

# A Nerve-theoretic Result on the Problem of a Topological Obstruction in Reach Control

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**Abstract**— This paper deals with a necessary condition for the solvability of the Reach Control Problem (RCP) by continuous state feedback. The RCP seeks to find a continuous state feedback which drives a control system defined on a simplex to leave the simplex through a predetermined facet. The problem of a topological obstruction has been previously identified as a strong necessary condition for the solvability of the RCP, and finding a characterization of a topological obstruction has been an ongoing effort. Using the methodology of homotopy theory and nerve theory, this paper provides a new necessary condition for the existence of a topological obstruction. This condition removes the gap that exists between currently available necessary conditions and current sufficient conditions, and hence results in a sufficient and necessary condition for the existence of a topological obstruction in the RCP.

## I. INTRODUCTION

The Reach Control Problem (RCP) studies the behaviour of an affine control system on a simplex. It seeks, for a given system, to design a control feedback law such that all system trajectories starting from an initial point in the simplex leave the simplex through a predetermined *exit facet*. The interest in the RCP is primarily driven by reach control theory, which is an approach, inspired by studies of reachability (see, e.g., [20]), to meet complex control objectives on a constrained state space. Reach control theory and the RCP have been the subject of significant theoretical and practical research in the past decade (see, e.g., [3], [7], [9], [19], [21]).

One of the fundamental difficulties of reach control theory is in determining whether the RCP, for a given system and on a given simplex, is solvable. Recent research effort has been devoted to finding suitable classes of feedback to solve the RCP. For affine control laws, a set of necessary and sufficient conditions for the solvability of the RCP has been identified [9], [19]. These conditions are of limited use in practice. For a given candidate feedback, they can be utilized to verify whether that particular feedback solves the RCP. However, they cannot be easily used to determine whether there *exists* an affine feedback which solves the RCP. In the case of continuous state feedback, analogous to the problem of continuous feedback stabilization [6], the existence of a *topological obstruction* has been identified as a fundamental obstacle to the solvability of the RCP. The goal of this line of research is to obtain an elegant and computable characterization for the existence of such an obstruction, thus providing a practically useful necessary condition for the solvability of the RCP.

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The topological obstruction problem has been examined in a number of recent papers: [8], [15], [16], [17]. A similar problem of an affine obstruction in the RCP has also been examined in [11], but that problem deals with affine feedbacks. The methods and results of [8], [15], [16] are significantly different from this work. In particular, [8] deals with a very specific case where the possible set of equilibria of the control system has a special geometric structure. The assumptions of this paper are far more relaxed. Papers [15], [16] limit the dimensions of the state space and the number of control inputs, respectively. This paper contains no such limits.

This paper is a direct successor of [17]. That paper established a strong sufficient condition and a strong necessary condition for the existence of a topological obstruction in the RCP, with limited assumptions on the geometric structure and dimensions of the system. In [17], there a minor gap between the presented necessary condition and the presented sufficient condition. In particular, [17] shows that there is a topological obstruction in the system if a certain union of open convex cones in  $\mathbb{R}^m$  equals the entire  $\mathbb{R}^m$ . On the other hand, it shows that if a union of *larger*, closed, cones does not equal all of  $\mathbb{R}^m$ , then there is no topological obstruction. In other words, in a degenerate case where the smaller union of open cones does not equal  $\mathbb{R}^m$ , but the larger union of closed cones does equal  $\mathbb{R}^m$ , the results of [17] could not be used to determine whether there exists a topological obstruction for a given control system.

This paper has two contributions. First of all, it closes the described gap from [17]. In particular, we will show that there exists a topological obstruction in the system *if and only if* the above union of larger closed convex cones equals  $\mathbb{R}^m$ . The second contribution is in the method used to close this gap. This paper invokes the machinery of nerve theory, which was not used in any of the previous papers involving a topological obstruction in the RCP. Moreover, to the authors' knowledge, it has never been used in control theory before.

