A Graph-theoretic Approach to the Reach Control Problem

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***Abstract*— This paper establishes new necessary and sufficient conditions for the solvability of the Reach Control Problem (RCP). The RCP seeks to drive the trajectories of a control system defined on a simplex to leave this simplex through a predetermined facet. This paper takes a novel approach to the RCP, transforming it into a problem in positive system theory. Using the notions of Z-matrices and graph theory, this results in a number of new necessary and sufficient conditions for the solvability of the RCP. In parallel, we also examine open-loop equilibria in the RCP, and provide a number of necessary and sufficient conditions for their existence.**

I. Introduction

The focus of this paper is on the solvability of the Reach Control Problem (RCP) by affine feedback. The RCP, defined in its current form in [10], [21], seeks to find a feedback control that drives trajectories of an affine control system defined on a simplex $\mathcal{S}$ to exit $\mathcal{S}$ through a predetermined facet, without previously leaving through any other facets. The RCP is a building block of reach control theory, which is an approach to satisfying complex control specifications on a constrained state space. The entire theory of reach control is outside the scope of this paper; for more details on reach control, the RCP, and the applications, we direct the reader to, e.g., [7], [10], [12], [21], [23], and a particularly comprehensive set of references in [18].

While it has been extensively researched, the theory of reach control has not yet been fully related to other directions and topics of control research. In particular, its fundamental building block, the RCP, concerns the behaviour of a control system on a simplex $\mathcal{S}$. With an appropriate coordinate transformation, $\mathcal{S}$ can be taken to lie in the positive orthant, with edges aligned with coordinate axes. Thus, a tempting approach to the RCP is to restate it in terms of positive systems theory, and draw from the apparatus developed therein [5], [9]. This paper is the first step in such a direction.

The contributions of this paper are as follows: we begin by transforming the RCP into a problem in terms of positive systems. This results in a set of sufficient and necessary conditions for solvability of the RCP by affine feedback posed in terms of existence of positive solutions of a linear equation. We then examine the set $\mathcal{E}_{\mathcal{S}}$ of open-loop equilibria of the underlying affine system, and provide graph-theoretic sufficient and necessary conditions for $\mathcal{E}_{\mathcal{S}} \neq \emptyset$. This is then expanded to a set of sufficient and necessary conditions for the solvability of the RCP by affine feedback. Finally, we focus on two strategies for the triangulation of the state space previously used in the reach control literature. In the first of these two strategies, under an additional technical assumption, we provide a third set of sufficient and necessary conditions for $\mathcal{E}_{\mathcal{S}} = \emptyset$. In the second strategy, we provide a novel strong sufficient condition for the solvability of the RCP by affine feedback, and show that this condition is computationally easy to verify.

Notation: $\mathbb{1}_n \in \mathbb{R}^n$ is a vector consisting solely of 1's. $e_i \in \mathbb{R}^n$ is the i-th coordinate vector: it consists solely of 0's, except for a 1 at the i-th position. The set of all real $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$. If A is a matrix, then $[A]_{ij}$ is the (i,j)-th element of A. If A is a block-matrix, then A_{ij} is the (i,j)-th block of A. If P is a matrix, P is *nonnegative* (denoted by $P \geq 0$) if $[P]_{ij} \geq 0$ for all i,j. It is *semipositive* (denoted by $P > 0$) if $P \geq 0$ and $P \neq 0$, and it is *positive* (denoted by $P \gg 0$) if $[P]_{ij} > 0$ for all i,j. If $a \in \mathbb{R}^n$ is a block-vector with p blocks, $\mathrm{supp}(a) = \{i \in \{1,\ldots,p\} \mid a_i \neq 0\}$. If $\mathcal{X}$ is a set, $\partial\mathcal{X}$ denotes its (relative) boundary, and $\mathrm{int}(\mathcal{X})$ its (relative) interior. Given points $v_1,\ldots,v_n$, their span is denoted by $\mathrm{span}\{v_1,\ldots,v_n\}$, their affine hull is denoted by $\mathrm{aff}\{v_1,\ldots,v_n\}$, and their convex hull is denoted by $\mathrm{co}\{v_1,\ldots,v_n\}$. A continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *affine* if $f(x) = Kx + g$ for some $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$.

II. Reach Control Theory

Let $\mathcal{S}$ be a simplex with vertices $v_0,\ldots,v_n$ and facets $\mathcal{F}_0,\ldots,\mathcal{F}_n$, each indexed by the vertex it does not contain. For each $i \in \{0,\ldots,n\}$, let h_i be a normal vector of $\mathcal{F}_i$ pointing outside $\mathcal{S}$. We consider the system

$$\dot{x} = Ax + Bu + a, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $a \in \mathbb{R}^n$. Define $\mathcal{B} = \mathrm{Im}(B)$.

Let $u : \mathbb{R}^n \to \mathbb{R}^m$. Let $\phi_u(\cdot, x_0)$ be the trajectory generated by system (1), with $\phi_u(0,x_0) = x_0$. (We assume such a trajectory is unique, which is true for all classes of control laws currently investigated for the RCP; for more, see [19].) We say that *the Reach Control Problem is solvable for $(A,B,a,\mathcal{S})$ by a class of functions $\mathfrak{F}$* if there exists a feedback control $u \in \mathfrak{F}$ such that for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\varepsilon > 0$ for which the following holds:

(i) $\phi_u(t,x_0) \in \mathcal{S}$ for all $t \in [0,T]$,
(ii) $\phi_u(T,x_0) \in \mathcal{F}_0$,
(iii) $\phi_u(T+t,x_0) \notin \mathcal{S}$ for all $t \in (0,\varepsilon)$.

