On a Topological Obstruction in the Reach Control Problem

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Abstract This paper explores aspects of the Reach Control Problem (RCP) to drive the states of an affine control system to a facet of a simplex without first exiting from other facets. In analogy with the problem of nonlinear feedback stabilization, we investigate a topological obstruction that arises in solving the RCP by continuous state feedback. The problem is fully solved in this paper for the case of two and three dimensions.

1 Problem Statement

This paper studies a topological obstruction that arises in solving the Reach Control Problem (RCP) using continuous state feedback. We consider a simplex $\mathcal{S} := \mathrm{co}\{v_0,\ldots,v_n\}$ with vertices $\{v_0,\ldots,v_n\}$ and facets $\{\mathcal{F}_0,\ldots,\mathcal{F}_n\}$. Each facet is indexed according to the vertex it does not contain. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $\mathbf{h_j}$ be the normal vector to facet $\mathcal{F}_j$ pointing outside $\mathcal{S}$. Define $I = \{1,\ldots,n\}$. Let $I(x) \subseteq \{0,1,\ldots,n\}$ be the minimal set of indices such that $x \in \mathrm{co}\{v_i, \mid i \in I(x)\}$. That is, x is in the interior of $\mathrm{co}\{v_i \mid i \in I(x)\}$.

We consider the affine control system on $\mathcal{S}$

$$\dot{x} = Ax + Bu + a\,, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ where $1 \le m < n$. Let $\mathcal{B} := \mathrm{Im}(B)$ and $\mathcal{O} := \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$. Let $\phi_u(t,x_0)$ denote the trajectory of (1) starting at x_0 under a feedback $u(x)$.

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The problem is to find a state feedback $u(x)$ such that all trajectories $\phi_u(\cdot,x_0)$ starting in $\mathcal{S}$ exit $\mathcal{S}$ through $\mathcal{F}_0$ in finite time without first leaving $\mathcal{S}$. The RCP has been extensively studied [6, 4, 5]. The purpose of this paper is to announce a topological obstruction in the RCP, paralleling the analogous problem arising in the problem of continuous state feedback stabilization [3], and to give preliminary results on the problem for low dimensional systems.

A parallel study of the same problem was made in [10] (some preliminary work appeared in [8]). The contributions of this paper significantly differ from [10]. This paper primarily uses retraction theory to study the case of $\dim(\mathcal{O}_{\mathcal{S}}) = n-1$. This leads to a simple solution in low dimensions of n. Studies of low-dimensional systems are prevalent [1]. On the other hand, [10] uses homotopy theory to study the case of $\dim(\mathcal{B}) = 2$. While the conclusions of [10] could also potentially lead to a solution in low dimensions of n, this situation is not explored in [10]. Furthermore, the elegant cone condition for the topological obstruction developed in this paper, $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}}) = \mathbf{0}$, does not make an appearance at all in [10]; its place is taken by a more involved result based on null-homotopic maps on a circle.

A supplement to this paper is found in [11] where we present supporting results and a study of the case of an obstruction using affine feedback. The results of this paper for the case of $n = 2,3$, Theorems 3 and 4, could be obtained from [11]. However, while it is tempting to forgo more advanced topological methods in favour of the brute force linear algebra arguments in [11], we insist on the importance of the topological approach in order to have a hope of generalizing the results to higher dimensions.

For each $x \in \mathcal{S}$, we define the cone

$$\mathcal{C}(x) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{h_j} \cdot \mathbf{y} \leq 0, j \in I \backslash I(x)\}\,. \tag{2}$$

In other words, $\mathcal{C}(x)$ is the set of all vectors $\mathbf{y}$ which, when attached at x, point into $\mathcal{S}$ or through the exit facet $\mathcal{F}_0$. (We note that if $x \in \mathrm{Int}(\mathcal{S})$, $\mathcal{C}(x) = \mathbb{R}^n$.) In order for the trajectory $\phi_u(t,x_0)$ to not leave $\mathcal{S}$ through any facet except the exit facet, we require [6]:

$$\frac{d\phi_u}{dt} = Ax + Bu(x) + a \in \mathcal{C}(x)\,, \quad x \in \mathcal{S}\,. \tag{3}$$

In addition to this necessary condition, if $u(x)$ solves the RCP then there are no closed-loop equilibria in $\mathcal{S}$. The equilibria of an affine system can only lie in the affine space $\mathcal{O}$, and for all $x \in \mathcal{S} \cap \mathcal{O}$ and $u \in \mathbb{R}^m$, $Ax + Bu + a \in \mathcal{B}$. Defining the closed-loop vector field $f(x) = Ax + Bu(x) + a$, the previous statements suggest that a necessary condition to solve the RCP by continuous state feedback is: there exists a non-vanishing continuous map $f(x)$ on the set $\mathcal{S} \cap \mathcal{O}$ such that $f(x) \in \mathcal{B} \cap \mathcal{C}(x)$. Motivated by Brockett's work [3], we will say that if such a function does not exist, the system contains a *topological obstruction*.

Define $\mathcal{O}_{\mathcal{S}} = \mathcal{S} \cap \mathcal{O}$. We define $\mathrm{cone}(\mathcal{O}_{\mathcal{S}}) = \cap_{x \in \mathcal{O}_{\mathcal{S}}} \mathcal{C}(x)$. For the remainder of the paper we assume that $\mathcal{O}_{\mathcal{S}} \neq \emptyset$. We will also assume $v_0 \notin \mathcal{O}_{\mathcal{S}}$, as well as $1 \leq \dim(\mathcal{O}_{\mathcal{S}}) \leq n-1$. The cases of $\dim(\mathcal{O}_{\mathcal{S}}) = 0, n$ and $v_0 \in \mathcal{O}_{\mathcal{S}}$ are trivial to analyze.