*Notation:* If  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a map, and  $\mathcal{A} \subset \mathcal{X}$ , then  $f|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{Y}$  denotes the restriction of  $f$  to  $\mathcal{A}$ .  $id_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X}$  denotes an identity map on  $\mathcal{X}$ .  $\mathbb{B}^n$  denotes an  $n$ -dimensional unit ball,  $\mathbb{S}^n$  denotes an  $n$ -dimensional unit sphere, and  $\Delta^n$  denotes an  $n$ -dimensional simplex.  $\text{co}\{v_1, \dots, v_k\}$  denotes the convex hull of points  $v_1, \dots, v_k$ . If  $\mathcal{X}$  is a set, then  $\text{int}(\mathcal{X})$  denotes its (relative) interior, and  $\partial\mathcal{X}$  its (relative) boundary.

## II. MATHEMATICAL BACKGROUND

Note that the results and definitions of Section II-A are also contained in a number of previous works on the topological obstruction in the RCP, most notably [17]. However, we repeat them here to introduce the reader to the topological machinery used in the paper. The definitions and results of Section II-B have, however, not appeared in previous related papers, as they serve as foundation for the use of nerve theory.

### A. Homotopy Theory and Topology of Spaces

Continuous maps  $f, g : \mathcal{X} \rightarrow \mathcal{Y}$  are *homotopic* if there exists a continuous function  $F : \mathcal{X} \times [0, 1] \rightarrow \mathcal{Y}$  such that  $F(\cdot, 0) \equiv f$ ,  $F(\cdot, 1) \equiv g$ . A continuous map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is *null-homotopic* if it is homotopic to a constant map  $c(x) = y_0$ ,  $y_0 \in \mathcal{Y}$ .

Topological spaces  $\mathcal{X}$  and  $\mathcal{Y}$  are *homotopy equivalent*, denoted by  $\mathcal{X} \simeq \mathcal{Y}$ , if there exist two continuous maps  $f : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $g : \mathcal{Y} \rightarrow \mathcal{X}$  such that  $f \circ g$  and  $g \circ f$  are homotopic to  $id_{\mathcal{Y}}$  and  $id_{\mathcal{X}}$ , respectively. A topological space  $\mathcal{X}$  is *contractible* if the identity map  $id : \mathcal{X} \rightarrow \mathcal{X}$  is null-homotopic. A topological space  $\mathcal{X}$  is *locally contractible* if for every  $x \in \mathcal{X}$  and every open subset  $x \in \mathcal{V}_\varepsilon \subset \mathcal{X}$ , there exists an open subset  $x \in \mathcal{V}_\delta \subset \mathcal{V}_\varepsilon$  which is contractible in the subspace topology inherited from  $\mathcal{X}$ .

Topological spaces  $\mathcal{X}$  and  $\mathcal{Y}$  are *homeomorphic*, denoted by  $\mathcal{X} \cong \mathcal{Y}$ , if there is a continuous map  $f : \mathcal{X} \rightarrow \mathcal{Y}$  which is bijective and has a continuous inverse. This  $f$  is called a *homeomorphism*. Note that homeomorphic spaces are trivially homotopy equivalent, as  $f \circ f^{-1} = id_{\mathcal{Y}}$  and  $f^{-1} \circ f = id_{\mathcal{X}}$ . Informally, for the purposes of topology, homeomorphic spaces are considered to be essentially the same.

Finally, the following definition is given in [1]. We note that it substitutes the usual definition of absolute extensors by absolute retracts. However, these notions are in the case of metrizable spaces equivalent (see [18]). The same definition is also used in [17].

*Definition 1:* A metrizable space  $\mathcal{X}$  is an *absolute retract* (AR) if for every metrizable space  $\mathcal{Y}$  and every closed set  $\mathcal{A} \subset \mathcal{Y}$ , each continuous map  $f : \mathcal{A} \rightarrow \mathcal{X}$  is extendable to a continuous map  $F : \mathcal{Y} \rightarrow \mathcal{X}$ .

*Proposition 2* ([5], [1]): If  $\mathcal{X}$  is (homeomorphic to) a compact, contractible and locally contractible finite-dimensional metric space, it is AR.