For simplicity, instead of stating that the RCP is solvable for $(A,B,a,\mathcal{S})$ by the class of feedback controls $\mathfrak{F}$, it is usually only stated that *the RCP is solvable by* $\mathfrak{F}$. A number of different classes $\mathfrak{F}$ have been investigated in previous work

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on the RCP, including affine feedback [21], [18], continuous state feedback [7], and piecewise affine feedback [8].

For the purpose of computational work with the RCP, it would clearly be advantageous to assume that the vertices of $\mathcal{S}$ lie on the coordinate axes. This assumption was previously taken, e.g., in [20]. Indeed, by observing (1), it is clear that an affine coordinate transformation can be applied in order to transform $\mathcal{S}$ into a so-called standard orthogonal simplex. However, the full methodology for such a transformation was never rigorously presented; we provide it here.

Definition 1: A simplex $\mathcal{S} \subset \mathbb{R}^n$ with vertices $v_0, \ldots, v_n$ is a *standard orthogonal simplex* if $v_0 = 0$ and $v_i = e_i$ for all $i \in \{1, \ldots, n\}$. It has the following properties:

- $\mathcal{S} = \{x \in \mathbb{R}^n \mid x \geq 0, \mathbb{1}_n^T x \leq 1\}$,
- $h_0 = \mathbb{1}_n$, and $h_i = -e_i$ for all $i \in \{1, \ldots, n\}$,
- $\mathcal{F}_0 = \{x \in \mathbb{R}^n \mid x \geq 0, \mathbb{1}_n^T x = 1\}$, and $\mathcal{F}_i = \{x \in \mathbb{R}^n \mid x \geq 0, \mathbb{1}_n^T x \leq 1, [x]_i = 0\}$ for all $i \in \{1, \ldots, n\}$.

Let $\mathfrak{F}$ be a class of functions defined on $\mathbb{R}^n$. $\mathfrak{F}$ is *affine-invariant* if $f \in \mathfrak{F} \Rightarrow f \circ p \in \mathfrak{F}$ for all invertible affine functions $p : \mathbb{R}^n \to \mathbb{R}^n$.

Remark 2: Clearly, affine functions, continuous functions, and piecewise affine functions are all affine-invariant.

Lemma 3: Let $\mathcal{S} \subset \mathbb{R}^n$ be a simplex, and let $\Delta \subset \mathbb{R}^n$ be the standard orthogonal simplex. Let $\mathfrak{F}$ be an affine-invariant class of feedback controls $u : \mathbb{R}^n \to \mathbb{R}^m$. Let $p : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible affine map $p(x) = Kx + g$ with $K(e_i) = v_i$ for all $i \in \{0, \ldots, n\}$, and $g = v_0$. Then, RCP is solvable for $(A, B, a, \mathcal{S})$ by $\mathfrak{F}$ if and only if it is solvable for $(K^{-1}AK, K^{-1}B, K^{-1}a + K^{-1}Ag, \Delta)$.
The proof of Lemma 3 is contained in the Appendix.

Lemma 3 ensures that we can always assume without loss of generality that $\mathcal{S}$ is the standard orthogonal simplex, and discuss the solvability of the RCP in that setting. Hence, in the remainder of the paper it is assumed that $\mathcal{S}$ is the standard orthogonal simplex.

A. Conditions for Solvability

There are two clear necessary conditions for solvability of the RCP by an affine feedback u. First, if an affine feedback u solves the RCP for system (1), there cannot be any equilibria in $\mathcal{S}$. In other words, we must have

$$Ax + Bu(x) + a \neq 0 \tag{2}$$

for all $x \in \mathcal{S}$. In relation to that, we define the sets $\mathcal{E} = \{x \in \mathbb{R}^n \mid Ax + a = 0\}$ and $\mathcal{E}_\mathcal{S} = \mathcal{E} \cap \mathcal{S}$, i.e., the set of equilibria of (1) on the entire Euclidean space with null control input, and the set of equilibria of (1) in $\mathcal{S}$ with null control input, respectively. We also define the sets $\mathcal{O} = \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}$ and $\mathcal{O}_\mathcal{S} = \mathcal{O} \cap \mathcal{S}$. It can easily be shown that $\mathcal{O}_\mathcal{S}$ is a convex polytope, and that all equilibria of (1) with any control input u lie in $\mathcal{O}$ (for details, see, e.g., [22]). Thus, if $\mathcal{O}_\mathcal{S} = \emptyset$, the no-equilibrium condition (2) is trivially satisfied. In the remainder of this paper, we assume $\mathcal{O}_\mathcal{S} \neq \emptyset$.

The other necessary condition consists of the *invariance conditions* [21]. In informal terms, the velocity vector $Ax + Bu(x) + a$ cannot point outside of simplex $\mathcal{S}$ at any point $x \in \partial\mathcal{S} \backslash \mathcal{F}_0$. Otherwise, the trajectory $\phi_u(\cdot, x)$ would leave $\mathcal{S}$ by exiting through a wrong facet, thus breaking condition (ii) in the RCP above. Formally, this is encoded by defining $I(x) \subset \{0, 1, \ldots, n\}$ to be the smallest set such that $x \in \text{co}\{v_i \mid i \in I(x)\}$, and defining $\mathcal{C}(x) = \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, j \in \{1, 2, \ldots, n\} \backslash I(x)\}$ with the agreed convention that $\mathcal{C}(x) = \mathbb{R}^n$ if $\{1, 2, \ldots, n\} \backslash I(x) = \emptyset$. An illustration of the two-dimensional standard orthogonal simplex with corresponding cones, modified from [20], is given in Figure 1.