For the sake of completeness, this analysis was formally done in Lemma 9, Lemma 10 and Corollary 11 of [11].

We study the following problem.

Problem 1. Let $\mathcal{S}$, $\mathcal{B}$, $\mathcal{O}$, and $\mathcal{O}_{\mathcal{S}}$ be as above. Does there exist a continuous map $f : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}\backslash\{\mathbf{0}\}$ such that for every $x \in \mathcal{O}_{\mathcal{S}}$, $f(x) \in \mathcal{C}(x)$?

2 Main Results

This section presents the main results on solving Problem 1. We show that if $\dim(\mathcal{O}_{\mathcal{S}}) = n-1$, then it is possible to characterize the solution of Problem 1 in terms of a smaller polytope $\mathcal{O}'_{\mathcal{S}}$, and $\mathcal{O}'_{\mathcal{S}}$ will be amenable to a complete analysis of the problem in low dimensions. The main result is presented in Theorem 1. The consequences of Theorem 1 to low dimensional systems are presented in Theorem 3.

Let us assume $\mathcal{O}_{\mathcal{S}}$ is $(n-1)$-dimensional. According to [7, 9], this means $\mathcal{S}$ is cut by $\mathcal{O}$ into two parts: one part containing v_0 and $p \geq 0$ other vertices, and the other containing the other $n-p \geq 1$ vertices of $\mathcal{S}$. W.l.o.g. we assume $\{v_0, v_1, \ldots, v_p\}$ are on one side of $\mathcal{O}_{\mathcal{S}}$ and $\{v_{p+1}, \ldots, v_n\}$ are on the other side, where we assume vertices of $\mathcal{S}$ on $\mathcal{O}_{\mathcal{S}}$ are in the set $\{v_{p+1}, \ldots, v_n\}$. The vertices of $\mathcal{O}_{\mathcal{S}}$ lie on those edges of $\mathcal{S}$ connecting v_i's which are on different sides of $\mathcal{O}_{\mathcal{S}}$. Thus, we employ the notation o_{ij} to denote a vertex of $\mathcal{O}_{\mathcal{S}}$ with $I(o_{ij}) = \{i, j\}$. If there are no vertices of $\mathcal{S}$ on $\mathcal{O}_{\mathcal{S}}$, then $\mathcal{O}_{\mathcal{S}}$ has $(p+1)(n-p)$ vertices [7], but if $\mathcal{O}_{\mathcal{S}}$ contains r vertices of $\mathcal{S}$, then $\mathcal{O}_{\mathcal{S}}$ has $(p+1)(n-p) - pr$ vertices. At this point we introduce a mild abuse of notation with the convention that if $v_j \in \mathcal{O}_{\mathcal{S}}$, then $o_{ij} = v_j$ for all $i = 0, \ldots, p$.

Let us introduce the following notation. Let

$$\{i_1, i_2, \ldots, i_k | j_1, j_2, \ldots, j_l\} = \mathrm{co}\{o_{i_\alpha j_\beta} : 1 \leq \alpha \leq k, 1 \leq \beta \leq l\}.$$

We observe that since $I(o_{ij}) = \{i, j\}$, if $x \in \{i_1, i_2, \ldots, i_k | j_1, j_2, \ldots, j_l\}$ then $I(x) \subseteq \{i_1, i_2, \ldots, i_k\} \cup \{j_1, j_2, \ldots, j_l\}$. Also observe that $\mathcal{O}_{\mathcal{S}} = \{0, \ldots, p | p+1, \ldots, n\}$.

Lemma 1. *Let* $\mathcal{O}_{\mathcal{S}} = \{0, \ldots, p | p+1, \ldots, n\}$, $\mathcal{A} = \{i_1, \ldots, i_k | j_1, \ldots, j_l\} \subseteq \mathcal{O}_{\mathcal{S}}$, *and* $\mathcal{A}' = \{i'_1, \ldots, i'_{k'} | j'_1, \ldots, j'_{l'}\} \subseteq \mathcal{O}_{\mathcal{S}}$. *Let* $L = \{i_1, \ldots, i_k\} \cap \{i'_1, \ldots, i'_{k'}\}$. *Analogously, let* $R = \{j_1, \ldots, j_l\} \cap \{j'_1, \ldots, j'_{l'}\}$. *Then,* $\mathcal{A} \cap \mathcal{A}' = \{L|R\}$.

Proof. By definition every vertex of $\{L|R\}$ is a vertex of $\mathcal{A}$ and of $\mathcal{A}'$, so $\{L|R\} \subseteq \mathcal{A} \cap \mathcal{A}'$. Conversely, suppose $x \in \mathcal{A} \cap \mathcal{A}'$. Since $x \in \mathcal{A}$, $I(x) \subseteq \{i_1, \ldots, i_k, j_1, \ldots, j_l\}$ and since $x \in \mathcal{A}'$, $I(x) \subseteq \{i'_1, \ldots, i'_{k'}, j'_1, \ldots, j'_{l'}\}$. Hence, $I(x) \subseteq L \cup R$, where we use the fact that $\{i_1, \ldots, i_k\} \cap \{j'_1, \ldots, j'_{l'}\} = \emptyset$ and $\{i'_1, \ldots, i'_{k'}\} \cap \{j_1, \ldots, j_l\} = \emptyset$. It follows $x \in \{L|R\}$. □

Before getting to the crux of the problem, let us introduce the notions of a homeomorphism and a retraction. Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be topological spaces. $\mathcal{X}$ and $\tilde{\mathcal{X}}$ are *homeomorphic* if there exists a continuous bijection $h : \mathcal{X} \to \tilde{\mathcal{X}}$ which has a continuous inverse. Furthermore, if $\mathcal{A}$ is a subspace of $\mathcal{X}$, a continuous map $r : \mathcal{X} \to \mathcal{A}$ is a *retraction* if $r|_{\mathcal{A}} \equiv id$.