### B. Simplicial Complexes

Let  $J$  be a finite index set.  $\mathcal{D} \subset 2^J \setminus \{\emptyset\}$  is a (finite) *abstract simplicial complex* if

$$(L \subset K \wedge K \in \mathcal{D}) \Rightarrow L \in \mathcal{D}, \quad K, L \in 2^J \setminus \{\emptyset\}. \quad (1)$$

Geometrically, an abstract simplicial complex  $\mathcal{D}$  can be realized with the singletons in  $J$  playing the role of vertices, and with  $k$ -subsets of  $J$  being the  $k$ -dimensional faces of  $\mathcal{D}$ . It is well-known (see, e.g., Remark 1.3.4 in [14]) that we can indeed represent  $\mathcal{D}$  as a usual geometric simplicial complex instead of an abstract one, without any loss of functionality.

Let  $\mathcal{D}$  be a finite simplicial complex with vertices denoted by  $1, 2, \dots, l$ . The (*first*) *barycentric subdivision* of  $\mathcal{D}$  divides each  $k$ -dimensional simplex  $\mathcal{P} \subset \mathcal{D}$  into  $(k+1)!$  smaller simplices,  $k = 1, \dots, l-1$ . Each of these smaller simplices consists of one vertex  $x_0 \in \{1, 2, \dots, l\}$  of the original simplex, one vertex in the middle of some edge  $\text{co}\{x_0, x_1\}$ , one vertex in the middle of some two-dimensional simplex  $\text{co}\{x_0, x_1, x_2\}$ , etc. The *barycentric center* of  $\mathcal{P}$  is the vertex of its barycentric subdivision contained in the interior of  $\mathcal{P}$ . The *barycentric star*  $\text{bst}(j)$  in  $\mathcal{D}$  consists of all simplices in the first barycentric subdivision of  $\mathcal{D}$  which contain  $\{j\}$ . See Fig. 1 for an example of one simplicial complex and its first barycentric subdivision.

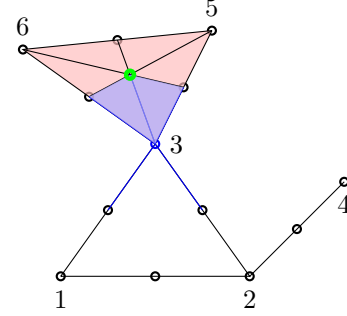


Fig. 1: The barycentric subdivision of the simplicial complex  $\mathcal{D} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{3, 6\}, \{5, 6\}, \{3, 5, 6\}\}$ . The barycentric star of vertex 3 is marked in blue, and the barycentric center of the two-dimensional simplex with vertices 3, 5, 6 is marked in green.

Finally, we give a technical lemma that will be crucial in our main result.

*Lemma 3* (Lemma 4.3, [2]): Let  $\Delta$  be a simplex, and let  $f : \Delta \rightarrow \Delta$  be a continuous map such that  $f(\mathcal{P}) \subset \mathcal{P}$  holds for each face  $\mathcal{P} \subset \Delta$ . Then,  $f$  is surjective.

*Corollary 4:* Let  $\Delta$  be a simplex. There does not exist a continuous map  $f : \Delta \rightarrow \partial\Delta$  such  $f(\mathcal{P}) \subset \mathcal{P}$  holds for each face  $\mathcal{P} \subset \Delta$ .

*Proof:* Assume otherwise. Then, by taking the codomain of  $f$  to be  $\Delta$  instead of  $\partial\Delta$ , such a map  $f$  satisfies the conditions of Lemma 3 with  $\text{Im}(f) \subset \partial\Delta$ . However, by Lemma 3,  $\text{Im}(f) = \Delta$ . ■

## III. A SHORT TUTORIAL ON NERVE THEORY

The notion of a *set cover* is a central notion of topology. To remind the reader, a cover of a set  $\mathcal{X}$  is a set  $\{\mathcal{X}_j \mid j \in J\}$  such that  $\cup_{j \in J} \mathcal{X}_j = \mathcal{X}$ . If all the sets  $\mathcal{X}_j$  are open (in a topology on  $\mathcal{X}$ ),  $\{\mathcal{X}_j \mid j \in J\}$  is an *open cover*. If all  $\mathcal{X}_j$  are closed,  $\{\mathcal{X}_j \mid j \in J\}$  is a *closed cover*. If the index set  $J$  is finite, the corresponding cover is a *finite cover*.