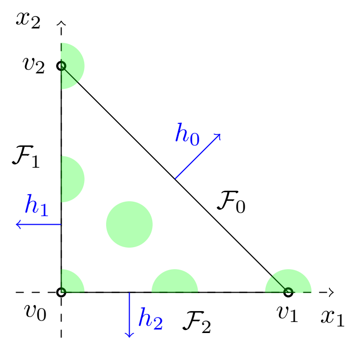

Fig. 1. An illustration of the standard orthogonal simplex $\mathcal{S} \subset \mathbb{R}^2$, with cones $\mathcal{C}(x)$ depicted as green cones at several points $x \in \mathcal{S}$. Normal vectors h_i are denoted by blue arrows.

We say that a feedback control u *satisfies the invariance conditions* if $Ax + Bu(x) + a \in \mathcal{C}(x)$ for all $x \in \mathcal{S}$. If u is an affine function, the invariance conditions are satisfied if and only if they are satisfied for all v_i, $i \in \{0, \ldots, n\}$. In other words,

$$Av_i + Bu(v_i) + a \in \mathcal{C}(v_i) \tag{3}$$

needs to hold for all $i \in \{0, \ldots, n\}$.

The following result unifies the above two necessary conditions into a complete set of sufficient and necessary conditions:

Theorem 4 ([10], [21]): The RCP is solvable by an affine feedback u if and only if both conditions (2) and (3) hold.

While the conditions in Theorem 4 are easily checkable for a given candidate feedback u, they are not directly useful to determine whether RCP is solvable by the class of affine feedbacks. Additionally, determining that the conditions of Theorem 4 are not satisfied does not provide any geometric intuition as to why the RCP is not solvable. While other sufficient, necessary, or sufficient and necessary conditions for solvability of the RCP have been developed in the hope of gaining further intuition (see, e.g., [14], [18]), all of these conditions were tailor-made for the RCP and did not serve to immerse the RCP theory into wider work on control. The results of Section IV amend this by posing the RCP as a problem in the theory of positive systems.

III. PRELIMINARIES

A. Matrix Theory

We say that $P \in \mathbb{R}^{n \times n}$ is a *Z-matrix* if $[P]_{ij} \leq 0$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. The set of all $n \times n$ Z-matrices is denoted by $\mathcal{Z}_n$. By the Perron-Frobenius theorem (e.g., [17]), Z-matrices have at least one real eigenvalue. If P is a Z-matrix, we denote its smallest real eigenvalue by $l(P)$. Matrix P is an *M-matrix* if it is a Z-matrix with $l(P) \geq 0$. For a detailed survey of Z- and M-matrices, and multiple different characterizations of M-matrices, see [4].

We define

$$M := -(A + a\mathbb{1}_n^T). \tag{4}$$

The following lemma is the key in all the work presented further in the paper.

Lemma 5: Let $\mathcal{S}$ be the standard orthogonal simplex. Then, $Av_i + a \in \mathcal{C}(v_i)$, $i \in \{0, \dots, n\}$, if and only if the following two conditions hold:

(i) M is a Z-matrix,

(ii) $a \geq 0$.

Proof: We note that, since $\mathcal{S}$ is the standard orthogonal simplex, $h_j \cdot (Av_i + a) = -([A]_{ji} + [a]_j) = [M]_{ji}$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Hence, for any $i \in \{1, \dots, n\}$, the invariance condition $Av_i + a \in \mathcal{C}(v_i)$ holds if and only if $[M]_{ji} \leq 0$ for all $j \neq i$. By going through all $i \in \{1, \dots, n\}$, we obtain the definition of a Z-matrix.

For (ii), we notice that the invariance condition $Av_0 + a \in \mathcal{C}(v_0)$ holds if and only if $h_j \cdot (Av_0 + a) = -e_j \cdot a = -[a]_j \leq 0$ for all $j \in I$. That is, if and only if $a \geq 0$. ■

We note that conditions (i) and (ii) of Lemma 5 are standard conditions for positive invariance of an affine dynamical system $\dot{x} = Ax + a$ [1]. Analogously, the invariance conditions (3) can just be interpreted as conditions for positive invariance of an affine control system (1).

B. Frobenius Normal Form

A matrix is *irreducible* if its rows and columns cannot be simultaneously permuted to obtain a lower triangular matrix. By simultaneously permuting its rows and columns, any matrix P can be written in a (lower triangular) *Frobenius normal form*

$$P = \begin{bmatrix} P_{11} & 0 & \cdots & 0 \\ P_{21} & P_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{p1} & P_{p2} & \cdots & P_{pp} \end{bmatrix},$$

where each block $P_{11}, \dots, P_{pp}$ is either irreducible or a 1×1 zero-matrix. For more background, see [3].

In the remainder of this section we largely use the notation and notions from [15], [16]. Assume that P is in Frobenius normal form, with p^2 blocks. Let $R(P) = (V, E)$ be a directed graph with the vertex set $V = \{1, \dots, p\}$ and edges $E = \{(i,j) \in V \times V \mid P_{ij} \neq 0, \, i \neq j\}$.

Definition 6: We say that i *accesses* j if $i = j$ or there is a path in $R(P)$ going from i to j. We denote this by $i \rightsquigarrow j$. If $W \subset \{1, \dots, p\}$, we say that W accesses j (denoted by $W \rightsquigarrow j$) if $i \rightsquigarrow j$ for some $i \in W$.

