have appeared in the literature, often with little distinction made between the two. Clearly, if a trajectory crosses a facet $\mathcal{F}_i$, then the trajectory exits $\mathcal{P}$ through $\mathcal{F}_i$. But the converse statement is not true. In fact, it is possible for a trajectory to exit $\mathcal{P}$ through one or more facets, but not cross any of its facets. The following example illustrates the dichotomy.

Example 5 *Let* $\mathcal{P} = \mathrm{co}\{v_0, v_1, v_2, v_3\}$*, with* $v_0 = (0, 0, 1)$*,* $v_1 = (1, 0, 0)$*,* $v_2 = (0, 1, 0)$*, and* $v_3 = (0, 0, 0)$*. Let the facets of* $\mathcal{P}$ *be* $\mathcal{F}_0, \ldots, \mathcal{F}_3$*, each facet indexed by the vertex it does not contain. Consider system* (1) *with* $A = 0$*,* $B = I$*,* $a = 0$*, and* $u(x) = f(x)$*, where*

$$f(x) := \begin{pmatrix} \frac{e^{-1/x_3^2}(x_3 \cos(1/x_3) - 2\sin(1/x_3) - 1.98)}{x_3^3} \\ \frac{e^{-1/x_3^2}(-2\cos(1/x_3) - x_3 \sin(1/x_3) - 1.98)}{x_3^3} \\ 1 \end{pmatrix},$$

with

$$f(\cdot, \cdot, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It can be verified that $f \in C^\infty$*. For the initial condition* $x_0 = 0$*, the solution to* (1) *is*

$$\phi(t, x_0) = \begin{pmatrix} -e^{-1/t^2}(\sin\frac{1}{t} + 0.99) \\ -e^{-1/t^2}(\cos\frac{1}{t} + 0.99) \\ t \end{pmatrix}, \quad t > 0, \qquad (5)$$

with $\phi(0, x_0) = 0$*.*

First, we will show that $\phi(\cdot, x_0)$ *exits* $\mathcal{P}$ *at time* $T = 0$ *in the sense of Definition 3. Since* $\mathcal{P} = \{x \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}$*, it suffices to prove that for each* $t > 0$*, at least one of the coordinates of* $\phi(t, x_0)$ *is always negative. Suppose otherwise. Then, by* (5)*, for some* $t > 0$*,* $-e^{-1/t^2}(\cos(1/t) + 0.99) \geq 0$ *and* $-e^{-1/t^2}(\sin(1/t) + 0.99) \geq 0$*. This implies* $\cos(1/t), \sin(1/t) \leq -0.99$*, which contradicts* $\cos^2(1/t) + \sin^2(1/t) = 1$*. Hence,* $\phi(\cdot, x_0)$ *exits* $\mathcal{P}$ *at time* $T = 0$*.*

On the other hand, $\phi(\cdot, x_0)$ *does not cross any facet* $\mathcal{F}_i$ *at* $T = 0$ *in the sense of Definition 4. Since* $\phi(0, x_0) = 0 \notin \mathcal{F}_3$*,* $\phi(\cdot, x_0)$ *clearly does not cross* $\mathcal{F}_3$ *at time* $T = 0$*. Observe that* $\mathcal{H}_0 = \{x \in \mathbb{R}^3 \mid x_3 \geq 0\}$*, and* $\mathcal{H}_i = \{x \in \mathbb{R}^3 \mid x_i \geq 0\}$ *for* $i = 1, 2$*. From* (5) *we note that* $\phi(t, x_0) \in \mathcal{H}_0$ *for all* $t \geq 0$*. Hence,* $\phi(\cdot, x_0)$ *does not cross* $\mathcal{F}_0$ *at* $T = 0$*.*

Next suppose that $\phi(\cdot, x_0)$ *crosses* $\mathcal{F}_1$ *at time* $T = 0$*. Then, by (iii) in Definition 4 and by* (4)*, there exists* $\varepsilon > 0$ *such that* $-e^{-1/t^2}(\cos(1/t) + 0.99) < 0$ *for all* $0 < t < \varepsilon$*. However, this is impossible: take* $t = 1/((2k+1)\pi)$ *with sufficiently large* $k \in \mathbb{N}$ *to obtain* $-e^{-1/t^2}(\cos(1/t) + 0.99) = 0.01e^{-1/t^2} > 0$*. Thus,* $\phi(\cdot, x_0)$ *does not cross* $\mathcal{F}_1$*.*

Analogously, there does not exist $\varepsilon > 0$ *such that* $-e^{-1/t^2}(\sin(1/t) + 0.99) < 0$ *for all* $0 < t < \varepsilon$*, by taking* $t = 1/((2k + 3/2)\pi)$ *for sufficiently large* $k \in \mathbb{N}$*. Hence,* $\phi(\cdot, x_0)$ *does not cross* $\mathcal{F}_2$*. We conclude that* $\phi(\cdot, x_0)$ *does not cross any facet of* $\mathcal{P}$*, yet,* $\phi(\cdot, x_0)$ *exits* $\mathcal{P}$ *at time* $T = 0$*.*

An illustration of Example 5 is provided in Figure 3.

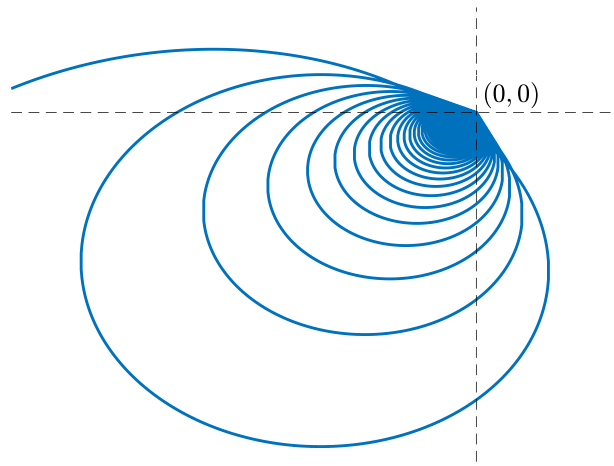

Fig. 3. An illustration of the trajectory in Example 5 projected on the xy-plane. To make the pertinent details more clearly visible, the trajectory has been slightly modified from the one given in (5).

While the example above falls into the class of systems (1) with $A = 0$, $a = 0$, $B = I$, and $u(x) = f(x)$, the above example is clearly rather artificial. It was obtained by finding two smooth real functions which both oscillate between positive and negative values when approaching 0, but are never simultaneously positive. This results in the situation where the state trajectory constantly switches between $\mathcal{H}_1 \backslash \mathcal{H}_2$ and $\mathcal{H}_2 \backslash \mathcal{H}_1$. We will call such a pathological behaviour *chattering*.

Definition 6 *We say the trajectory* $\phi(\cdot, x_0)$ chatters *if it exits* $\mathcal{P}$*, but does so without crossing any of its facets.*

The primary motivation for exploring chattering has been briefly mentioned in the introduction: the method of reach control relies on designing a sequence of polytopes, with a control law on each polytope, such that the trajectory of a system is driven from a starting point in the first polytope to an ending point in the final polytope. An example of that is given in Figure 1. The RCP, which just concerns driving a system out of a polytope through a given facet, is a building block for this method. The underlying assumption needed to relate the RCP to reach control is that, if the exit facet of one polytope connects that polytope to the next one in the sequence, leaving through an exit facet will force the system to automatically enter the next polytope. However, this assumption is not encoded anywhere in the statement of the RCP, and chattering is one obvious pathology that results in a system leaving a polytope, but not entering the next one in sequence.

We note that chattering occurs because the definition of crossing a facet requires the existence of an open interval on which the trajectory is in the same halfspace. Example 5 shows that with a smooth vector field, due to oscillating trigonometirc terms in the equation of the trajectory, no such interval exists. Thus, the trajectory chatters. A question of practical interest in design, as motivated by the above paragraph, is to determine which classes of feedback controls used to solve the RCP guarantee that no chattering occurs.