The properties of a set cover reveal a great deal of information about the underlying space: for more details, see [12]. This motivates the need to examine the structure of the intersections of elements of a set cover. Nerve theory seeks to provide a comprehensive methodology for doing so.

A nerve of a cover of a space is just a list of all subsets of  $\{\mathcal{X}_j \mid j \in J\}$  whose elements all have a nonempty intersection. Formally, let  $J$  be an finite index set, and let  $\{\mathcal{X}_j \mid j \in J\}$  be a family of subsets of a topological space  $\mathcal{X}$ . The *nerve*  $\mathcal{N}$  of  $\{\mathcal{X}_j \mid j \in J\}$  with respect to  $\mathcal{X}$  is the set of non-empty subsets  $K$  of  $J$  given by

$$K \in \mathcal{N} \Leftrightarrow \bigcap_{k \in K} \mathcal{X}_k \neq \emptyset. \quad (2)$$

While other related objects can be described and investigated within the purview of nerve theory (e.g., the *nerve graph* introduced and used in [10]), due to the confines of the format of this paper, we omit such a discussion. We refer the reader to [10], [13], and other references regarding nerve theory contained at the end of this paper.

The structure of a nerve has remarkable geometric properties. First of all, by (1),  $\mathcal{N}$  is an abstract simplicial complex. Thus,  $\mathcal{N}$  has a natural geometric interpretation: see Fig. 2 for an example.

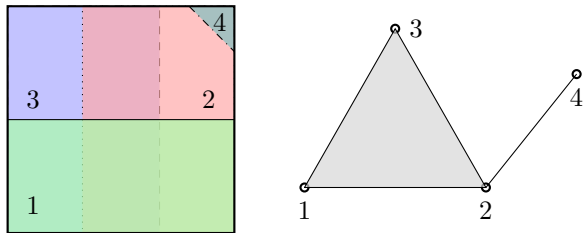


Fig. 2: The figure on the left gives an example of a closed cover of a unit square  $\mathcal{X} = [0, 1] \times [0, 1]$ . The square is covered by four sets: one (green, denoted by 1) covering the bottom half, one (red, denoted by 2) covering the right two thirds of the square, another (blue, denoted by 3) covering the left two thirds of the square, and the fourth one (gray, denoted by 4) covering the upper right corner of the square. The figure on the right gives a geometric realization of the corresponding nerve  $\mathcal{N}$ .

A far more remarkable result concerning nerves is the nerve theorem. It states that, under certain conditions, a space and a nerve of its cover are homotopy equivalent. This is remarkable because a given space can clearly admit a variety of different covers, which are thus shown to share the same homotopic properties.

We note that nerve  $\mathcal{N}$  depends on  $\mathcal{X}$  and  $\{\mathcal{X}_j \mid j \in J\}$ . In the remaining sections of this text, however, we will not explicitly note this. This is for two reasons. First, we will be using nerve theory exclusively to investigate a nerve of a single cover on one particular space. Secondly, the nerve theorem exactly states that the homotopic properties of  $\mathcal{N}$  are preserved regardless of the underlying cover.

Although the statement of the nerve theory can be proved under varying conditions, the form given in [4] and [13] is most applicable to our work.

*Theorem 5 (Theorem 3.3, [13]):* Let  $\mathcal{N}$  be the nerve of a closed finite cover  $\{\mathcal{X}_j \mid j \in J\}$  with respect to  $\mathcal{X}$ . Assume

that the following holds:  $\bigcap_{j \in K} \mathcal{X}_j$  is AR for all  $\emptyset \neq K \subset J$ . Then,  $\mathcal{X} \simeq \mathcal{N}$ .

If a closed cover  $\{\mathcal{X}_j \mid j \in J\}$  satisfies this assumption of Theorem 5, we say that it is *regular*. We note that the nerve in Fig. 2 is indeed homotopy equivalent its corresponding space (unit square), confirming Theorem 5.

#### IV. REACH CONTROL PROBLEM

In the RCP, we consider an  $n$ -dimensional simplex  $\mathcal{S} \subset \mathbb{R}^n$ , with vertices  $v_0, \dots, v_n$  and facets  $\mathcal{F}_0, \dots, \mathcal{F}_n$ , where each facet is indexed by the vertex that it does not contain.