Define

$$\text{above}(W) = \{j \in \{1, \dots, p\} \mid W \rightsquigarrow j\}. \tag{5}$$

Let P be a Z-matrix in Frobenius normal form. Then, all blocks P_{ii} are Z-matrices as well. We define

$$\begin{aligned} S &= \{i \in \{1, \dots, p\} \mid P_{ii} \text{ is singular}\}, \\ T &= \{i \in \{1, \dots, p\} \mid l(P_{ii}) < 0\}. \end{aligned} \tag{6}$$

The following two known propositions will be the key to a graph-theoretic characterization of the solvability of the RCP:

Proposition 7 (Corollary 5.13, [15]): Suppose P is a Z-matrix in Frobenius normal form. The linear equation $Py = 0$ has a solution $y > 0$ if and only if $S \backslash \text{above}(T) \neq \emptyset$.

Proposition 8 (Theorem 3.11, [16]): Suppose P is a Z-matrix in Frobenius normal form, and $a \geq 0$. The linear equation $Py = a$ has a solution $y \geq 0$ if and only if $\text{supp}(a) \cap \text{above}(S \cup T) = \emptyset$.

IV. Main Results

As alluded to in the introduction, solvability of the RCP by affine feedback is intimately connected with the geometry of the set $\mathcal{E}_\mathcal{S}$. Suppose that the RCP is solvable by an affine feedback $u(x) = Kx + g$. By Theorem 4, $\{x \in \mathcal{S} \mid (A + BK)x + (a + Bg) = 0\} = \emptyset$. However, this set is exactly $\mathcal{E}_\mathcal{S}$ for a new affine system

$$\dot{x} = \tilde{A}x + \tilde{a}, \tag{7}$$

with $\tilde{A} = A + BK$, $\tilde{a} = a + Bg$.

In order to examine the set $\mathcal{E}_\mathcal{S}$, we partition it into disjoint sets $\mathcal{E}_\mathcal{S} \cap \mathcal{F}_0$ and $\mathcal{E}_\mathcal{S} \backslash \mathcal{F}_0$. This has two benefits: first, we will show that it is possible to easily obtain necessary and sufficient graph-theoretical conditions for $\mathcal{E}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$ and $\mathcal{E}_\mathcal{S} \backslash \mathcal{F}_0 = \emptyset$. Secondly, this partition naturally follows from the previous work on the RCP. Because the choice of a triangulation of the state space into simplices is left to the designer, the designer can choose to triangulate the state space into simplices so that for each simplex in the triangulation, the set of potential equilibria $\mathcal{O}_\mathcal{S}$ is either empty or lies in the exit facet $\mathcal{F}_0$ of the simplex. Alternatively, the triangulation can be chosen so that $\mathcal{O}_\mathcal{S}$ is either empty or $\mathcal{O}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$. Since $\mathcal{E}_\mathcal{S} \subset \mathcal{O}_\mathcal{S}$, choosing $\mathcal{O}_\mathcal{S} \subset \mathcal{F}_0$ automatically ensures that $\mathcal{E}_\mathcal{S} \backslash \mathcal{F}_0 \subset \mathcal{O}_\mathcal{S} \backslash \mathcal{F}_0 = \emptyset$. On the other hand, choosing $\mathcal{O}_\mathcal{S} \subset \mathcal{S} \backslash \mathcal{F}_0$ would ensure that $\mathcal{E}_\mathcal{S} \cap \mathcal{F}_0 = \emptyset$. We discuss the results obtained by these two triangulation strategies in Section V.

We now examine the sets $\mathcal{E}_\mathcal{S} \cap \mathcal{F}_0$ and $\mathcal{E}_\mathcal{S} \backslash \mathcal{F}_0$.

Proposition 9: Let $\mathcal{S}$ be the standard orthogonal simplex. Let M be as in (4). There exists $x \in \mathcal{E}_\mathcal{S} \cap \mathcal{F}_0$ if and only if the linear equation $My = 0$ has a solution $y > 0$.

Proof: Assume that there exists $x \in \mathcal{E}_\mathcal{S} \cap \mathcal{F}_0$. Then, by the definition of $\mathcal{E}_\mathcal{S}$, $Ax + a = 0$. Also, by Definition 1, $x > 0$ and $\mathbb{1}_n^T x = 1$. Thus, $Ax + a(\mathbb{1}_n^T x) = 0$, i.e., $Mx = 0$.

Conversely, assume that $My = 0$ has a solution $y > 0$. Define $x = y/(\mathbb{1}_n^T y)$. We note that $x > 0$, and $\mathbb{1}_n^T x = \mathbb{1}_n^T y/(\mathbb{1}_n^T y) = 1$. Hence, by Definition 1, $x \in \mathcal{F}_0$. Additionally, $Mx = 0$, i.e., $(A + a\mathbb{1}_n^T)x = Ax + a = 0$. Thus, $x \in \mathcal{E}_\mathcal{S}$. ■

Proposition 10: Let $\mathcal{S}$ be the standard orthogonal simplex. Then, there exists $x \in \mathcal{E}_\mathcal{S} \backslash \mathcal{F}_0$ if and only if the linear equation $My = a$ has a solution $y \geq 0$.

Proof: Assume that there exists $x \in \mathcal{E}_\mathcal{S} \backslash \mathcal{F}_0$. Then, by the definition of $\mathcal{E}_\mathcal{S}$, $Ax + a = 0$. Hence, $Ax + a\mathbb{1}_n^T x = a(\mathbb{1}_n^T x - 1)$, i.e., $Mx = a(1 - \mathbb{1}_n^T x)$. Also, by Definition 1,

$x \geq 0$ and $\mathbb{1}_n^T x < 1$. Thus, if we define $y = x/(1 - \mathbb{1}_n^T x)$, we obtain that $y \geq 0$ is a solution to $My = a$.