Problem 7 *Find conditions on the vector field in* (1) *such that no trajectory of that system chatters.*

Example 5 shows that smoothness is not sufficient to satisfy Problem 7. In Section 3 we will show that analyticity of the vector field is such a sufficient condition. In turn, this finding answers Problem 7 for standard classes of feedback used in the RCP. In Section 3.2 we extend the results to continuous piecewise affine vector fields. We use this extension to provide the missing proof of Lemma 13 from Kroeze and Broucke (2016) in Section 3.3. Additionally, using the result that affine feedback guarantees there is no chattering, we complete the proof of Lemma 3 from Roszak and Broucke (2006) in Section 3.4. Finally, in Section 4 we revisit the work of Broucke and Ganness (2014) on discontinuous piecewise affine feedback. We identify an issue with the control law proposed in that paper, propose an amended control law, and prove that chattering does not occur in the context of that control law.

3 Main Results

We begin our investigation of Problem 7 by identifying the key property of a vector field that closes the gap between Definition 3 and Definition 4 regarding how trajectories exit $\mathcal{P}$. Motivated by Example 5, we consider the system

$$\dot{x} = f(x) \tag{6}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

We use the following standard definition of analytic functions (see, e.g., Krantz and Parks (2002) for a longer discussion of analyticity):

Definition 8 *Map* $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ *is* real analytic *at* x_0 *if there exists a neighbourhood* $\mathbb{R}^n \supset U \ni x_0$ *such that, for each* $y \in U$*, each* f_i *can be represented by a convergent power series in some neighbourhood of* y*.*

Lemma 9 *Let* $\mathcal{P}$ *be an* n*-dimensional convex polytope. Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be analytic at* x_0*, and let* $\phi(\cdot, x_0)$ *be the unique solution of* (6) *with* $\phi(0, x_0) = x_0$*. Then for every* $h \in \mathbb{R}^n$*, there exists* $\varepsilon > 0$ *such that either* $h \cdot (\phi(t, x_0) - x_0) \leq 0$ *for all* $t \in (0, \varepsilon)$ *or* $h \cdot (\phi(t, x_0) - x_0) > 0$ *for all* $t \in (0, \varepsilon)$*. Moreover, if* $\phi(\cdot, x_0)$ *exits* $\mathcal{P}$ *at time* $T = 0$*, then there exists a facet* $\mathcal{F}_i$ *of* $\mathcal{P}$ *such that* $\phi(\cdot, x_0)$ *crosses* $\mathcal{F}$ *at* $T = 0$*.*

PROOF. By the Cauchy-Kowalevski theorem (see, e.g., Courant and Hilbert (1962); Garabedian (1964)), $\phi(t, x_0)$ is guaranteed to be analytic in t on some interval $(-\varepsilon_1, \varepsilon_1)$, where $\varepsilon_1 > 0$. Considering any $h \in \mathbb{R}^n$, $g(\cdot) := h \cdot (\phi(\cdot, x_0) - x_0)$ is a real analytic function of one variable. Clearly, $g(0) = 0$. Additionally, since the zeros of a non-zero real analytic functions on an interval are isolated (see, e.g., the discussion around the Weierstrass Preparation Theorem in Krantz and Parks (2002)), we obtain that there exists an interval $(0, \varepsilon)$ such that one of the following holds:

(i) $g(t) < 0$ for all $t \in (0, \varepsilon)$,
(ii) $g(t) > 0$ for all $t \in (0, \varepsilon)$,
(iii) $g(t) = 0$ for all $t \in [0, \varepsilon)$.

This proves our first claim.

For the second claim, suppose it is incorrect. That is, $\phi(\cdot, x_0)$ leaves $\mathcal{P}$ at time $T = 0$, but does not cross any facets $\mathcal{F}_i \subset \mathcal{P}$. Thus, for all facets $\mathcal{F}_i$ such that $\phi(0, x_0) = x_0 \in \mathcal{F}_i$, there is no interval $(0, \varepsilon)$ such that $h_i \cdot (\phi(t, x_0) - x_0) > 0$. By our first claim, this implies that there exists some $\varepsilon > 0$ such that for all $\mathcal{F}_i$ such that $x_0 \in \mathcal{F}_i$, $h_i \cdot (\phi(t, x_0) - x_0) \leq 0$ for all $t \in (0, \varepsilon)$. We also know that $x_0 \in \mathcal{P}$, so for all $\mathcal{F}_i$ such that $x_0 \notin \mathcal{F}_i$, $h_i \cdot (\phi(t, x_0) - x_0) < 0$ for all $t \in (0, \varepsilon')$, for some sufficiently small $\varepsilon' > 0$. Combining the previous two statements, $\phi(t, x_0) \in \mathcal{P}$ for all $t \in (0, \min\{\varepsilon, \varepsilon'\})$. This contradicts that $\phi(\cdot, x_0)$ leaves $\mathcal{P}$ at time $T = 0$. $\square$

3.1 Chattering Under Feedback

Lemma 9 provides the mathematical foundation to resolve the issue of chattering in previous work. We review the different classes of feedback control previously investigated in the literature and indicate how Lemma 9 applies to each class.

- **Affine feedback** was investigated by Habets and van Schuppen (2001, 2004); Roszak and Broucke (2005, 2006); Habets et al. (2006); Semsar-Kazerooni and Broucke (2014); Moarref et al. (2016); Wu and Shen (2016). Because affine maps are analytic, Lemma 9 applies directly and shows that *chattering cannot occur* in the RCP with affine feedback. In Section 3.4 we further discuss the exit behaviour of trajectories under affine feedback.

- **Continuous feedback** was investigated by Broucke (2010); Semsar-Kazerooni and Broucke (2014); Ornik and Broucke (2017). As shown in Example 5, Lemma 9 cannot apply to continuous maps in general, or even to smooth maps. Thus, *chattering can occur* in the RCP with continuous feedback.
- **Continuous piecewise affine feedback** was explored by Habets and van Schuppen (2001, 2004); Helwa and Broucke (2013, 2015). Unlike affine feedback, piecewise affine feedback results in the vector field f having a non-analytic structure. However, we will show in Section 3.2 that Lemma 9 is extendable to continuous piecewise affine vector fields. Hence, *chattering cannot occur* in the RCP with continuous piecewise affine feedback. This class of feedback arises in the *Output Reach Control Problem* (ORCP) investigated by Kroeze and Broucke (2016), which we discuss in Section 3.3.
- **Discontinuous piecewise affine feedback** was explored by Broucke and Ganness (2014). While the control law proposed in that paper suffers from minor inconsistencies discussed and resolved in Section 4, it can be shown that *chattering cannot occur* in that setting as well. We provide a proof of that claim in Section 4.
- **Multi-affine feedback** was explored by Belta et al. (2002); Helwa and Broucke (2014). Multi-affine functions have the form

$$u(x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_n \in \{0,1\}} c_i x_1^{i_1} \cdots x_n^{i_n},$$

  so they are a special case of polynomials in $\mathbb{R}^n$. Hence, multi-affine feedback is analytic, and Lemma 9 proves that *chattering cannot occur* in this case as well.
- **Time-varying feedback** was explored by Ashford and Broucke (2013). Time-varying feedback does not fall into the setting explored in this paper. However, Theorem 16 of Ashford and Broucke (2013) expresses the time-varying feedback proposed to solve the RCP in that paper as a multi-affine feedback on the extended state space $\mathcal{S} \times [0, 1]$, where $\mathcal{S}$ is the original state space. Thus, Lemma 9 guarantees that *chattering cannot occur in this extended system* using the feedback given by Ashford and Broucke (2013).

3.2 Continuous Piecewise Affine Feedback

Because piecewise affine maps are not analytic at points where switches between two pieces occur, piecewise affine feedback generally results in the vector field f lacking an analytic structure. Hence, Lemma 9 does not apply directly. However, continuous piecewise affine maps still contain much more structure than general continuous maps. We will show that they admit a piecewise analytic structure, as defined by Sussmann (1982). By invoking Theorem II of Sussmann (1982), we will prove that Lemma 9 can be extended to continuous piecewise affine maps. We recount the definition of a proper map and the definition of a subanalytic set given by Sussmann (1982).