We consider the system

$$\dot{x} = Ax + Bu + a, \quad (3)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  with  $\text{rank}(B) = m$ ,  $a \in \mathbb{R}^n$ . Let  $\mathcal{B} = \text{Im}(B)$ .

Assume that  $u : \mathcal{S} \rightarrow \mathbb{R}^m$  is a continuous function. Let  $\phi_u(\cdot, x_0)$  be the solution of (3), with  $\phi_u(0, x_0) = x_0$ . The Reach Control Problem, given in its current version in [9], [19], is as follows:

*Problem 6 (Reach Control Problem (RCP)):* Given system (3) on  $\mathcal{S}$ , determine whether there exists a control  $u : \mathcal{S} \rightarrow \mathbb{R}^m$  such that for all  $x_0 \in \mathcal{S}$  there exist  $T \geq 0$  and  $\varepsilon > 0$  such that the following holds:

- (i)  $\phi_u(t, x_0) \in \mathcal{S}$  for all  $t \in [0, T]$ ,
- (ii)  $\phi_u(T, x_0) \in \mathcal{F}_0$ ,
- (iii)  $\phi_u(t, x_0) \notin \mathcal{S}$  for all  $t \in (T, T + \varepsilon)$ .

There are two trivial necessary conditions for the solvability of Problem 6 by continuous state feedback  $u$ . The first condition is that system (3) cannot contain any equilibria. Related to that condition, we note that any equilibrium point  $x$  would satisfy  $Ax + a = -Bu(x) \in \mathcal{B}$  and hence define  $\mathcal{O}_{\mathcal{S}} = \{x \in \mathcal{S} \mid Ax + a \in \mathcal{B}\}$ . The following two statements trivially hold:

- (E1) Any equilibrium of (3) necessarily lies in  $\mathcal{O}_{\mathcal{S}}$ .
- (E2) Velocity vectors  $Ax + Bu(x) + a$  are contained in  $\mathcal{B}$  for all  $x \in \mathcal{O}_{\mathcal{S}}$ .

The second necessary condition for the solvability of the RCP is that the velocity vectors  $Ax + Bu(x) + a$  cannot point outside of the simplex at facets other than  $\mathcal{F}_0$ . Suppose otherwise:  $d\phi(0, x_0)/dt = Ax_0 + Bu(x_0) + a$  points outside of  $\mathcal{S}$  for some  $x_0 \in \partial\mathcal{S} \setminus \mathcal{F}_0$ . Then, the trajectory  $\phi(\cdot, x_0)$  will leave simplex  $\mathcal{S}$  by exiting through a facet other than  $\mathcal{F}_0$ , breaking conditions (ii) and (iii) of Problem 6.

The rigorous statement of the second condition is as follows. We denote  $I = \{1, \dots, n\}$ . Let  $h_j$  be the outward pointing normal to the facet  $\mathcal{F}_j$  for  $j \in I$ . For each  $x \in \mathcal{S}$ , let  $I(x) \subset \{0, 1, \dots, n\}$  be the smallest set such that  $x \in \text{co}\{v_i \mid i \in I(x)\}$ . Inward-pointing cones  $\mathcal{C}(x)$ ,  $x \in \mathcal{S}$ , are defined by  $\mathcal{C}(x) = \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, j \in I \setminus I(x)\}$ , with the convention that  $\mathcal{C}(x) = \mathbb{R}^n$  if  $I \setminus I(x) = \emptyset$ .

The above necessary condition for the solvability of the RCP by continuous state feedback  $u$  can now be stated as  $Ax + Bu(x) + a \in \mathcal{C}(x)$  for all  $x \in \mathcal{S}$ . These are the so-called *invariance conditions*. For more details on these conditions, we refer the reader to [9], [19].