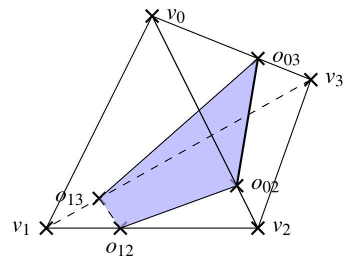

Fig. 1 The set $\mathcal{O}_\mathcal{S}$ for Example 1. The edge from o_{02} to o_{03} forms $\mathcal{O}'_\mathcal{S}$.

Example 1. Let us consider the case of $n = 3$, with $\mathcal{O}_\mathcal{S}$ a quadrilateral with vertices $o_{02} \in \mathrm{co}\{v_0, v_2\}$, $o_{03} \in \mathrm{co}\{v_0, v_3\}$, $o_{12} \in \mathrm{co}\{v_1, v_2\}$, and $o_{13} \in \mathrm{co}\{v_1, v_3\}$. By definition, $\mathcal{C}(o_{02}) \subseteq \mathcal{C}(o_{12})$ and $\mathcal{C}(o_{03}) \subseteq \mathcal{C}(o_{13})$. In fact, one can easily show that the cone of any point on the edge $\overline{o_{12}o_{02}}$ will be larger than $\mathcal{C}(o_{02})$, and analogously for the edge $\overline{o_{13}o_{03}}$. Thus, we reach the idea that if a continuous function satisfying Problem 1 exists on the convex set $\mathrm{co}\{o_{02}, o_{03}\}$ containing the most restrictive cones, then that function can easily be extended to the entire $\mathcal{O}_\mathcal{S}$.

Theorem 1 will serve to prove the above claim. The procedure outlined in the proof of Theorem 1, adapted to this example, is as follows. If a function f satisfying Problem 1 can be defined on the edge $\overline{o_{02}, o_{03}}$, we can also define it on the edge $\overline{o_{12}o_{02}}$ by $f(x) = f(o_{02})$ and on the edge $\overline{o_{13}o_{03}}$ by $f(x) = f(o_{03})$. We note that such f is non-zero and satisfies the cone condition $f(x) \in \mathcal{C}(x)$ because $\mathcal{C}(o_{02}) \subseteq \mathcal{C}(o_{12})$ and $\mathcal{C}(o_{03}) \subseteq \mathcal{C}(o_{13})$. So far f has been defined on three edges of $\mathcal{O}_\mathcal{S}$. Then f can be defined on the remainder of $\mathcal{O}_\mathcal{S}$, which consists of its interior as well as the relative interior of the edge $\overline{o_{12}o_{13}}$, by using a retraction r — a continuous map from $\mathcal{O}_\mathcal{S}$ to the three edges of $\mathcal{O}_\mathcal{S}$ on which f is already defined, such that r is identity on those three edges. More formally, it can be shown, as in Theorem 1, that the function $f(x) = f(r(x))$ exists and solves Problem 1.

Theorem 1 (Dimension Reduction). *Let* $\dim \mathcal{O}_\mathcal{S} = n-1$*,* $v_0 \notin \mathcal{O}_\mathcal{S}$*, and* $p > 0$*. Define* $V'_{\mathcal{O}_\mathcal{S}} = \{o \in V_{\mathcal{O}_\mathcal{S}} \mid (\exists j \in \{1,\ldots,n\}) o \in \mathrm{co}\{v_0, v_j\}\}$*, and let* $\mathcal{O}'_\mathcal{S} = \mathrm{co}(V'_{\mathcal{O}_\mathcal{S}})$*. Then the answer to Problem 1 is affirmative if and only if it is affirmative for* $\mathcal{O}'_\mathcal{S}$*.*

Proof. Since $V'_{\mathcal{O}_\mathcal{S}} \subseteq V_{\mathcal{O}_\mathcal{S}}$ it follows that $\mathcal{O}'_\mathcal{S} \subseteq \mathcal{O}_\mathcal{S}$. Thus, if there exists $f : \mathcal{O}_\mathcal{S} \to \mathcal{B} \backslash \{\mathbf{0}\}$ solving Problem 1 then $f|_{\mathcal{O}'_\mathcal{S}} : \mathcal{O}'_\mathcal{S} \to \mathcal{B} \backslash \{\mathbf{0}\}$ also solves Problem 1. Conversely, suppose there exists $f' : \mathcal{O}'_\mathcal{S} \to \mathcal{B} \backslash \{\mathbf{0}\}$ solving Problem 1. From our notational convention, $\mathcal{O}_\mathcal{S} = \{0, 1, \ldots, p \mid p+1, \ldots, n\}$, $V'_{\mathcal{O}_\mathcal{S}} = \{o_{0(p+1)}, \ldots, o_{0n}\}$, and $\mathcal{O}'_\mathcal{S} = \{0 \mid p+1, \ldots, n\}$.

We now proceed with the main topological argument. Informally, we build a skeleton of $\mathcal{O}_\mathcal{S}$, starting with $\mathcal{O}'_\mathcal{S}$, and adding in each step additional edges and faces of $\mathcal{O}_\mathcal{S}$ until in the last step all of $\mathcal{O}_\mathcal{S}$ is added. We then use topological methods to show that Problem 1 for the set obtained in each step can be reduced to the same problem applied to the set from the previous step, thus going back from $\mathcal{O}_\mathcal{S}$ to $\mathcal{O}'_\mathcal{S}$.