Definition 10 *Let $\mathcal{X}, \mathcal{Y}$ be topological spaces, and $\mathcal{Z} \subset \mathcal{X}$. Map $f : \mathcal{X} \to \mathcal{Y}$ is* proper *on $\mathcal{Z}$ if for every compact set $A \subset \mathcal{Z}$, $f^{-1}(A)$ is a compact set.*

Definition 11 *The class of* subanalytic sets *is the smallest class $\mathfrak{C}$ of subsets E of finite-dimensional real analytic manifolds $\mathcal{M}$ such that:*

(i) *$\mathfrak{C}$ contains all sets $\{x \in \mathbb{R}^n \mid g(x) = 0\}$ and $\{x \in \mathbb{R}^n \mid g(x) > 0\}$, where $g : \mathbb{R}^n \to \mathbb{R}$ is analytic.*
(ii) *$\mathfrak{C}$ is closed under complementation, locally finite unions and intersections.*
(iii) *$\mathfrak{C}$ is closed under an inverse image of an analytic map.*
(iv) *$\mathfrak{C}$ is closed under an image $f(E)$ of an analytic map $f : \mathcal{M} \to \mathcal{N}$ which is proper on $\overline{E}$.*

We also use the following definition of a piecewise affine function.

Definition 12 *Let L be a finite index set and let $\{\mathcal{A}_l \mid l \in L\}$ be a finite polyhedral partition of $\mathbb{R}^n$. That is, each $\mathcal{A}_l$ consists of finite unions and intersections of sets $\{x \in \mathbb{R}^n \mid g(x) = 0\}$ and $\{x \mid g(x) > 0\}$, where $g : \mathbb{R}^n \to \mathbb{R}$ are affine functions. We say that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a* piecewise affine *(PWA) function if $f|_{\mathcal{A}_l}$ is an affine function for all $l \in L$.*

Remark 13 *We note that a PWA function is not necessarily continuous, as there is no requirement of continuity on the boundary between two polyhedra $\mathcal{A}_l$. Thus, a general PWA function is merely piecewise continuous. We further deal with discontinuous PWA functions in Section 4, while the remainder of this section is devoted to continuous PWA feedbacks.*

Remark 14 *For the sake of simplicity we consider PWA functions $f : \mathbb{R}^n \to \mathbb{R}^n$, defined on all of $\mathbb{R}^n$, even if the system* (1) *is defined only a polytope $\mathcal{P} \subset \mathbb{R}^n$. It was shown by Blanchini and Pellegrino (2007) that any PWA function on $\mathcal{P}$ can be extended to a PWA function on $\mathbb{R}^n$, and if the original function was continuous PWA, the extension can be continuous PWA as well.*

The setting explored by Sussmann (1982) is quite general. It deals with an extendably piecewise analytic vector field defined on a locally finite subanalytic partition of a real analytic manifold. We now give a version of Theorem II of Sussmann (1982) adapted to our needs. In our case, the sets $\mathcal{A}_j$ from Definition 12 are finite unions and intersections of sets in the form of case (i) of Definition 11. Thus, by property (ii) of Definition 11, they are subanalytic. The real analytic manifold that we deal with is $\mathbb{R}^n$ itself. Additional conditions (A1)-(A5) of Theorem II of Sussmann (1982) can be trivially verified to hold in the case of continuous PWA functions. Hence, we forgo the general statement of Sussmann (1982) in favour of a more wieldy version used in our setting.

Theorem 15 *Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous PWA function. Then for every compact set $K \subset \mathbb{R}^n$ and every $T > 0$ there exists a positive integer $N(K,T)$ which satisfies the following: if $x_0 \in K$ and $\phi(\cdot, x_0) : [0,T) \to K$ is the trajectory solving* (6) *with $\phi(0, x_0) = x_0$, then $\phi(\cdot, x_0)$ is a concatenation of at most $N(K,T)$ curves $\phi_1, \ldots, \phi_p$ such that for each $i \in \{1, \ldots, p\}$ there exists $l \in L$ with $\phi_i \in A_l$.*

Theorem 15 is proved by Sussmann (1982) in its entirety. We are now able to prove an extension of Lemma 9 to piecewise affine systems.

Lemma 16 *Let $\mathcal{P}$ be an n-dimensional convex polytope. Consider the system* (6) *and suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and PWA. Let $\phi(\cdot, x_0)$ be the unique solution of* (6) *with $\phi(0, x_0) = x_0$. Then for every $h \in \mathbb{R}^n$, there exists $\varepsilon > 0$ such that either $h \cdot (\phi(t, x_0) - x_0) \leq 0$ for all $t \in (0, \varepsilon)$ or $h \cdot (\phi(t, x_0) - x_0) > 0$ for all $t \in (0, \varepsilon)$. Moreover, if $\phi(\cdot, x_0)$ exits $\mathcal{P}$ at time $T = 0$, then there exists a facet $\mathcal{F}$ of $\mathcal{P}$ such that $\phi(\cdot, x_0)$ crosses $\mathcal{F}$ at $T = 0$.*

PROOF. Let us first note that the existence of $\phi(\cdot, x_0)$ is guaranteed on some interval $(-\varepsilon', \varepsilon')$, where $\varepsilon' > 0$, by the Picard-Lindelöf theorem, because f is Lipschitz continuous on a neighbourhood of x_0 (see, e.g., Blanchini (1995)). Because $\phi(\cdot, x_0)$ is continuous, it is bounded on the interval $[0, \varepsilon'/2]$. Let K be any closed ball such that $\{\phi(t, x_0) \mid t \in [0, \varepsilon'/2]\} \subset K$.

Let $\mathbb{R}^n$ be partitioned into finitely many sets $\{A_l \mid l \in L\}$ as in Definition 12. By Theorem 15, trajectory $\phi(t, x_0)$, $t \in [0, \varepsilon'/2]$ is a concatenation of finitely many curves $\phi_1, \ldots, \phi_p$ such that $\phi_i \in A_l$ for some $l \in L$. Thus, there exist $0 < \delta < \varepsilon'/2$ and $l \in L$ such that $\phi(t, x_0) \in A_l$ for all $t \in (0, \delta)$. Hence, on the interval $[0, \delta)$, $\phi(\cdot, x_0)$ is governed by an affine system $\dot{x} = f|_{\overline{A_l(x)}}$. As an affine function is certainly analytic, the claim now holds by Lemma 9. □

3.3 Output Reach Control Problem

The Output Reach Control Problem (ORCP) was introduced in Kroeze and Broucke (2016). The goal of the ORCP is to drive output trajectories starting in a given simplex $\mathcal{S}$ in the output space through a predetermined facet $\mathcal{F}_0$. The approach is to use viability theory to construct a polytope $\mathcal{P}$ in the state space such that if a state trajectory exits $\mathcal{P}$ through an exit facet $\mathcal{F}_0^{\mathcal{P}}$ of $\mathcal{P}$, then the output trajectory exits $\mathcal{S}$ through its exit facet $\mathcal{F}_0$. This problem is of particular relevance to our present investigation because a state trajectory exiting $\mathcal{P}$ through $\mathcal{F}_0^{\mathcal{P}}$ in the sense of Definition 3 does not imply the output trajectory exits $\mathcal{S}$. Instead it is necessary that state trajectories exit $\mathcal{P}$ in the sense of Definition 4. Resolving this dichotomy is the goal of Lemma 13 of Kroeze and Broucke (2016). A proof was not provided in that paper. Here we provide a complete proof.

Problem 17 (Output Reach Control Problem (ORCP)) *Consider system (1) defined on $\mathbb{R}^n$. Let $C \in \mathbb{R}^{n \times p}$, and let $y(\cdot, x_0) = C\phi(\cdot, x_0)$ for all $x_0 \in \mathbb{R}^n$. Let $\mathcal{S} \subset \mathbb{R}^p$ be a simplex. Find a state feedback $u(x)$ such that for every $x_0 \in \mathbb{R}^n$ such that $Cx_0 \in \mathcal{S}$, there exist $T \geq 0$ and $\varepsilon > 0$ such that*

(i) $y(t, x_0) \in \mathcal{S}$ *for all* $t \in [0, T]$,
(ii) $y(T, x_0) \in \mathcal{F}_0$, *and*
(iii) $y(t, x_0) \notin \mathcal{S}$ *for all* $t \in (T, T + \varepsilon)$.