The problem of a topological obstruction [8], [15], [16], [17] seeks to combine the two above conditions into a strong and operable necessary condition for the solvability of the RCP by continuous state feedback. In particular, we first assume that the invariance conditions hold for all  $x \in \mathcal{O}_S$ . We know from (E1) that outside of  $\mathcal{O}_S$  there will not be any equilibria of (3), no matter which feedback control we use. Thus, we are interested in whether there exists a feedback control  $u$  on  $\mathcal{O}_S$  such that  $f(x) := Ax + Bu + a$  does not vanish on  $\mathcal{O}_S$ . Since  $f(x) \in \mathcal{B}$  for all  $x \in \mathcal{O}_S$  by (E2), this problem can be given as follows:

*Problem 7 (Topological Obstruction):* Does there exist a continuous function  $f : \mathcal{O}_S \rightarrow \mathcal{B} \setminus \{0\}$  such that  $f(x) \in \mathcal{C}(x)$  for all  $x \in \mathcal{O}_S$ ?

From the above discussion, an affirmative answer to Problem 7 is a necessary condition for the solvability of the RCP. Our goal is to find an elegant necessary and sufficient condition for an affirmative answer to Problem 7.

Problem 7 can easily be transformed into a form more amenable to topological investigation. This requires employing a linear transformation that maps  $\mathcal{B} \setminus \{0\}$  into  $\mathbb{R}^m \setminus \{0\}$  and noticing that  $f$  solves is a solution to Problem 7 if and only if  $\tilde{f}(x) = f(x)/\|f(x)\|$  is a solution to Problem 7; we refer the reader to [15], [17] for all details. The above transformations also require us to redefine the cone  $\mathcal{C}(x)$ . We finally obtain the following problem, which is equivalent to the original Problem 7.

*Problem 8:* Let

$$\mathcal{C}(x) = \{y \in \mathbb{S}^{m-1} \mid h_j \cdot y \leq 0, j \in I \setminus I(x)\}. \quad (4)$$

Does there exist a continuous map  $f : \mathcal{O}_S \rightarrow \mathbb{S}^{m-1}$  which satisfies  $f(x) \in \mathcal{C}(x)$  for all  $x \in \mathcal{O}_S$ ?

In line with the exposition in [17], we also define cones

$$\mathcal{C}_j = \{y \in \mathbb{S}^{m-1} \mid h_j \cdot y \leq 0\}, \quad j \in I. \quad (5)$$

From (4) and (5), we have  $\mathcal{C}(x) = \bigcap_{j \notin I(x)} \mathcal{C}_j$  for all  $x \in \mathcal{O}_S$ .

In the remainder of the text, we make the following assumptions, which are the same as the assumptions in [17]:

*Assumption 9:*

- (A1) The pair  $(A, B)$  is controllable,
- (A2)  $2 \leq m \leq n - 1$ ,
- (A3) For any index set  $\emptyset \neq I' \subset I$ , if  $\mathcal{Y} = \bigcap_{j \in I'} \mathcal{C}_j \neq \emptyset$ , then  $\mathcal{Y} \cong \mathbb{B}^\rho$  for some  $\rho \in \{0, \dots, m - 1\}$ ,
- (A4)  $\mathcal{O}_S = \text{co}\{o_1, \dots, o_{\kappa+1}\}$  is a  $\kappa$ -dimensional simplex with vertices  $o_1, \dots, o_{\kappa+1}$ ,
- (A5)  $v_0 \notin \mathcal{O}_S$ ,
- (A6)  $\mathcal{O}_S \cap \text{int}(\mathcal{S}) \neq \emptyset$ ,
- (A7)  $\mathcal{C}(o_i) \neq \emptyset$  for all  $i \in \{1, \dots, \kappa + 1\}$ .

In the interest of conserving space, we refer the reader to [17] for a discussion on the meaning of the stipulations contained in Assumption 9. Although some of the stipulations can be relaxed for the results that we are presenting here, this is not the focus of this paper, and in the interest of clarity we choose to remain by the assumptions of [17].

The following result can be easily derived from Assumption 9:

*Lemma 10 (Lemma 3(i), [17]):* If the pair  $(A, B)$  is controllable and  $\mathcal{O}_S \cap \text{int}(\mathcal{S}) \neq \emptyset$ , then  $\dim(\mathcal{O}_S) = m$ .