Conversely, assume that $y \geq 0$ is a solution to $My = a$. Define $x = y/(1 + \mathbb{1}_n^T y)$. We note that $x \geq 0$ and

$$\mathbb{1}_n^T x = (\mathbb{1}_n^T y)/(1 + \mathbb{1}_n^T y) < 1. \tag{8}$$

Hence, $x \in \mathcal{S} \backslash \mathcal{F}_0$. Additionally, $Mx = My/(1 + \mathbb{1}_n^T y) = a/(1+\mathbb{1}_n^T y)$. Thus, $Ax + a\mathbb{1}_n^T x = -a/(1+\mathbb{1}_n^T y)$, i.e., $Ax + a = a(1 - \mathbb{1}_n^T x - 1/(1 + \mathbb{1}_n^T y))$. From (8) it follows that $Ax + a = 0$, i.e., $x \in \mathcal{E}$. We are done. ∎

At this point, we can provide a general characterization of solvability of the RCP for the case of affine feedback.

Theorem 11: Let $\mathcal{S}$ be the standard orthogonal simplex. Then, RCP is solvable by affine feedback if and only if there exist $K \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$ such that the following conditions are satisfied:

(i) $-((A + BK) + (a + Bg)\mathbb{1}_n^T)$ is a Z-matrix,

(ii) $a + Bg > 0$,

(iii) neither of the following equations:

(iii.a) $-((A + BK) + (a + Bg)\mathbb{1}_n^T)y = 0$,

(iii.b) $-((A + BK) + (a + Bg)\mathbb{1}_n^T)y = a + Bg$

admits a solution $y > 0$.

If the above conditions are satisfied for some K, g, then $u = Kx + g$ solves the RCP.

Proof: Assume that the RCP is solvable by affine feedback. Let $u = Kx+g$ solve the RCP. Then, (i) holds by Lemma 5 and the discussion at the beginning of this section. Also by Lemma 5, $a + Bg \geq 0$. However, $a + Bg \neq 0$, because $a + Bg = 0$ would imply that $v_0 = 0$ is an equilibrium of (1), which is prohibited by Theorem 4. Hence, (ii) holds as well. Finally, if the RCP is solvable by affine feedback, then by Theorem 4 it neither has an equilibrium in $\mathcal{F}_0$ nor in $\mathcal{S}\backslash\mathcal{F}_0$. Hence, by Proposition 9, equation (iii.a) admits no solutions $y > 0$, and by Proposition 10, (iii.b) admits no solutions $y \geq 0$.

Conversely, assume that (i), (ii) and (iii) hold. Then, from (i), (ii) and Lemma 5, we obtain that the invariance conditions (3) hold for $u = Kx + g$. Additionally, from (iii.a) and Proposition 9, we obtain that the set of equilibria of (1) contained in $\mathcal{F}_0$ is empty. We note that (ii) and (iii.b) imply that the equation in (iii.b) also admits no solutions $y \geq 0$. Thus, by Proposition 9, system (1) has no equilibria in $\mathcal{S}\backslash\mathcal{F}_0$. Hence, as noted in Theorem 4, the RCP is solvable by affine feedback u. ∎

Finally, let us dig into the graph-theoretic conditions for solvability of RCP implied by Proposition 9 and Proposition 10. Let us permute the rows and columns of matrix M from (4) so that M becomes a block lower triangular matrix in a Frobenius normal form with blocks M_{ij}, $i, j \in \{1, \ldots, p\}$. Hence, from now on we assume that $M_{ij} = 0$ for $i < j$, and each block M_{ii} is either irreducible or a 1×1 block with $M_{ii} = 0$. The components of a are permuted according to the same permutation, with blocks $a_1, \ldots, a_p$. This process simply corresponds to the relabeling of vertices $v_1, \ldots, v_n$.

In order to connect the results of Proposition 9 and Proposition 10 with results on graph theory from Section III-B, we need to ensure that M is a Z-matrix. As we saw previously, this property follows naturally from invariance conditions. If conditions (3) are solvable, then one can always pre-apply a control $u(x) = K'x + g'$ such that (3) holds, and then define $\tilde{A} = A + BK'$ and $\tilde{a} = a + Bg'$. Then, $\tilde{A}v_i + \tilde{a} \in \mathcal{C}(v_i)$ for all $i \in \{0, \ldots, n\}$. If conditions (3) are not solvable, then the RCP will not be solvable at all, as discussed in Section II-A. Hence, by abusing notation and removing the tilde's from $\tilde{A}$ and $\tilde{a}$, we can without loss of generality make the following assumption:

Assumption 12: For all $i \in \{0, \ldots, n\}$, $Av_i + a \in \mathcal{C}(v_i)$. With this assumption, by Lemma 5, M is a Z-matrix.

The following theorem is an immediate consequence of previous work, Proposition 7 and Proposition 8:

Theorem 13: Assume that $\mathcal{S}$ is the standard orthogonal simplex, and suppose Assumption 12 holds. Additionally, assume without loss of generality that vertices $v_1, \ldots, v_n$ are relabeled so that $M = -(A + a\mathbb{1}_n^T)$ is in Frobenius normal form. Let S and T be defined as in (6) with respect to matrix M.

Then, $\mathcal{E}_{\mathcal{S}} = \emptyset$ if and only if both following statements hold:

(i) $S \subset \text{above}(T)$,

(ii) $\text{supp}(a) \cap \text{above}(T) \neq \emptyset$.