The main result of Kroeze and Broucke (2016) is that solvability of the RCP on a particularly chosen bounded polytope $\mathcal{P} \subseteq \{x \in \mathbb{R}^n \mid Cx \in \mathcal{S}\}$, with some additional technical conditions, implies solvability of the ORCP on $\mathcal{S}$. As mentioned in Section 1, a key part of the proof is that trajectories exit a polytope through a restricted facet only if they also cross a non-restricted facet at the same time. This result is given in Lemma 13 of Kroeze and Broucke (2016). However, the proof is not provided.

It turns out that the issue of chattering is a salient one in proving Lemma 13. In other words, for the proof to go through, a non-chattering assumption should be made. Kroeze and Broucke (2016) make note of this, but do not actually explicitly state whether a particular non-chattering condition is assumed, and do not discuss why chattering does not occur in the context of that paper. In this paper, we clarify this inconsistency. While this is not explicitly stated in their paper, the viability approach used by Kroeze and Broucke (2016) relies on continuous PWA feedback control, thus resulting in a continuous PWA dynamical system. Our Lemma 16 now proves that chattering indeed does not occur in that case.

The statement of the claim that we prove is also slightly different than the statement of Lemma 13 by Kroeze and Broucke (2016). The original lemma states that a trajectory that exits $\mathcal{P}$ will cross a restricted facet only if it also crosses $\mathcal{F}_0$. Since we proved in Lemma 16 that a trajectory exiting $\mathcal{P}$ will necessarily cross at least one facet, this statement is equivalent to saying that a trajectory exiting $\mathcal{P}$ will necessarily cross $\mathcal{F}_0$. This is the form in which we present the lemma. The proof is adapted from the standard proof of the Nagumo/Bony-Brezis theorem (for instance, see Brezis (1970)).

Lemma 18 *Consider an n-dimensional convex polytope $\mathcal{P}$ with facets $\mathcal{F}_0, \ldots, \mathcal{F}_r$, such that $0 \in \mathcal{F}_0 \cap \cdots \cap \mathcal{F}_k$ for some $k < r$. Consider the affine system* (1) *and let $u(x)$ be a continuous PWA feedback such that*

$$h_i \cdot (Ax + Bu(x) + a) \leq 0, \quad x \in \text{aff}(\mathcal{F}_i), \quad i = 1, \ldots, r. \tag{7}$$

Suppose $\phi(\cdot, 0)$ is the unique closed-loop trajectory of (1) *with $\phi(0, 0) = 0$. If $\phi(\cdot, 0)$ exits $\mathcal{P}$ at time $T = 0$, then it does so by crossing $\mathcal{F}_0$.*

Remark 19 *Note that Lemma 18 does not say that trajectories do not cross $\mathcal{F}_1, \ldots, \mathcal{F}_k$ at 0.*

Remark 20 *Condition (7) is essentially the same as imposing cone conditions (3) on facets $\mathcal{F}_1, \ldots, \mathcal{F}_r$. We use $\mathrm{aff}(\mathcal{F}_i)$ instead of $\mathcal{F}_i$ because of a minor technicality in the proof. However, $\mathrm{aff}(\mathcal{F}_i) \backslash \mathcal{F}_i$ is entirely contained in $\mathbb{R}^n \backslash \mathcal{P}$, and the difference thus only concerns those parts of the controller which are actually outside our polytope of interest. In addition, as is clear from the proof below, we can additionally relax the requirement (7) into*

$$h_i \cdot (Ax + Bu(x) + a) \leq 0, \quad x \in \mathcal{F}_i^N, \quad i = 1, \ldots, r,$$

where $\mathcal{F}_i^N$ is any neighbourhood of $\mathcal{F}_i$ in the subspace topology on $\mathrm{aff}(\mathcal{F}_i)$. We do not make such a relaxation in order not to burden the reader with further technical details in the proof of Lemma 18.

PROOF. Suppose not; that is, $\phi(\cdot, 0)$ does not cross $\mathcal{F}_0$ at $T = 0$. Observe that (1) with a continuous piecewise affine feedback satisfies the requirements of Lemma 16. By Lemma 16, $\phi(\cdot, 0)$ crosses at least one facet $\mathcal{F}_i$.

Without loss of generality, let us assume that for some $1 \leq l \leq k$, $\phi(\cdot, 0)$ crosses facets $\mathcal{F}_1, \ldots, \mathcal{F}_l$, and it does not cross $\mathcal{F}_0, \mathcal{F}_{l+1}, \ldots, \mathcal{F}_k$. Additionally, let $1 \leq p \leq l$ be such that $\{h_1, \ldots, h_p\}$ is a basis for the span of $\{h_1, \ldots, h_l\}$. Through an invertible linear transformation (under which $\mathcal{P}$ remains a convex polyhedron), we may assume that $\{h_1, \ldots, h_p\}$ is orthonormal.

By Lemma 16, there exists $\varepsilon > 0$ such that

$$\phi(t, 0) \in \mathcal{H}_0 \cap \left(\bigcap_{i=1}^{l} \mathcal{H}_i^c \right) \cap \left(\bigcap_{j=l+1}^{k} \mathcal{H}_j \right) =: \mathcal{G}, \quad t \in (0, \varepsilon).$$

Define $\mathcal{H} = \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_l$. Note that $\mathcal{H}$ is closed as an intersection of closed sets. Hence we can define the point to set distance $d_{\mathcal{H}}(x) = \min_{z \in \mathcal{H}} \|x - z\|$.

Since $h_0, \ldots, h_p$ are orthonormal, every $x \in \mathbb{R}^n$ can be uniquely expressed as $x = \lambda_1(x) h_1 + \cdots + \lambda_l(x) h_p + \tilde{x}$, where $h_i \cdot \tilde{x} = 0$, $i = 1, \ldots, p$. Also, $h_i \cdot x = \lambda_i(x)$, $i = 1, \ldots, p$. Thus, if $z' \in \mathcal{H}$, then $z' = \lambda_1(z') h_1 + \ldots + \lambda_l(z') h_p + \tilde{z}'$, with $\lambda_i(z') \leq 0$, $i = 1, \ldots, p$. Now consider $x \in \mathcal{G}$, which means $\lambda_i(x) > 0$, $i = 1, \ldots, p$. Then

$$\begin{aligned} \|x - z'\|^2 &= \sum_{i=1}^{p} |\lambda_i(x) - \lambda_i(z')|^2 + \|\tilde{x} - \tilde{z}\|^2 \\ &\geq \sum_{i=1}^{p} \lambda_i(x)^2. \end{aligned} \tag{8}$$

Now, take the unique $z \in \mathbb{R}^n$ which satisfies

$$\lambda_i(z) = 0,\ i = 1, \ldots, p, \text{ and } \tilde{z} = \tilde{x}. \tag{9}$$

This z clearly satisfies $h_i \cdot z = \lambda_i(z) = 0$ for all $i = 1, \ldots, p$. Additionally, as all $h_{p+1}, \ldots, h_l$ are linear combinations of $h_1, \ldots, h_p$, it also satisfies $h_i \cdot z = 0$ for all $i = p+1, \ldots, l$. Hence, $z \in \mathcal{H}$. Additionally, from (9), $\|x - z\| = \lambda_1(x)^2 + \ldots + \lambda_p(x)^2$. Thus, from (8), $\|x - z\| = d_{\mathcal{H}}(x)$.

With this choice, for all $x \in \mathcal{G}$,

$$d_{\mathcal{H}}^2(x) = \lambda_1^2(x) + \cdots + \lambda_p^2(x) = (h_1 \cdot x)^2 + \cdots + (h_p \cdot x)^2. \tag{10}$$

Because $\phi(\cdot, 0)$ is a solution of a continuous system (1), it is differentiable in t. Since $\phi(t, 0) \in \mathcal{G}$ for all $t \in (0, \varepsilon)$ by assumption, $d_{\mathcal{H}}^2(\phi(t, 0))$ is differentiable on $(0, \varepsilon)$ by (10). Additionally, $d_{\mathcal{H}}^2(\phi(t, 0))$ is continuous from the right at 0 by letting $x \to 0$ in (10) and noting that $d_{\mathcal{H}}^2(0) = 0$.