Finally, we introduce some additional notation from [17]. Let  $\mathcal{F}_1^\mathcal{O}, \dots, \mathcal{F}_{\kappa+1}^\mathcal{O}$  be the facets of  $\mathcal{O}_S$ , again denoted by the vertex  $o_j$  they do not contain. It is easy to verify that for any facet  $\mathcal{F}_j^\mathcal{O}$  and any  $x \in \text{int}(\mathcal{F}_j^\mathcal{O})$ ,

$$I(x) = \bigcup_{\substack{1 \leq i \leq \kappa+1 \\ i \neq j}} I(o_i).$$

Thus, by (4),  $\mathcal{C}(x)$  is the same for all  $x \in \text{int}(\mathcal{F}_j^\mathcal{O})$ . We denote

$$\mathcal{H}_j = \mathcal{C}(x), \quad x \in \text{int}(\mathcal{F}_j^\mathcal{O}). \quad (6)$$

The following lemma was proved in [17].

*Lemma 11 (Lemma 6, [17]):* At most one of the sets  $\mathcal{H}_1, \dots, \mathcal{H}_{\kappa+1}$  equals  $\mathbb{S}^{m-1}$ .

Finally, we introduce the notation

$$\mathcal{H} = \bigcup_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \mathcal{H}_j, \quad \mathcal{H}^* = \bigcup_{\mathcal{H}_j \neq \mathbb{S}^{m-1}} \text{int}(\mathcal{H}_j).$$

## V. INFORMAL RESULT OUTLINE

The main result of [17] was the following:

*Theorem 12 ([17]):* If  $\mathcal{H} \neq \mathbb{S}^{m-1}$ , Problem 8 has an affirmative answer. On the other hand, if  $\mathcal{H}^* = \mathbb{S}^{m-1}$ , Problem 8 has a negative answer.

As discussed before, this leaves a gap between the two stated conditions. If  $\mathcal{H}^* \subsetneq \mathbb{S}^{m-1} = \mathcal{H}$ , Theorem 12 does not answer Problem 8. We will now amend that gap and show that the condition  $\mathcal{H}^* = \mathbb{S}^{m-1}$  above can be replaced by  $\mathcal{H} = \mathbb{S}^{m-1}$ . Our paper does not directly focus on the condition  $\mathcal{H} \neq \mathbb{S}^{m-1}$  for an affirmative answer. For more details on this condition, we invite the reader to see [17]. We provide an example outlining our method of proof that  $\mathcal{H} = \mathbb{S}^{m-1}$  implies that Problem 8 has a negative answer. This example, along with Fig. 3, is modified from [17].

*Example 13:* Suppose  $n = 3$ ,  $m = 2$ , and  $\mathcal{O}_S = \text{co}\{o_1, o_2, o_3\}$ , where  $o_i$  lies in the interior of the segment  $\overline{v_0 v_i}$  for all  $i \in \{1, 2, 3\}$ . This is illustrated on the left side of Fig. 3. For any  $x$  in the interior of  $\overline{o_j o_k}$ , we have  $I(x) = \{0, j, k\}$ . Hence, by (5) and (6),  $\mathcal{H}_i = \mathcal{C}_i$  for all  $i \in \{1, 2, 3\}$ . Assume that cones  $\mathcal{H}_i$  are given as on the right side of Fig. 3.

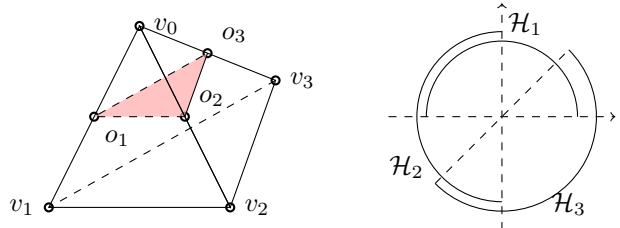


Fig. 3: Configuration of the set  $\mathcal{O}_S$  and cones  $\mathcal{H}_i$  from Example 13.

By looking at the right side of Fig. 3, we notice that we can split  $\mathbb{S}^1$  into six (non-disjoint) arcs:  $\mathcal{H}_1$ ,  $\mathcal{H}_1 \cap \mathcal{H}_2$ ,  $\mathcal{H}_2$ ,  $\mathcal{H}_2 \cap \mathcal{H}_3$ ,  $\mathcal{H}_3$ , and  $\mathcal{H}_3 \cap \mathcal{H}_1$ . This is shown in Fig. 4.