Proof: We note that, by Lemma 5, M is a Z-matrix and $a \geq 0$. Also, $\mathcal{E}_{\mathcal{S}} = \emptyset$ if and only if $\mathcal{E}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$ and $\mathcal{E}_{\mathcal{S}} \backslash \mathcal{F}_0 = \emptyset$.

First consider $\mathcal{E}_{\mathcal{S}} \cap \mathcal{F}_0$. By Proposition 9, $\mathcal{E}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$ if and only if there does not exist $y > 0$ such that $My = 0$. By Proposition 7, this is equivalent to $S \backslash \text{above}(T) = \emptyset$, i.e., $S \subset \text{above}(T)$. This gives (i).

Next consider $\mathcal{E}_{\mathcal{S}} \backslash \mathcal{F}_0$. By Proposition 10, $\mathcal{E}_{\mathcal{S}} \backslash \mathcal{F}_0 = \emptyset$ if and only if there does not exist $y \geq 0$ such that $My = a$. By Proposition 8, this is equivalent to $\text{supp}(a) \cap \text{above}(S \cup T) \neq \emptyset$. However, using (i), if $S \subset \text{above}(T)$, then $\text{above}(S \cup T) = \text{above}(S) \cup \text{above}(T) = \text{above}(T)$. This gives (ii). ∎

Theorem 13 can also be rephrased in terms of solvability of the RCP, as follows:

Corollary 14: Assume that $\mathcal{S}$ is the standard orthogonal simplex. Then, the RCP is solvable by affine feedback if and only if there exist $K \in \mathbb{R}^{m \times n}$ and $g \in \mathbb{R}^n$ such that:

(i) $\tilde{M} = -(A + BK) - (a + Bg)\mathbb{1}_n^T$ is a Z-matrix,

(ii) $\tilde{a} = a + Bg \geq 0$,

(iii) the conditions of Theorem 13 are satisfied, with $\tilde{M}$ instead of M, and $\mathcal{E}_{\mathcal{S}}$ being defined with respect to $\tilde{A}$ and $\tilde{a}$.

Proof: Let $\tilde{A} = A + BK$ and $\tilde{a} = a + Bg$. Then, the RCP is solved for system (1) by affine feedback $u = Kx+g$ if and only if it is solvable for (7), which is a system without control. By Lemma 5, the invariance conditions (3) for that system are equivalent to (i) and (ii). Lack of equilibria in system (7) is characterized by the conditions of Theorem 13, i.e., by (iii). Thus, system (7) will satisfy the invariance conditions and will not contain any equilibria in $\mathcal{S}$ if and only if conditions (i)-(iii) hold. By Theorem 4, this is equivalent to the solvability of the RCP for system (7). ∎

V. Necessary and Sufficient Conditions for Special Triangulation Strategies

In the remainder of this paper, we give some results on solvability of the RCP using affine feedback for two particular cases of the geometric structure of $\mathcal{O}_{\mathcal{S}}$. As described in Section IV, since the triangulation strategy in reach control theory is left to the designer, it can be performed in a way which ensures that $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$, or, alternatively, that $\mathcal{O}_{\mathcal{S}} \subset \mathcal{F}_0$. The former case was investigated in, e.g., [13], while the latter was explored in [6], [8]. We cover these cases in the remainder of this paper.

A. First Triangulation Strategy

In this section, we assume $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$. Here we give yet another set of necessary and sufficient conditions for the set $\mathcal{E}_{\mathcal{S}}$ to be empty.

The following result is a consequence of Proposition 8 and Proposition 10.

Proposition 15: Assume that $\mathcal{S}$ is the standard orthogonal simplex with $\mathcal{O}_{\mathcal{S}} \cap \mathcal{F}_0 = \emptyset$. Additionally, suppose that Assumption 1 holds and that $a \gg 0$. Then, $\mathcal{E}_{\mathcal{S}} \neq \emptyset$ if and only if M is a nonsingular M-matrix.

Proof: Since $a \gg 0$, by Proposition 8 and Proposition 10 we know that $\mathcal{E}_{\mathcal{S}} \neq \emptyset$ if and only if above$(S \cup T) = \emptyset$. By (5), this is equivalent to $S \cup T = \emptyset$. Now, first assume that $S \cup T = \emptyset$. Consider any $i \in \{1, \ldots, p\}$. Since $i \notin S \cup T$, matrix M_{ii} satisfies $l(M_{ii}) > 0$ by (6). We recount that M is a block triangular matrix with blocks M_{ii}, $i \in \{1, \ldots, p\}$, on the diagonal. Thus, the eigenvalues of M are the union of eigenvalues of all M_{ii}. Hence, $l(M) > 0$, i.e., M is a non-singular M-matrix.

Conversely, if M is a non-singular M-matrix, $l(M) > 0$. Thus, since M is a block triangular matrix, $l(M_{ii}) > 0$ for all $i \in \{1, \ldots, p\}$. Thus, by (6), $S \cup T = \emptyset$, which means that $\mathcal{E}_{\mathcal{S}} \neq \emptyset$. ■

Remark 16: Proposition 15 can also be proved more directly from Proposition 10, by invoking properties (I_{28}) and (N_{39}) of non-singular M-matrices from [4]. However, we chose to present a proof which uses Proposition 8 as it once again illustrates the graph-theoretic nature of M-matrices.

B. Second Triangulation Strategy

In this section, we assume $\mathcal{O}_{\mathcal{S}} \subset \mathcal{F}_0$. The primary contribution of this section is an easily checkable strong sufficient condition for solvability of the RCP. Let M be given by (4). Furthermore, let $M + \mathcal{B} \subset \mathbb{R}^{n \times n}$ denote the set of matrices $M + X$ where all the columns of X are in $\mathcal{B}$. Equivalently, $M + \mathcal{B} = \{M + BK \mid K \in \mathbb{R}^{m \times n}\}$.