Let $f(x) = Ax + Bu(x) + a$. We have from (1) and (10) that, for all $t \in (0, \varepsilon)$,

$$\frac{d}{dt} \left[d_{\mathcal{H}}^2(\phi(t, 0)) \right] = \sum_{i=1}^{p} 2(h_i \cdot \phi(t, 0)) h_i \cdot f(\phi(t, 0)). \tag{11}$$

If $z(t)$, in analogy to the point z in the first part of the proof, is the point in $\mathcal{H}$ closest to $\phi(t, 0)$, by (9) we have $\phi(t, 0) - z(t) = \sum_{i=1}^{p} \lambda_i(\phi(t, 0)) h_i = \sum_{i=1}^{p} (h_i \cdot \phi(t, 0)) h_i$. Substituting into (11), we get

$$\frac{d}{dt} \left[d_{\mathcal{H}}^2(\phi(t, 0)) \right] = 2(\phi(t, 0) - z(t)) \cdot f(\phi(t, 0)). \tag{12}$$

Since $\lambda_i(z(t)) = h_i \cdot z(t) = 0$ for all $i = 1, \ldots, l$ by (9) and the subsequent discussion, and since $0 \in \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_l$,

$$z(t) \in \bigcap_{i=1}^{l} \mathrm{aff}(\mathcal{F}_i). \tag{13}$$

Let us rewrite (12) as

$$\begin{aligned} \frac{d}{dt} \left[d_{\mathcal{H}}^2(\phi(t, 0)) \right] &= 2(\phi(t, 0) - z(t)) \cdot f(z(t)) + \\ & 2(\phi(t, 0) - z(t)) \cdot \left[f(\phi(t, 0)) - f(z(t)) \right]. \end{aligned} \tag{14}$$

From (9) we know $\phi(t, 0) - z(t) = \sum_{i=1}^{p} \lambda_i(\phi(t, 0)) h_i$. Also, because $\phi(t, 0) \in \mathcal{G}$ for all $t \in (0, \varepsilon)$,

$$\lambda_i(\phi(t, 0)) = h_i \cdot \phi(t, 0) > 0, \quad i = 1, \ldots, l. \tag{15}$$

Now, by (13), $z(t)$ is in the intersection of $\mathrm{aff}(\mathcal{F}_i)$ for all $i \in \{1, \ldots, l\}$. Hence, by (7) and (15) we get

$$2(\phi(t, 0) - z(t)) \cdot f(z(t)) = 2 \sum_{i=1}^{p} \lambda_i(\phi(t, 0)) h_i \cdot f(z(t)) \leq 0.$$

Thus, by (14)

$$\frac{d}{dt}\left[d^2_{\mathcal{H}}(\phi(t,0))\right] \le 2(\phi(t,0)-z(t))\cdot(f(\phi(t,0))-f(z(t))). \tag{16}$$

We note that f is continuous and piecewise affine. Hence, it can easily be shown (see, e.g., Blanchini (1995)) that it is Lipschitz continuous on some neighbourhood of 0. Let $L > 0$ be the Lipschitz constant of f in that neighbourhood. We can reduce ε such that both $\phi(t,0)$ and $z(t)$ are in this neighbourhood for all $t \in (0,\varepsilon)$. Then from (16)

$$\frac{d}{dt}\left[d^2_{\mathcal{H}}(\phi(t,0))\right] \le 2L\|\phi(t,0)-z(t)\|^2 = 2Ld^2_{\mathcal{H}}(\phi(t,0)) .$$

Using this result, we find $\frac{d}{dt}\left[\mathrm{e}^{-2Lt}d^2_{\mathcal{H}}(\phi(t,0))\right] = -2L\mathrm{e}^{-2Lt}d^2_{\mathcal{H}}(\phi(t,0)) + \mathrm{e}^{-2Lt}\frac{d}{dt}\left[d^2_{\mathcal{H}}(\phi(t,0))\right] \le 0$. Thus, $\mathrm{e}^{-2Lt}d^2_{\mathcal{H}}(\phi(t,0))$ is a non-increasing function on the interval $(0,\varepsilon)$. It is also nonnegative and continuous from the right at $t = 0$. Thus, since $\mathrm{e}^0 d^2_{\mathcal{H}}(\phi(0,0)) = 0$, we have $d^2_{\mathcal{H}}(\phi(t,0)) = 0$ for all $t \in [0,\varepsilon)$. This is in contradiction with our assumption that $\phi(t,0) \in \mathcal{G} \subset \mathcal{H}^c$. □

3.4 Affine Feedback

The results of Section 3.2 can be applied to the special case of affine feedback. We consider the situation investigated by Roszak and Broucke (2006) on the use of affine feedback to solve the RCP on simplices. The following result was stated by Roszak and Broucke (2006), with a partial proof provided by Roszak and Broucke (2005).

Lemma 21 *Consider the affine system* (1) *and consider an n-dimensional simplex $\mathcal{S}$ with facets $\mathcal{F}_0,\ldots,\mathcal{F}_n$. Suppose that $\mathcal{F}_0$ is the exit facet and $\mathcal{F}_1,\ldots,\mathcal{F}_n$ are restricted facets. Let $u(x) = Kx + g$ be an affine feedback such that*

$$Ax + Bu(x) + a \in \mathcal{C}(x)\,, \qquad \textit{for all } x \in \mathcal{S}\,. \tag{17}$$

Then all trajectories originating in $\mathcal{S}$ that exit $\mathcal{S}$ do so through $\mathcal{F}_0$.

Using Lemma 18 we are able to prove the following stronger result, showing that a trajectory exiting $\mathcal{S}$ does so not only by exiting through $\mathcal{F}_0$, but also by crossing this facet in the sense of Definition 4.

Lemma 22 *Consider the affine system* (1) *and consider an n-dimensional simplex $\mathcal{S}$ with facets $\mathcal{F}_0,\ldots,\mathcal{F}_n$. Suppose that $\mathcal{F}_0$ is the exit facet and $\mathcal{F}_1,\ldots,\mathcal{F}_n$ are restricted facets. Let $u(x) = Kx + g$ be an affine feedback such that* (17) *holds. Then all trajectories originating in $\mathcal{S}$ that exit $\mathcal{S}$ do so by crossing $\mathcal{F}_0$.*

PROOF. Assume otherwise: a trajectory $\phi(\cdot,x_0)$ with $x_0 \in \mathcal{S}$ exits $\mathcal{S}$, but does so without crossing $\mathcal{F}_0$. Without loss of generality, we may assume that $\phi(\cdot,x_0)$ exits $\mathcal{S}$ at time $T = 0$, i.e., at point x_0. We distinguish between two cases: if $x_0 \in \mathcal{F}_0$, the conditions of Lemma 18 are satisfied, and it is impossible to exit $\mathcal{S}$ without crossing $\mathcal{F}_0$. If $x_0 \notin \mathcal{F}_0$, then all the facets that x_0 is contained in are restricted. Exactly the same proof as in Lemma 18 works, just without any mention of the unrestricted facet $\mathcal{F}_0$: if we assume that $\phi(\cdot,x_0)$ exits $\mathcal{S}$ at time $T = 0$, we obtain a contradiction. □

Remark 23 *Lemma 22 also holds for continuous piecewise affine feedback controls, with the same proof.*

Analogously to Remark 19, Lemma 22 does not guarantee that a trajectory exiting $\mathcal{S}$ does not cross any restricted facets. The following example shows that even if the invariance conditions are solvable, and $f(x) \in \mathcal{C}(x)$ for all $x \in \mathcal{S}$, solutions may cross a restricted facet.