We build a skeleton of $\mathcal{O}_\mathcal{S}$ as follows. Let $\mathcal{O}^1_\mathcal{S} = \mathcal{O}'_\mathcal{S}$, and for all $2 \le k \le n$, let

$$\mathcal{O}_{\mathcal{S}}^{k} = \mathcal{O}_{\mathcal{S}}^{k-1} \cup \bigcup_{\substack{0<i_1<\ldots<i_\alpha \leq p,\\ p<j_1<\ldots<j_\beta \leq n,\\ \alpha+\beta=k,\alpha,\beta\geq 1}} \{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}.$$

Observe that each $\mathcal{H} = \{0,i_1,\ldots,i_\alpha \mid j_1,\ldots,j_\beta\}$ is a closed, convex polytope of some dimension d, so it is homeomorphic to the closed ball $\mathbb{B}^d$, and its boundary is homeomorphic to the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ [2]. We claim that

$$\partial\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\} \setminus \mathcal{O}_{\mathcal{S}}^{k-1} = \mathrm{Int}(\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}). \tag{4}$$

There are three points to the proof of the claim.

(i) $\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\} \in \partial\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}$. To show that, we note that if $x \in \{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}$, then

$$I(x) \subseteq \{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta\} \subseteq \{0,i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta\},$$

so $x \in \{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}$. Moreover, $\partial\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}$ consists of points with $|I(x)| \leq \alpha+\beta$.

(ii) $\mathrm{Int}(\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}) \cap \mathcal{O}_{\mathcal{S}}^{k-1} = \emptyset$. Assume $x \in \mathrm{Int}(\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\})$. Then $0 \notin I(x)$ and $I(x) = \alpha+\beta = k$. However, if $x \in \mathcal{O}_{\mathcal{S}}^{k-1}$, then either $0 \in I(x)$ or $|I(x)| \leq k-1$.

(iii) $\partial\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\} \setminus \mathrm{Int}(\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}) \subseteq \mathcal{O}_{\mathcal{S}}^{k-1}$. This follows because if $x \in \partial\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\} \setminus \mathrm{Int}(\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\})$, then either $0 \in I(x)$ and $|I(x)| \leq k$ so $x \in \mathcal{O}_{\mathcal{S}}^{k-1}$; or $0 \notin I(x)$ and $|I(x)| \leq k-1$, so again $x \in \mathcal{O}_{\mathcal{S}}^{k-1}$.

What we have shown so far is that $\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\}$ is homeomorphic to a closed ball, and $\partial\{0,i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\} \cap \mathcal{O}_{\mathcal{S}}^{k-1}$ is homeomorphic to its boundary sphere $\mathbb{S}^{d-1}$ with an open connected set $\mathrm{Int}(\{i_1,\ldots,i_\alpha|j_1,\ldots,j_\beta\})$ of dimension $d-1$ cut out of it. Now we show there exists a retraction from $\mathbb{B}^d$ to the punctured sphere $\mathbb{S}^{d-1} \setminus \mathcal{P}$, where $\mathcal{P}$ is homeomorphic to an open ball of dimension $d-1$. Since $\mathcal{P}$ has dimension $d-1$, $\mathbb{S}^{d-1}$ is split into two connected parts: $\mathcal{P}$ and $\mathbb{S}^{d-1} \setminus \mathcal{P}$. The retraction argument is standard in topology. We provide the proof since it is integral to our results.

First, let us note that $\mathbb{B}^d$ is homeomorphic to the upper half-ball $\mathbb{B}^{+d} = \{x \in \mathbb{B}^d : x_1 \geq 0\}$. The precise homeomorphism is not difficult to find, but one can simply imagine taking the ball and flattening its lower half. Now, our sphere $\mathbb{S}^{d-1}$ was mapped by this to the boundary of $\mathbb{B}^{+d}$. Furthermore, without loss of generality[1], we can assume that the closed part $\mathbb{S}^{d-1} \setminus \mathcal{P}$ makes up the bottom of the half-ball: $\{x \in \mathbb{B}^d : x_1 = 0\}$, while the open part $\{x \in \mathbb{B}^d : x_1 > 0\}$ corresponds to $\mathcal{P}$.

Let us define the function $r'_{\mathcal{H}} : \mathbb{B}^{+d} \to \{x \in \mathbb{B}^d : x_1 = 0\}$ by $r'_{\mathcal{H}}(x_1,x_2,\ldots,x_n) = (0,x_1,x_2,\ldots,x_n)$. Clearly, this is a valid retraction and thus, we have obtained a

[1] Really, this is done through another homeomorphism: this time, imagine, before flattening the ball, choosing the part that needs to be flattened to be $\mathbb{S}^{d-1} \setminus \mathcal{P}$.

retraction from $\mathbb{B}^{+d}$ to $\{x \in \mathbb{B}^d : x_1 = 0\}$. Now, using the fact that $\mathbb{B}^{+d}$ is homeomorphic to $\mathbb{B}^d$, while the same homeomorphism takes $\{x \in \mathbb{B}^d : x_1 = 0\}$ to $\mathbb{S}^{d-1} \backslash \mathcal{P}$, we know there thus exists a retraction $r''_{\mathcal{H}} : \mathbb{B}^d \to \mathbb{S}^{d-1} \backslash \mathcal{P}$. Finally, reminding ourselves that there exists a homeomorphism between $\mathcal{H}$ and $\mathbb{B}^d$ which takes $\mathbb{S}^{d-1} \backslash \mathcal{P}$ to $\partial \mathcal{O}_{\mathcal{S}} \cap \mathcal{O}_{\mathcal{S}}^{k-1}$, by "pushing" $r''_{\mathcal{H}}$ through that homeomorphism, we obtain a retraction $r_{\mathcal{H}} : \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\} \to \partial \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\} \cap \mathcal{O}_{\mathcal{S}}^{k-1}$.