Theorem 17: Assume that $\mathcal{S}$ is the standard orthogonal simplex, $a > 0$, and $\mathcal{O}_{\mathcal{S}} \subset \mathcal{F}_0$. Let $\mathcal{D} = \mathcal{Z}_n \cap (M + \mathcal{B})$. Suppose that there exists a non-singular matrix $D \in \mathcal{D}$. Then, the RCP is solvable by affine feedback.

Proof: Let $D \in \mathcal{D}$. Then, $D = M - BK$ for some matrix $K \in \mathbb{R}^{m \times n}$. We observe system (1), and claim that the closed-loop feedback $u(x) = Kx + g$ with $g = 0$ solves the RCP. We will do that by showing that the conditions of Theorem 11 are satisfied.

First, we note that $-((A + BK) + (a + Bg)\mathbb{1}_n^T) = -(A + a\mathbb{1}_n^T) - BK = M - BK = D \in \mathcal{Z}_n$. Thus, condition (i) of Theorem 11 is satisfied. Since $a + Bg = a > 0$, condition (ii) is satisfied as well. Since D is non-singular, condition (iii.a) is satisfied. Finally, let us consider condition (iii.b). By Proposition 10 and the discussion in the proof of Theorem 11, this condition is equivalent to (1) not containing an equilibrium in $\mathcal{S} \backslash \mathcal{F}_0$. However, all possible equilibria of (1) are contained in $\mathcal{O}_{\mathcal{S}}$ [22], and $\mathcal{O}_{\mathcal{S}} \subset \mathcal{F}_0$. Hence, condition (iii.b) is automatically verified. Thus, $u(x)$ indeed solves the RCP. ■

We note that the set $\mathcal{D}$ from Theorem 17 is either empty or a polyhedron, as $\mathcal{Z}_n$ is defined by linear inequalities, and $M + \mathcal{B} \subset \mathbb{R}^{n \times n}$ is an affine space. Hence, it is easy to compute $\mathcal{D}$. Verifying that there exists a non-singular matrix in a polyhedron is computationally easy as well. This is shown by the following two corollaries.

Corollary 18: Let $\mathcal{D} \neq \emptyset$. Then, there exists a non-singular matrix $D \in \mathcal{D}$ if and only if there exists a non-singular matrix $D' \in \text{aff}(\mathcal{D})$.

Proof: One direction is obvious. In the other direction, let $D' \in \text{aff}(\mathcal{D})$ be non-singular, and assume $\det(D) = 0$ for all $D \in \mathcal{D}$. Take any $D_0 \in \text{int}(\mathcal{D})$. Then, $\text{aff}(\mathcal{D}) = \{D_0 + \alpha_1 D_1 + \ldots + \alpha_k D_k \mid \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$, where $D_1, \ldots, D_k$ are some matrices, not necessarily in $\mathcal{D}$. Now, $p(\alpha_1, \ldots, \alpha_k) = \det(D_0 + \alpha_1 D_1 + \ldots + \alpha_k D_k)$ is a multivariate polynomial in $\mathbb{R}[\alpha_1, \ldots, \alpha_k]$, and because $\det(D') \neq 0$, it is not always 0. However, $p(\alpha_1, \ldots, \alpha_k)$ is always zero for some small ball $\mathbb{B}^k$ around $(0, 0, \ldots, 0)$, because $D_0 \in \text{int}(\mathcal{D})$, so $D_0 + \alpha_1 D_1 + \ldots + \alpha_k D_k \subset \mathcal{D}$ for small α_i. Thus, the Taylor expansion of p around $(0, 0, \ldots, 0)$ is zero. Since the Taylor expansion of p is p itself, all coefficients of p are 0. Hence, $\det(D') = 0$, which is a contradiction. ■

Corollary 19: Assume that $\mathcal{S}$ is the standard orthogonal simplex, $a > 0$, and $\mathcal{O}_{\mathcal{S}} \subset \mathcal{F}_0$. Let $D_0 \in \mathcal{D}$, and let $D_1, \ldots, D_k \in \mathbb{R}^{n \times n}$ be the basis elements of the vector space $\{D' - D_0 \mid D' \in \text{aff}(\mathcal{D})\}$. Then, if $\det(D_0 + \alpha_1 D_1 + \ldots + \alpha_k D_k)$ is not a zero polynomial in $\alpha_1, \ldots, \alpha_k$, the RCP is solvable.

Proof: Assume that $\det(D_0 + \alpha_1 D_1 + \ldots + \alpha_k D_k)$ is not a zero polynomial. Then, there exist $\alpha'_1, \ldots, \alpha'_k \in \mathbb{R}^n$ such that $\det(D_0 + \alpha'_1 D_1 + \ldots + \alpha'_k D_k) \neq 0$. Let $D' = D_0 + \alpha'_1 D_1 + \ldots + \alpha'_k D_k \in \text{aff}(\mathcal{D})$. Then, $\det(D') \neq 0$. By Corollary 18, there exists $D \in \mathcal{D}$ such that $\det(D) \neq 0$. By Theorem 17, the RCP is solvable. ■

Corollary 19 shows that the sufficient condition for the solvability of the RCP in Theorem 17 can be verified by just checking whether all coefficients of a certain easily computable polynomial are 0. Additionally, this is a strong condition, in the following sense: assume that the RCP is solvable by affine feedback. Let $u(x) = Kx + g$ be the affine feedback that solves the RCP. Then, by Theorem 11, $-((A + BK) + (a + Bg)\mathbb{1}_n^T) = M - B(K + g\mathbb{1}_n^T) \in \mathcal{Z}_n$. Additionally, $M - B(K + g\mathbb{1}_n^T) \in M + \mathcal{B}$. Hence, $\mathcal{D} = \mathcal{Z}_n \cap (M + \mathcal{B}) \neq \emptyset$. Then, either the condition of Theorem 17 is satisfied, or the entire polyhedron $\mathcal{D}$ consists solely of singular matrices.