Example 24 *Let us consider $\mathcal{S} \subset \mathbb{R}^2$ as the two-dimensional simplex with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Let $\mathcal{F}_0 = \{(x_1,0) \mid 0 \le x_1 \le 1\}$, $\mathcal{F}_1 = \{(x_1,1-x_1) \mid 0 \le x_1 \le 1\}$ and $\mathcal{F}_2 = \{(0,x_2) \mid 0 \le x_2 \le 1\}$. We consider the dynamics given by the system*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -1 \end{pmatrix}. \tag{18}$$

We note that at $\mathcal{F}_0$ the vector field generated by this system points exactly down, through $\mathcal{F}_0$, while at $\mathcal{F}_1$ and $\mathcal{F}_2$ it points into the half-spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence, $f(x) \in \mathcal{C}(x)$, with the definition of $\mathcal{C}(x)$ from (2). *Thus, the invariance conditions are solvable, with $\mathcal{F}_0$ as the exit facet and $\mathcal{F}_1$ and $\mathcal{F}_2$ as restricted facets.*

Now, let us consider the trajectory $\phi(\cdot,0)$ generated by (18). *We can easily calculate that*

$$\phi(t,0) = (-t^2/2, -t)$$

for all $t \ge 0$. Hence, this trajectory exits $\mathcal{S}$ by crossing $\mathcal{F}_0$, as guaranteed by Lemma 22, but it also crosses $\mathcal{F}_2$ at the same time.

The situation presented in Example 24 is illustrated in Figure 4.

4 Chattering with Discontinuous Piecewise Affine Feedback

In this section we examine the question of chattering for the class of discontinuous PWA feedbacks defined on a possibly non-convex polytope formed by a so-called *chain of simplices* (Helwa and Broucke, 2015). This class of feedbacks was also studied by Broucke and Ganness (2014) when the polytope is itself a simplex. We improve the results of

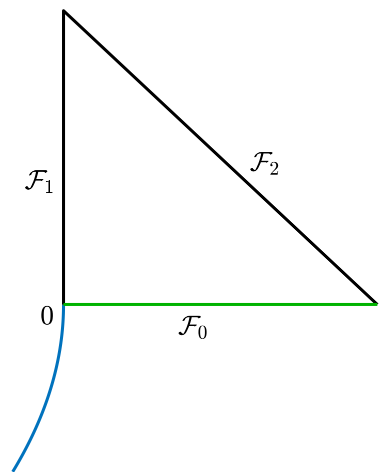

Fig. 4. An illustration of Example 24. The trajectory is drawn in blue, the exit facet $\mathcal{F}_0$ is drawn in green, while the restricted facets are drawn in black.

Broucke and Ganness (2014) in several ways. First, we relax the requirement on open-loop controls that they satisfy the invariance conditions. Instead, we *prove* that there exists a set of open-loop controls satisfying the invariance conditions. Given this result, we then present a modified discrete supervisor rule based on the one in Broucke and Ganness (2014). More details are given below. Finally, we prove that the discontinuous affine feedback given here does not exhibit chattering. First we review the definition of a triangulation (Lee, 2004).

Definition 25 *A triangulation $\mathbb{T} = \{\mathcal{S}^1, \ldots, \mathcal{S}^\chi\}$ of a polytope $\mathcal{P}$ is a subdivision of $\mathcal{P}$ into full dimensional simplices $\mathcal{S}^1, \cdots, \mathcal{S}^\chi$ such that the following conditions hold: (i) $\mathcal{P} = \cup_{i=1}^{\chi} \mathcal{S}_i$, (ii) For all $i,\ j \in \{1, \cdots, \chi\}$ with $i \neq j$, the intersection $\mathcal{S}^i \cap \mathcal{S}^j$ is either empty or a common face of $\mathcal{S}^i$ and $\mathcal{S}^j$.*

Definition 26 *Let $L := \{1, \ldots, \chi\}$. Let $\{\mathcal{S}^j \mid j \in L\}$ be a collection of n-dimensional simplices. Define $\mathcal{P} := \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^\chi$. We say that $\mathcal{P}$ is a* chain *if the following hold:*

(i) $\mathbb{T} = \{\mathcal{S}^1, \ldots, \mathcal{S}^\chi\}$ is a triangulation of $\mathcal{P}$ such that, for all $k \in L$, $k \geq 2$, $\mathcal{S}^k \cap \mathcal{S}^{k-1}$ is a facet of $\mathcal{S}^k$ and of $\mathcal{S}^{k-1}$.
(ii) $\mathcal{F}_0^k := \mathcal{S}^k \cap \mathcal{S}^{k-1}$ is designated to be the exit facet of $\mathcal{S}^k$, for each $k = 2, \ldots, \chi$.
(iii) The exit facet of $\mathcal{P}$ is designated to be $\mathcal{F}_0 := \mathcal{F}_0^1$, the exit facet of $\mathcal{S}^1$, and it is not a facet of any other simplex $\mathcal{S}^k$, $k \neq 1$.

See Figure 5, adapted from Helwa and Broucke (2015), for an example of a chain, with the exit facet given by $\overline{v_1 v_2}$. For the sake of formality, we note that Definition 26 introduces a slight abuse of notation: since the structure of the chain depends on its constituent simplices and their exit facets, it would formally make more sense to say that a sequence $(\mathcal{S}^1, \mathcal{F}_0^1), \ldots, (\mathcal{S}^\chi, \mathcal{F}_0^\chi)$ is a chain. However, for

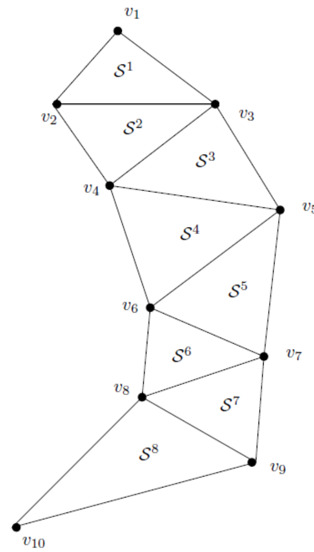

Fig. 5. A chain of simplices.

convenience, in the remainder of the text we assume that this underlying structure inherent in the notion of a chain is known, and simply continue stating that $\mathcal{P}$ is a chain.

Since we allow discontinuous controls, a slightly stronger version of the RCP is needed.

Problem 27 (Discontinuous RCP (DRCP)) *Consider system (1) defined on a chain $\mathcal{P}$. Find a state feedback $u(x)$ such that:*

(i) For every $x_0 \in \mathcal{P}$, there exist $T \geq 0$ and $\varepsilon > 0$ such that
(i.1) $\phi(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$,
(i.2) $\phi(T, x_0) \in \mathcal{F}_0$, and
(i.3) $\phi(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \varepsilon)$.
(ii) There exists $\gamma > 0$ such that for every $x \in \mathcal{P}$, $\|Ax + Bu(x) + a\| > \gamma$.

Remark 28 *As already discussed by Broucke and Ganness (2014), the new condition (ii) appearing in Problem 27 compared to Problem 1 is a robustness requirement. It circumvents a vanishingly small closed-loop vector field that could result in the appearance of an equilibrium in $\mathcal{P}$ if the system parameters are perturbed. Condition (ii) holds automatically when continuous feedbacks are used. We must include it explicitly in the statement of the DRCP only since we require discontinuous PWA feedbacks to solve the RCP when it is not solvable by continuous state feedback (Broucke, 2010).*

We consider the following class of discontinuous PWA feedbacks for solving the DRCP on chains.

Definition 29 *Consider a chain $\mathcal{P} = \mathcal{S}^1 \cup \cdots \mathcal{S}^\chi$. Let $u_k(x) := K_k x + g_k$, $k \in L$, be a set of affine feedbacks where $K_k \in \mathbb{R}^{m \times n}$ and $g_k \in \mathbb{R}^m$. Define*

$$k(x) := \min\{k \in L \mid x \in \mathcal{S}^k\}\,. \tag{19}$$

We say $u(x)$ is a PWA feedback on $\mathcal{P}$ *if*

$$u(x) = u_{k(x)}(x) \,. \tag{20}$$

That is, $u(x)$ *is affine on the interior of each simplex, and at a point* $x \in \mathcal{P}$ *belonging to more than one simplex, the affine controller for the simplex with the smallest index is used.*

The next result shows that the class of discontinuous PWA affine feedbacks introduced above can be used to solve the DRCP on a chain $\mathcal{P}$, assuming the constituent affine feedbacks solve a local RCP on each simplex.