Now we glue these retractions to each other. In order to do that, we need to know that for $\mathcal{H}$'s with constant $\alpha + \beta = k$, all the different retractions $r_{\mathcal{H}} : \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\} \to \partial \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\} \cap \mathcal{O}_{\mathcal{S}}^{k-1}$ agree on the intersections of their domains. That is, if $\mathcal{H} \neq \mathcal{H}'$, then $r_{\mathcal{H}}|_{\mathcal{H} \cap \mathcal{H}'} \equiv r_{\mathcal{H}'}|_{\mathcal{H} \cap \mathcal{H}'}$. (The claim is obvious if $\mathcal{H} = \mathcal{H}'$.) Let $\mathcal{H} = \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\}$. Analogously, let $\mathcal{H}' = \{0, i'_1, \dots, i'_{\alpha'} | j'_1, \dots, j'_{\beta'}\}$. We noted in Lemma 1 that $\mathcal{H} \cap \mathcal{H}' = \{0, \{i_1, \dots, i_\alpha\} \cap \{i'_1, \dots, i'_{\alpha'}\} | \{j_1, \dots, j_\beta\} \cap \{j'_1, \dots, j'_{\beta'}\}\}$. Since we assumed that $\mathcal{H} \neq \mathcal{H}'$, there needs to be an element in $\{i_1, \dots, i_\alpha, j_1, \dots, j_\beta\}$ which is not an element of the set $\{i'_1, \dots, i'_{\alpha'}, j'_1, \dots, j'_{\beta'}\}$ and vice versa (note that both of those sets have k elements, so one cannot be a subset of the other).

Thus, $\mathcal{H} \cap \mathcal{H}'$ will not contain more than $k-1$ non-zero vertices of $\mathcal{S}$ in its notation, and hence it will be in both $\mathcal{O}_{\mathcal{S}}^{k-1}$ (by the definition of $\mathcal{O}_{\mathcal{S}}^{k-1}$), and in $\partial \mathcal{H}$ (as none of its elements can be in the interior of $\mathcal{H}$: the expansion as a convex sum of every element in the interior needs to contain every vertex mentioned in the notation of $\mathcal{H}$). Analogously, $\mathcal{H} \cap \mathcal{H}' \in \partial \mathcal{H}' \cap \mathcal{O}_{\mathcal{S}}^{k-1}$, which is the image of the retraction $r_{\mathcal{H}'}$.

Hence, we know that $r_{\mathcal{H}'}|_{\mathcal{H} \cap \mathcal{H}'}$ is an identity map, and so is $r_{\mathcal{H}}|_{\mathcal{H} \cap \mathcal{H}'}$. Thus, these two retractions can indeed be glued together. By iterating this procedure for all $\mathcal{H}$, we obtain a glued retraction $r^k : \mathcal{O}_{\mathcal{S}}^k \to \mathcal{O}_{\mathcal{S}}^{k-1}$ which takes each k-dimensional edge in $\mathcal{O}_{\mathcal{S}}^k$ to its boundary. Let us note what this retraction does. For every point $x \in \mathcal{O}_{\mathcal{S}}^k$, if x is also in $\mathcal{O}_{\mathcal{S}}^{k-1}$, it will not do anything. Thus, $\mathcal{C}(r^k(x)) = \mathcal{C}(x)$.

If $x \notin \mathcal{O}_{\mathcal{S}}^{k-1}$, then either x is in the interior of some $\mathcal{H} = \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\}$ that is being added to $\mathcal{O}_{\mathcal{S}}^k$, or it is in the interior of $\{i_1, \dots, i_\alpha | j_1, \dots, j_\beta\}$. Now, if x is in the interior of $\{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\}$, r^k maps it to a point in the boundary of $\mathcal{H}$. In that case, it is easy to verify that $\mathcal{C}(r^k(x)) \subseteq \mathcal{C}(x)$. This has formally been done in Lemma 6 of [11].

If x is in the interior of $\{i_1, \dots, i_\alpha | j_1, \dots, j_\beta\}$, then $I(x) = \{i_1, \dots, i_\alpha, j_1, \dots, j_\beta\}$. On the other hand, $\mathcal{H} = \{0, i_1, \dots, i_\alpha | j_1, \dots, j_\beta\}$, so for any point $y \in \mathcal{H}$, $I(y) \subseteq \{i_1, \dots, i_\alpha, j_1, \dots, j_\beta\}$. Thus, $\mathcal{C}(y) \subseteq \mathcal{C}(x)$. Thus, specifically $\mathcal{C}(r^k(x)) \subseteq \mathcal{C}(x)$.