Remark 20: This paper exposes a previously unexplored correspondence between reach control theory and the theory of positive systems. Nevertheless, some mathematical constructs that are ubiquitous in positive systems have also been previously applied in the RCP. Most notably, Z-matrices and M-matrices were used in [2], [6], [8] to develop the reach control indices, and syntheses of time-varying affine and piecewise affine feedbacks.

While it is outside the scope of this paper to delve into the machinery of the reach control indices, we remark that the methods introduced in this paper shed new light on the indices. It can be shown (we defer all details to a follow-up paper) that, supposing that Assumption 12 holds, and under the additional assumptions used by the theory of reach control indices, the Frobenius form of M is given in block-form by

$$\left[\begin{array}{c|c} M_0 & 0 \\ \hline * & M_1 \end{array}\right]. \tag{9}$$

In (9), M_0 is a lower triangular matrix in the Frobenius normal form, and M_1 is block-diagonal, with blocks M_{kk}, $k = 1, \ldots, p$. Additionally, each M_{kk} is an $r_k \times r_k$ matrix, where $\{r_1, \ldots, r_p\}$ are the reach control indices, and, moreover, each M_{kk} is a singular, irreducible M-matrix.

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Appendix

Proof of Lemma 3: We note that p maps each vertex of Δ into a corresponding vertex of $\mathcal{S}$. It also maps Δ into $\mathcal{S}$ and the exit facet $\mathcal{F}_0^\Delta$ of Δ, into the exit facet $\mathcal{F}_0^\mathcal{S}$ of $\mathcal{S}$.

Let u be any feedback in $\mathfrak{F}$, and let $\phi_u(\cdot, x_0)$ be the trajectory generated by system (1), with the initial condition $\phi_u(0, x_0) = x_0$. Consider the feedback $u' := u \circ p$. We first note that $u' \in \mathfrak{F}$ by the definition of affine-invariance. Now, consider the system $\dot{y} = K^{-1}AKy + K^{-1}Bu'(y) + K^{-1}a + K^{-1}Ag$. Let $\phi'_{u'}(\cdot, y_0)$ be the trajectory generated by this system, with $\phi'_{u'}(0, y_0) = y_0$. We claim that

$$p \circ \phi'_{u'}(\cdot, y_0) = \phi_u(\cdot, p(y_0)). \tag{10}$$

This is easily shown: $d(p \circ \phi'_{u'}(t, y_0))/dt = Kd(\phi'_{u'}(t, y_0))/dt = AK\phi'_{u'}(t, y_0) + Bu'(\phi'_{u'}(t, y_0)) + a + Ag = A(p \circ \phi'_{u'}(t, y_0)) + Bu(p \circ \phi'_{u'}(t, y_0)) + a$. Thus, $p \circ \phi'_{u'}(\cdot, y_0)$ satisfies (1), and $p \circ \phi'_{u'}(0, y_0) = p(y_0)$. Hence, $p \circ \phi_{u'}(\cdot, y_0) = \phi_u(\cdot, p(y_0))$.

Now, assume that the RCP is solvable for $(A, B, a, \mathcal{S})$ by $\mathfrak{F}$. Let $u \in \mathfrak{F}$ be the feedback that solves this RCP. We claim that $u' = u \circ p$ solves the RCP for $(K^{-1}AK, K^{-1}B, K^{-1}a + K^{-1}Ag, \Delta)$. Let $y_0 \in \Delta$. We verify conditions (i)-(iii) from the definition of the RCP:

(i) Since p maps Δ to $\mathcal{S}$, $\phi'_{u'}(t, y_0) \in \Delta$ for all $t \in [0, T]$ is equivalent to $p \circ \phi'_{u'}(t, y_0) \in \mathcal{S}$ for all $t \in [0, T]$. By (10), this is equivalent to $\phi_u(t, p(y_0)) \in \mathcal{S}$ for all $t \in [0, T]$, with $p(y_0) \in \mathcal{S}$. This is the condition (i) applied to the RCP on $(A, B, a, \mathcal{S})$.

(ii) Since p maps $\mathcal{F}_0^\Delta$ to $\mathcal{F}_0^\mathcal{S}$, $\phi'_{u'}(T, y_0) \in \mathcal{F}_0^\Delta$ if and only if $p \circ \phi'_{u'}(T, y_0) \in \mathcal{F}_0^\mathcal{S}$. This holds by (10) and condition (ii) applied to the RCP on $(A, B, a, \mathcal{S})$.

(iii) Since p maps $\mathbb{R}^n \backslash \Delta$ to $\mathbb{R}^n \backslash \mathcal{S}$, $\phi'_{u'}(t, y_0) \notin \Delta$ is equivalent to $\phi_u(t, p(y_0)) \notin \mathcal{S}$, by the same discussion as in (i). Thus, (iii) follows from the condition (iii) in the RCP applied to $(A, B, a, \mathcal{S})$.

Hence, the RCP is solvable for $(K^{-1}AK, K^{-1}B, K^{-1}a + K^{-1}Ag, \Delta)$ by $\mathfrak{F}$. The other direction is entirely analogous. ■