Theorem 30 *Consider the system* (1) *defined on a chain* $\mathcal{P} = \mathcal{S}^1 \cup \cdots \mathcal{S}^\chi$*. Let* $u(x)$ *be a PWA feedback on* $\mathcal{P}$ *as in* (19)-(20)*. Suppose that*

$$\mathcal{S}^k \xrightarrow{\mathcal{S}^k} \mathcal{F}_0^k \,, \qquad k \in L \,, \tag{21}$$

using $u = u_k(x)$ *in the sense of Problem 1. Then* $u(x)$ *solves Problem 27. In particular, for each* $x_0 \in \mathcal{P}$*, there exist* $T \geq 0$*,* $\varepsilon > 0$*,* $r \in \mathbb{N}$*, times* $0 < t_1 < \cdots < t_r = T$*, and a unique continuous solution* $\phi(\cdot, x_0) : [0, T+\varepsilon] \to \mathbb{R}^n$ *such that*

$$\frac{d}{dt}\phi(t, x_0) = A\phi(t, x_0) + Bu(\phi(t, x_0)) + a \tag{22}$$

for all $t \in (0, T+\varepsilon) \setminus \{t_1, \ldots, t_{r-1}\}$*. For all* $i \in \{1, \ldots, r-1\}$*, there exists* $j \geq 1$ *such that* $\phi(t_i, x_0) \in \mathcal{S}^j \cap \mathcal{S}^{j-1}$*. Also, (i)-(ii) of Problem 27 are satisfied, and* $\phi(\cdot, x_0)$ *exits* $\mathcal{P}$ *by crossing the facet* $\mathcal{F}_0 = \mathcal{F}_0^1$*.*

Remark 31 *As discussed in Section 2, the system* (1) *must be defined on a small neighbourhood* $\mathcal{P} \subset \mathcal{N}$*. In this case, we assume that the control law on* $\mathcal{N} \setminus \mathcal{P}$ *is given by* $u(x) = u_1(x)$*.*

PROOF. Let $k_0 := k(x_0)$ with k from (19). Assume that $k_0 > 1$ and $x_0 \in \mathcal{F}_0^{k_0}$. Then, by (ii) in Definition 26, $x_0 \in \mathcal{S}^{k_0 - 1}$, which is in contradiction to $k_0 = k(x_0)$. Thus, $x_0 \notin \mathcal{F}_0^{k_0}$ or $x_0 \in \mathcal{F}_0^1$. If the latter is true, part (i) of Problem 27 is trivially true by (21). Hence suppose $x_0 \notin \mathcal{F}_0^{k_0}$. By (21), there exists a $t_1 > 0$ and a unique trajectory $\phi_1(\cdot, x_0) : [0, t_1] \to \mathcal{S}^{k_0}$ with the following properties:

- (22) holds for all $t \in (0, t_1)$,
- $\phi_1(t_1, x_0) \in \mathcal{F}_0^{k_0}$,
- $\phi_1(t, x_0) \in \mathcal{S}^{k_0} \setminus \mathcal{F}_0^{k_0} \subset \mathcal{P}$ for all $t \in (0, t_1)$.

Consider now $x_1 := \phi_1(t_1, x_0)$. Since $x_1 \in \mathcal{F}_0^{k_0}$, either $k_0 = 1$ or $\min\{i \in L \mid x_1 \in \mathcal{S}^i\} \leq k_0 - 1$. If we define $k_1 := k(x_1)$, we have the same situation as from before: $x_1 \notin \mathcal{F}_0^{k_1}$ or $x_1 \in \mathcal{F}_0^1$. We proceed analogously and obtain a trajectory $\phi_2(\cdot, x_0) : [t_1, t_2) \to \mathcal{S}^{k_1}$. By iterating this procedure, we obtain a finite sequence of times $0 < t_1 < \cdots < t_r =: T$, indices $k_0 > k_1 > \cdots > k_r = 1$ and switching points $x_1, \ldots, x_r$, where $x_r \in \mathcal{F}_0^1$. By (21) there now exists a trajectory $\phi_{r+1}(\cdot, x_0)$ which leaves $\mathcal{S}_1$ through $\mathcal{F}_0^1$ in the sense of Problem 1. We define ϕ as the concatenation of trajectories $\phi_1, \ldots, \phi_{r+1}$. ϕ clearly satisfies part (i) of Problem 27.

For part (ii) of Problem 27, define $f(x) = Ax + Bu(x) + a$ and $f_k(x) = Ax + Bu_k(x) + a$ for all $k \in L$. We note that all f_k are continuous functions, and by (21), contain no zeros on $\mathcal{S}^k$. Assume otherwise: then, $\dot{x} = f_k(x)$ would contain an equilibrium in $\mathcal{S}^k$, which is in contradiction with u_k solving the RCP for $\mathcal{S}^k$ as stipulated by (21).

Hence, for all $x \in \mathcal{S}^k$, it holds that $\|f_k(x)\| > \gamma_k$ for some $\gamma_k > 0$. Now, take any $x \in \mathcal{P}$ and define $k' := \min_{1 \leq i \leq \chi}\{i \mid x \in \mathcal{S}^i\}$. Then, $\|f(x)\| = \|f_{k'}(x)\| > \gamma_{k'}$. Thus, part (ii) of Problem 27 is satisfied with $\gamma := \min\{\gamma_1, \ldots, \gamma_\chi\}$.

Finally, we showed above that ϕ exits $\mathcal{P}$ through facet $\mathcal{F}_0$. However, in the time interval $[T, T+\varepsilon)$ when ϕ is exiting $\mathcal{F}_0$, ϕ is, by Remark 31, governed by the affine feedback control law $u_1(x)$. Hence, Lemma 9 applies directly, implying that ϕ will also cross the facet $\mathcal{F}_0$. $\square$

Next we explore the extent to which PWA feedbacks on chains provide a complete solution to the DRCP; that is, if the DRCP is solvable by some reasonable class of open-loop controls, we want to show it is solvable by PWA feedback. We consider the special case, also investigated in Broucke and Ganness (2014), when $\mathcal{P}$ is itself a simplex denoted by $\mathcal{S}$.

We define the set of admissible open-loop controls for (1) as any measurable function $\mu : [0, \infty) \to \mathbb{R}^m$ that is bounded on compact time intervals. Solutions of (1) under an open-loop control μ are in the sense of Carathéodory. Thus, as noted in Section 2, they exist and are unique. We reuse the notation $\phi_\mu(\cdot, x_0)$ to denote a trajectory of (1) starting at x_0 under an open-loop control μ.

Definition 32 *Consider system* (1) *defined on a chain* $\mathcal{S}$*. We say the* DRCP is solved by open-loop controls *if there exists* $\gamma > 0$ *such that for each* $x \in \mathcal{S}$*, there exist* $\varepsilon_x > 0$*,* $T_x \geq 0$*, and an open-loop control* $\mu_x(\cdot) : [0, T_x + \varepsilon_x] \to \mathbb{R}^m$ *such that*

(i.1) $\phi_{\mu_x}(t, x) \in \mathcal{S}$ *for all* $t \in [0, T_x]$*,*
(i.2) $\phi_{\mu_x}(T_x, x) \in \mathcal{F}_0$*, and*
(i.2) $\phi_{\mu_x}(t, x) \notin \mathcal{S}$ *for all* $t \in (T_x, T_x + \varepsilon_x)$*.*
(ii) $\|A\phi_{\mu_x}(t, x) + B\mu_x(t) + a\| > \gamma$ *for all* $t \in [0, T_x]$*.*

In Broucke and Ganness (2014) an additional requirement on open-loop controls was that they satisfy the invariance conditions. Instead, in the next result we prove that there exists a set of open-loop controls that satisfy these conditions at $t = 0$.

Theorem 33 *If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls in the sense of Definition 32, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls that also satisfy:*

(iii) $Ax + B\mu_x(0) + a \in \mathcal{C}(x)$ *for all* $x \in \mathcal{S} \setminus \mathcal{F}_0$.