In all three cases, we deduce that $\mathcal{C}(r^k(x)) \subseteq \mathcal{C}(x)$. By composing $r = r^2 \circ r^3 \circ \dots \circ r^n$, we obtain a retraction $r : \mathcal{O}_{\mathcal{S}} = \mathcal{O}_{\mathcal{S}}^n \to \mathcal{O}_{\mathcal{S}}^1 = \mathcal{O}'_{\mathcal{S}}$. Define $f(x) = f'(r(x))$. We obtained a nowhere vanishing function f on $\mathcal{O}_{\mathcal{S}}$ such that $f(x) = f'(r^2(r^3(\dots(r^n(x))\dots)))$. Thus, $f(x)$ is contained in $\mathcal{C}(r^2(r^3(\dots(r^n(x))\dots))) \subseteq \mathcal{C}(r^2(r^3(\dots(r^{n-1}(x))\dots))) \subseteq \dots \subseteq \mathcal{C}(x)$. This function satisfies the conditions of Problem 1. $\square$

We now proceed to resolving Problem 1 for the case of $n = 2$ and $n = 3$. We have previously assumed that $\dim \mathcal{O}_{\mathcal{S}} \neq 0$ and $\dim \mathcal{O}_{\mathcal{S}} \neq n$, as these cases are simple to analyze. The case of $n = 2$ is thus reduced to $\dim \mathcal{O}_{\mathcal{S}} = 1$. As we have also required

$1 \le \dim \mathcal{B} < n$, we conclude that $\dim \mathcal{B} = 1$. However, the case of $\dim \mathcal{B} = 1$ is resolved by Theorem 1 in [12].

This resolves the case of $n = 2$, as well as $\dim \mathcal{B} = 1$. The only cases that remain are when $n = 3$, $\dim \mathcal{B} = 2$, and $\dim \mathcal{O}_{\mathcal{S}}$ is either 1 or 2. We will see that, when $\dim \mathcal{O}_{\mathcal{S}} = 1$, an argument based on linear algebra applies. On the other hand, a purely topological argument applies when $\dim \mathcal{O}_{\mathcal{S}} = 2$.

First we examine why a sufficiently high dimension for $\mathcal{B}$ resolves Problem 1.

Lemma 2. *Suppose $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{o_1, \ldots, o_{\kappa+1}\}$ where the o_i's are the vertices of $\mathcal{O}_{\mathcal{S}}$. If there exists a linearly independent set $\{\mathbf{b_i} \in \mathcal{B} \cap \mathcal{C}(o_i) \mid i = 1, \ldots, \kappa+1\}$, then the answer to Problem 1 is affirmative.*

Proof. Let $f : \mathcal{O}_{\mathcal{S}} \to \mathcal{B}$ be defined by $f(\sum_{i=1}^{\kappa+1} \alpha_i o_i) = \alpha_i \mathbf{b_i}$, where $\sum \alpha_i = 1$ and $\alpha_i \ge 0$. Necessarily $f(x) \neq \mathbf{0}$ for $x \in \mathcal{O}_{\mathcal{S}}$ for otherwise the $\mathbf{b_i}$'s would be linearly dependent. Also, by a standard convexity argument $f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_{\mathcal{S}}$. $\square$

The following is the key result in the case of $\dim \mathcal{O}_{\mathcal{S}} = 1$.

Lemma 3. *Let $n = 3$, $\dim \mathcal{B} = 2$, and let o_1 and o_2 be vertices of $\mathcal{O}_{\mathcal{S}}$. Then there exist linearly independent vectors $\{\mathbf{b_1}, \mathbf{b_2} \mid \mathbf{b_i} \in \mathcal{B} \cap \mathcal{C}(o_i)\}$. Moreover, if $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{o_1, o_2\}$, the answer to Problem 1 is affirmative.*

Proof. First we assume $o_1 \in \mathrm{Int}(\mathcal{F}_i)$ for some $i \in \{0,1,2,3\}$. By the definition of $\mathcal{C}(o_1)$, it is a closed half space or $\mathbb{R}^3$, so there exist linearly independent vectors $\mathbf{b_{11}}, \mathbf{b_{12}} \in \mathcal{B} \cap \mathcal{C}(o_1)$. We claim $\mathcal{B} \cap \mathcal{C}(o_2) \neq \mathbf{0}$. If $o_2 \in \mathrm{Int}(\mathcal{F}_i)$ for some $i \in \{0,1,2,3\}$ then the argument above proves the claim. Instead, assume w.l.o.g. that $o_2 \in \mathcal{F}_1 \cap \mathcal{F}_2$. Then $\mathcal{C}(o_2) = \{\mathbf{y} \in \mathbb{R}^3 | \mathbf{h_1} \cdot \mathbf{y} \le 0, \mathbf{h_2} \cdot \mathbf{y} \le 0\}$. Let $\mathcal{B} = \mathrm{Ker}(M^T)$ for some $M \in \mathbb{R}^{3 \times 1}$. Finding $\mathbf{0} \neq \mathbf{y} \in \mathcal{B} \cap \mathcal{C}(o_2)$ is equivalent to solving

$$\begin{bmatrix} \mathbf{h_1}^T \\ \mathbf{h_2}^T \\ M^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} s_1 \\ s_2 \\ 0 \end{bmatrix} \tag{5}$$

for some $s_1, s_2 \in \mathbb{R}_0^-$ and $\mathbf{y} \neq \mathbf{0}$. Because $\{\mathbf{h_1}, \mathbf{h_2}\}$ are linearly independent, $\mathrm{rank}(H) \ge 2$, where H is the matrix appearing on the left hand side of equation 5. If $\mathrm{rank}(H) = 3$, then let

$$[\mathbf{y_1} \ \ \mathbf{y_2}] = H^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Since $(-1,0,0)$ and $(0,-1,0)$ are linearly independent, $\mathbf{y_1}$ and $\mathbf{y_2}$ are linearly independent as well.