PROOF. Let $x \in \mathcal{S} \setminus \mathcal{F}_0$. By assumption there exists an open-loop control μ_x and a time $T_x > 0$ such that $\phi_{\mu_x}(t, x) \in \mathcal{S}$ for all $t \in [0, T_x]$. Since μ_x is an open-loop control, there exists $c \geq 0$ such that $\|\mu_x(t)\| \leq c$, for all $t \in [0, T_x]$. Define $\mathcal{Y}(z) := \{Az + Bw + a \mid w \in \mathbb{R}^m\}$ and $\mathcal{Y}_c(z) := \{Az + Bw + a \mid w \in \mathbb{R}^m, \|w\| \leq c\}$. Now take any sequence $\{t_i \in (0, T_x] \mid i \in \mathbb{N}\}$ such that $\lim_i t_i = 0$. Note that

$$\frac{\|\phi_{\mu_x}(t_i, x) - x\|}{t_i} = \frac{1}{t_i} \int_0^{t_i} \left[A\phi_{\mu_x}(\tau, x) + B\mu_x(\tau) + a\right] d\tau. \tag{23}$$

Thus, since $\{y \in \mathcal{Y}_c(z) \mid z \in \mathcal{S}\}$ is bounded, there exists $M > 0$ such that $\|\phi_{\mu_x}(t_i, x) - x\| \leq Mt_i$. Therefore $\left\{\frac{\phi_{\mu_x}(t_i,x)-x}{t_i}\right\}$ is a bounded sequence, and there exists a convergent subsequence (with indices relabeled) such that $\lim_{i\to\infty} \frac{\phi_{\mu_x}(t_i,x)-x}{t_i} =: v$. Since $\phi_{\mu_x}(t_i, x) \in \mathcal{S}$, by the definition of the Bouligand tangent cone, $v \in T_{\mathcal{S}}(x)$. On the other hand, by taking the limit in (23), we get $v = Ax + B\lim_{i\to\infty} \mu_x(t_i) + a \in \mathcal{Y}(x)$. Note that $\lim_{i\to\infty} \mu_x(t_i)$ exists by passing to a subsequence, if necessary, because μ_x is bounded on compact intervals. Since $T_{\mathcal{S}}(x) = \mathcal{C}(x)$ for $x \in \mathcal{S} \setminus \mathcal{F}_0$, we conclude that $v \in \mathcal{Y}(x) \cap \mathcal{C}(x)$.

Now we construct a modified set of open loop controls $\tilde{\mu}_x$ as follows. If $x \in \mathcal{F}_0$, then let $\tilde{\mu}_x := \mu_x$. If $x \in \mathcal{S} \setminus \mathcal{F}_0$, then let $\tilde{\mu}_x(t) := \mu_x(t)$ for all $t \neq 0$. At $t = 0$ let $\tilde{\mu}_x(0) := v \in \mathcal{Y}(x) \cap \mathcal{C}(x)$, as above. Since each open-loop control μ_x was changed at no more than a single point, the trajectory generated by $\tilde{\mu}_x$ is unchanged, and we obtain the desired result. □

A special triangulation of the state space was studied in Broucke and Ganness (2014) which has proved to be useful both in theory and in applications. Under this triangulation, a *Subdivision Algorithm* was presented in Broucke and Ganness (2014) that partitions the original simplex $\mathcal{S}$ into $p+1$ simplices $\{\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}\}$. By construction, the partition generates a chain $\mathcal{S} = \mathcal{S}^1 \cup \cdots \cup \mathcal{S}^{p+1}$. Our goal here is to apply our new result in Theorem 30 to solve the DRCP on this chain. As a byproduct we want to deduce there is no chattering. First, we introduce some assumptions from Broucke and Ganness (2014) in order to use existing results without duplication. It is sufficient for our purposes to say that these assumptions are primarily to set up the special triangulation. Further detailed explanations can be found in Broucke and Ganness (2014).

To that end, we define the affine space $\mathcal{O} := \{\, x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$ and the set $\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O}$. Note that closed-loop equilibria of (1) can only appear in $\mathcal{O}$.

Assumption 34 *Simplex $\mathcal{S}$ and system* (1) *satisfy the following conditions.*

- *(A1)* $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{v_1, \ldots, v_{\kappa+1}\}$, *with* $0 \leq \kappa < n$.
- *(A2)* $\mathcal{B} \cap \mathcal{C}(v_0) = \mathbf{0}$.
- *(A3) The maximum number of linearly independent vectors in any set* $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ *(with only one vector for each* $\mathcal{B}\cap\mathcal{C}(v_i)$*) is* $\widehat{m}$ *with* $0 \leq \widehat{m} < \kappa+1$.

Theorem 35 *Suppose Assumption 34 holds. If $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by open-loop controls in the sense of Definition 32, then Problem 27 is solvable by the discontinuous PWA feedback* (19)-(20)*. Moreover, all trajectories originating in $\mathcal{S}$ exit $\mathcal{S}$ by crossing $\mathcal{F}_0$.*

PROOF. By Theorem 33 there exists a set of open loop controls $\{\mu_x \mid x \in \mathcal{S}\}$ such that (i)-(ii) of Definition 32 and (iii) of Theorem 33 hold. Then with a minor variation of the proof in Broucke and Ganness (2014), Theorem 10 of Broucke and Ganness (2014) holds. It shows that for all $i \in \{1, \ldots, \kappa+1\}$,

$$\mathcal{B} \cap \mathcal{C}(v_i) \neq \mathbf{0}.$$

Combining this result with Assumption 34, we can invoke Theorem 23 of Broucke and Ganness (2014) on the construction of the reach control indices $\{r_1, \ldots, r_p\}$. Hence, the requirements to apply the Subdivision Algorithm of Broucke and Ganness (2014) are in place. This yields a chain $\{\mathcal{S}^1, \ldots, \mathcal{S}^{p+1}\}$, and guarantees the existence of affine feedbacks $u_k(x) = K_k x + g_k$ such that (21) holds in the sense of Problem 1. Now we construct the discontinuous PWA feedback given in (19)-(20). By Theorem 30, the resulting feedback $u(x)$ solves condition (i) of Problem 27. A trivial compactness argument gives condition (ii) of Problem 27. Finally, by Theorem 30 again, all trajectories exit $\mathcal{S}$ by crossing $\mathcal{F}_0$. □

Comparing our new result, Theorem 35, to the analogous result of Broucke and Ganness (2014), we have the following improvements. The requirement of Broucke and Ganness (2014) that the set of open-loop controls satisfy the invariance conditions on their interval of definition has been removed. This is partly a consequence of Theorem 33, which guarantees that there exist open-loop controls which satisfy the invariance conditions at the initial time. It turns out that this guarantee, more relaxed than the original requirement, is still sufficient to prove the result of Broucke and Ganness (2014), with minimal changes to the proof. Second, we show there is no chattering using our proposed discontinuous PWA controller; chattering was not discussed at all by Broucke and Ganness (2014). Finally and most importantly, we close a gap in the result of Broucke and Ganness (2014). Our discontinuous PWA feedback uses (19) to assign the affine feedback for the simplex with the smallest index at points that lie in more than one simplex of the chain. Instead the rule in Broucke and Ganness (2014) is to assign the affine feedback for the simplex with the largest index. In that case, the notion of a solution as defined by Theorem 30 becomes questionable. On one hand, if the stipulation of Remark 31 is kept and the dynamics in the neighbourhood of chain $\mathcal{S}$ are generated by the controller with the lowest index, a solution in the sense of Theorem 30 might not exist. On the other hand, if Remark 31 is changed so that the dynamics in the neighbourhood of $\mathcal{S}$ act differently, this opens the possibility of the pathology similar to Example 24, where a trajectory may cross an exit facet of one simplex, but not enter the next simplex of the chain.

5 Conclusion

This paper constitutes the first effort to provide a rigorous foundation for the switching theory that underlies the Reach Control Problem. The central question of that theory is to formally define what it means to exit a polytope through a predetermined facet and thereby transition to another polytope. We introduced a novel notion of crossing a facet, and identified a pathological behaviour that enables a trajectory to exit a polytope through a facet without crossing the said facet. Such behaviour can result in substantial undesirable consequences for design of appropriate control laws. The majority of this paper is dedicated to determining classes of closed-loop feedback that do not exhibit such behaviour. This paper, in particular, shows that analytic and continuous piecewise affine feedback are sufficient to guarantee desirable behaviour. Finally, we applied our results to the case of open-loop controls and discontinuous PWA on simplices, and corrected previous work in the area.

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