Next, assume $\mathrm{rank}(H) = 2$. In other words, $M = c_1 \mathbf{h_1} + c_2 \mathbf{h_2}$ for some $c_1, c_2 \in \mathbb{R}$. Then, by taking $s_1 = s_2 = 0$, equation (5) reduces to $[\mathbf{h_1} \ \mathbf{h_2}]^T y = 0$. By the rank-nullity theorem, there exists $\mathbf{y} \neq \mathbf{0}$ satisfying this equation. Moreover, if w.l.o.g. $v_0 = 0$, then $\mathbf{y} \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathrm{co}\{v_0, v_3\}$, and we can take $\mathbf{y} = v_3$. We have shown there exist linearly independent $\mathbf{b_{11}}, \mathbf{b_{12}} \in \mathcal{C}(o_1)$ and there exists $0 \neq \mathbf{b_2} \in \mathcal{C}(o_2)$. We claim at least one of the pairs $\{\mathbf{b_{11}}, \mathbf{b_2}\}$ and $\{\mathbf{b_{12}}, \mathbf{b_2}\}$ is linearly independent.

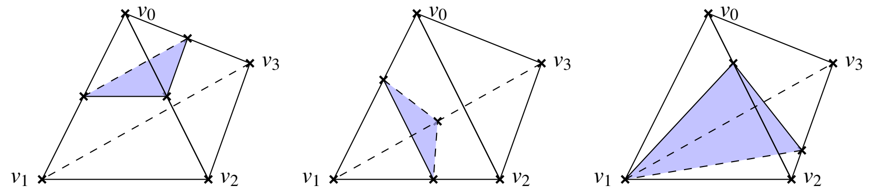

Fig. 2 Three of the four possible configurations of set $\mathcal{O}_\mathcal{S}$ for $n=3$ and $\dim \mathcal{O}_\mathcal{S} = 2$, with the fourth one given in Figure 1. The leftmost configuration is addressed by Theorem 2, while the other two can be reduced using Theorem 1.

For otherwise there exist $c_1, c_2 \in \mathbb{R}$ such that $\mathbf{b_2} = c_1\mathbf{b_{11}} = c_2\mathbf{b_{12}}$, implying $\mathbf{b_{11}}$ and $\mathbf{b_{12}}$ are linearly independent, a contradiction. We conclude there exists a linearly independent set $\{\mathbf{b_1}, \mathbf{b_2} \mid \mathbf{b_i} \in \mathcal{C}(o_i)\}$.

Next we assume neither o_1 nor o_2 lies in the interior of a facet. W.l.o.g. suppose $o_1 \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $o_2 \in \mathcal{F}_1 \cap \mathcal{F}_3$. If either $\mathcal{C}(o_1)$ or $\mathcal{C}(o_2)$ contains two linearly independent vectors, then by the previous argument, we are done. Otherwise, by the previous argument again $v_3 \in \mathcal{C}(o_1)$ and $v_2 \in \mathcal{C}(o_2)$. Since $\{v_2, v_3\}$ are linearly independent, we are done. Finally, if $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2\}$, then by Lemma 2 the answer to Problem 1 is affirmative. □

The remaining case to study is when $n=3$, $\dim \mathcal{O}_\mathcal{S} = 2$, and $\dim \mathcal{B} = 2$. Assuming $v_0 \notin \mathcal{O}_\mathcal{S}$ (which is a trivial case discussed in Lemma 10 of [11]), there are four topologically distinct cases for $\mathcal{O}_\mathcal{S}$, depending on the way $\mathcal{O}$ cuts $\mathcal{S}$. These are given in Figures 1 and 2. In Figure 1, $\mathcal{O}_\mathcal{S}$ is a quadrangle. In that case, $p=1$; that is, there are two vertices of $\mathcal{S}$ on each side of $\mathcal{O}$. Then we can apply Theorem 1 to reduce $\mathcal{O}_\mathcal{S}$ to $\mathcal{O}'_\mathcal{S}$, and according to the construction in the proof, $\mathcal{O}'_\mathcal{S}$ has dimension 1, and we can apply Lemma 3. Similarly, in the cases given in middle and the rightmost configuration of Figure 2, we can apply Theorem 1 to reduce $\mathcal{O}_\mathcal{S}$ to $\mathcal{O}'_\mathcal{S}$ with $\dim \mathcal{O}'_\mathcal{S}$ being either 0 or 1, respectively. Finally, in the situation given by the leftmost configuration of Figure 2, we draw upon a proof method already utilized in [4], which is based on Sperner's lemma. Here we employ a variant found in [13].

Lemma 4. *Let $\mathcal{P} = \mathrm{co}\{w_1, \ldots, w_{n+1}\}$ be an n-dimensional simplex. Furthermore, let $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{n+1}\}$ be a collection of sets covering $\mathcal{P}$ such that*

(P1) Vertex $w_i \in \mathcal{Q}_i$ and $w_i \notin \mathcal{Q}_j$ for $j \neq i$.
(P2) If w.l.o.g. $x \in \mathrm{co}\{w_1, \ldots, w_l\}$ for some $1 \le l \le n+1$, then $x \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_l$.

Then $\bigcap_{1=1}^{n+1} \overline{\mathcal{Q}}_i \neq \emptyset$.

Theorem 2. *Let $n=3$ and suppose $\mathcal{O}_\mathcal{S} = \mathrm{co}\{o_1, o_2, o_3\}$ with $v_0 \notin \mathcal{O}_\mathcal{S}$ and $o_i \in (v_0, v_i]$, $i = 1,2,3$. The answer to Problem 1 is affirmative if and only if*

$$\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}.$$

Proof. Sufficiency is clear: if $\mathbf{0} \neq \mathbf{b} \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, a constant function $f(x) = \mathbf{b}$ satisfies Problem 1. For necessity, suppose there exists $f : \mathcal{O}_\mathcal{S} \to \mathcal{B} \setminus \{\mathbf{0}\}$ such that

$f(x) \in \mathcal{C}(x)$, $x \in \mathcal{O}_\mathcal{S}$. By way of contradiction suppose $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) = \mathbf{0}$. Since $o_i \in (v_0, v_i]$, $i = 1,2,3$, we have

$$\mathrm{cone}(\mathcal{O}_\mathcal{S}) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{h_j} \cdot \mathbf{y} \leq 0\,, j = 1,2,3\}\,.$$

Define the sets

$$\mathcal{Q}_i := \{x \in \mathcal{O}_\mathcal{S} \mid \mathbf{h_i} \cdot f(x) > 0\}\,, \quad i = 1,2,3\,. \tag{6}$$

Now we verify the conditions of Lemma 4.

Firstly, we claim that $\{\mathcal{Q}_i\}$ cover $\mathcal{O}_\mathcal{S}$. For suppose not. Then there exists $x \in \mathcal{O}_\mathcal{S}$ such that $h_j \cdot f(x) \leq 0$, $j = 1,2,3$. Hence $f(x) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, so $f(x) = \mathbf{0}$, a contradiction to f being non-vanishing on $\mathcal{O}_\mathcal{S}$. Secondly, we verify property (P1). We claim that $o_i \in \mathcal{Q}_i$ for $i = 1,2,3$. For suppose not. Then $h_i \cdot f(x) \leq 0$. Additionally, because $f(o_i) \in \mathcal{C}(o_i)$, $h_j \cdot f(x) \leq 0$, $j \in \{1,2,3\} \setminus \{i\}$. We conclude $f(o_i) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, so $f(o_i) = \mathbf{0}$, a contradiction. Next we claim $o_i \notin \mathcal{Q}_j$, $j \neq i$. This is immediate since $f(o_i) \in \mathcal{C}(o_i)$ implies $h_j \cdot f(o_i) \leq 0$, $j \neq i$. Thirdly, we verify property (P2). Suppose w.l.o.g. (by reordering the indices $\{1,2,3\}$) $x \in \mathrm{co}\{o_1, \ldots, o_r\}$ for some $1 \leq r \leq 3$. We claim $x \in \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_r$. For suppose not. Then $h_j \cdot f(x) \leq 0$, $j = 1, \ldots, r$. Also, it is easily verified that $\mathcal{C}(x) = \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0\,, j = r+1, \ldots, 3\}$. Thus, $h_j \cdot f(x) \leq 0$, $j = r+1, \ldots, 3$. Hence, $f(x) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, so $f(x) = \mathbf{0}$, a contradiction to f being non-vanishing on $\mathcal{O}_\mathcal{S}$.

We have verified (P1)-(P2) of Lemma 4. Applying the lemma, there exists $\bar{x} \in \bigcap_{i=1}^{3} \overline{\mathcal{Q}}_i$; that is, $h_j \cdot f(\bar{x}) \geq 0$, $j = 1,2,3$. We conclude that $-f(\bar{x}) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S})$, so $f(\bar{x}) = \mathbf{0}$, a contradiction. □

The following result finally resolves Problem 1 in all cases of interest.

Theorem 3. *Let $\mathcal{S}$, $\mathcal{B}$ and $\mathcal{O}_\mathcal{S}$ be as above, and let $n \in \{2,3\}$. If $n = 3$, $\dim \mathcal{B} = 2$ and $\mathcal{O}_\mathcal{S}$ does not satisfy the conditions of Theorem 2, then the answer to Problem 1 is affirmative. Otherwise, the answer to Problem 1 is affirmative if and only if $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_\mathcal{S}) \neq \mathbf{0}$.*

Proof. The discussion prior to Lemma 2, as well as Lemma 3 and Theorem 2, covered all the cases except for the one where $\dim \mathcal{B} = \dim \mathcal{O}_\mathcal{S} = 2$ and $\mathcal{O}_\mathcal{S}$ does not satisfy the conditions of Theorem 2. However, in that case, as described prior to Lemma 4, Theorem 1 reduces $\mathcal{O}_\mathcal{S}$ to $\mathcal{O}'_\mathcal{S}$ of dimension 0 or 1. Applying Lemma 3, we now obtain that the answer to Problem 1 is affirmative. □

An interesting modification of Problem 1 requires that f be not only continuous, but also affine. This corresponds to a classic problem of designing an affine state feedback, applied to RCP. We note that in most of the configurations considered above, the same claims that work for Problem 1 can also be used in the affine case. The only significantly different case is $\dim \mathcal{O}_\mathcal{S} = \dim \mathcal{B} = 2$, i.e., the situation covered by Theorem 1. In this case, the full analysis of the problem of affine obstruction can be done using linear algebra. Such an analysis is not difficult, but is computationally long. It is presented in full in Section 3.2.2 of [11]. With the results contained in [11] (in fact, Theorem 16 in [11] is essentially the same as our Theorem 3), we reach the following theorem:

Theorem 4. *Let $n \in \{2,3\}$. The problem of affine obstruction is solvable if and only if Problem 1 is solvable.*

It is not known if such a result holds in general. Hence, we end this paper with the following conjecture.

Conjecture 1. *Let $n \geq 4$. The problem of affine obstruction is solvable if and only if Problem 1 is solvable.*

3 Conclusion

This paper introduces a topological obstruction to solving the RCP via continuous state feedback. The results show an interplay between linear algebra-based arguments regarding the number of control inputs and purely topological arguments regarding a cone condition on $\mathcal{B}$. We show that for $n = 2$ and $n = 3$ these two properties together fully characterize when a topological obstruction arises.

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