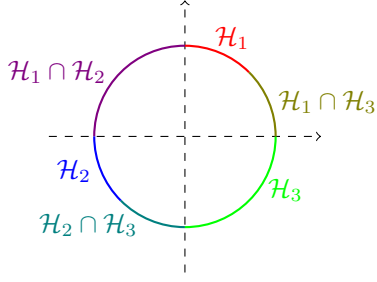


Fig. 4: Different areas of  $\mathbb{S}^1$  from Example 13. Set  $\mathcal{H}_1$  is denoted by red,  $\mathcal{H}_2$  by blue, and  $\mathcal{H}_3$  by green. The intersections of the sets are denoted by a mixture of the corresponding colours.

Let us contract  $\mathcal{H}_1 \cap \mathcal{H}_2$ ,  $\mathcal{H}_2 \cap \mathcal{H}_3$  and  $\mathcal{H}_1 \cap \mathcal{H}_3$  from Fig. 4 into single points, and “straighten out”  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$ . We are being imprecise here: this procedure will be formally justified in Theorem 19. We obtain a simplex as on the right side of Fig. 5.

Now, consider a function  $\partial f : \partial \mathcal{O}_S \rightarrow \mathbb{S}^1$  which satisfies  $f(x) \in \mathcal{C}(x)$  for all  $x \in \partial \mathcal{O}_S$ . Hence, by (4) and (6),  $\partial f$  satisfies  $\partial f(o_1) \in \mathcal{H}_2 \cap \mathcal{H}_3$ ,  $\partial f(o_2) \in \mathcal{H}_1 \cap \mathcal{H}_3$  and  $\partial f(o_3) \in \mathcal{H}_1 \cap \mathcal{H}_2$ . Also,  $\partial f$  satisfies  $\partial f(\mathcal{F}_i^\mathcal{O}) \subset \mathcal{H}_i$  for all  $i \in \{1, 2, 3\}$ . We obtain that  $\partial f$  is a map from one simplex ( $\mathcal{O}_S$ ) to the boundary of another (modified  $\mathbb{S}^1$ ) which preserves faces. This is illustrated in Fig. 5.

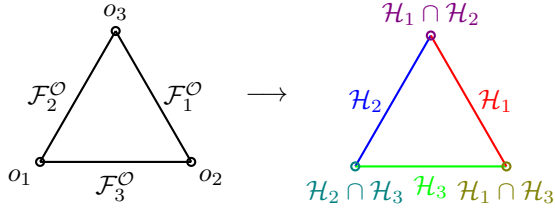


Fig. 5: An illustration of the proof from Example 13. On the left side is  $\mathcal{O}_S$ , while the right side contains a  $\mathbb{S}^1$  modified from Fig. 4 into a simplex, as outlined in the text.

However, by Corollary 4 such a map cannot be extended into a map on the entire  $\mathcal{O}_S$ .  $\square$

Example 13 gives an outline of the method that we use to prove that  $\mathcal{H} = \mathbb{S}^{m-1}$  implies a negative answer to Problem 8: we assume that there exists a solution to Problem 8, and transform it into a map from one simplex to another which preserves faces. To formalize this process we use a tool that automatically transforms sets  $\mathcal{H}_{i_1} \cap \dots \cap \mathcal{H}_{i_k}$  into faces of a simplex. This tool is nerve theory.

*Remark 14:* In Example 13,  $\mathcal{H}^* = \mathbb{S}^{m-1}$ . Thus, the problem of a topological obstruction in that case could have also been solved using the approach of [17]. As the situation in which  $\mathcal{H} = \mathbb{S}^{m-1} \neq \mathcal{H}^*$  is degenerate, any example which illustrates this gap would be highly inelegant.

## VI. MAIN RESULT

In order to solve Problem 8, we will reduce it to the following question:

*Problem 15:* Let  $\mathcal{F}_1^\mathcal{O}, \dots, \mathcal{F}_{\kappa+1}^\mathcal{O}$  be the facets of  $\mathcal{O}_S$ . Does there exist a continuous map  $f : \mathcal{O}_S \rightarrow \mathbb{S}^{m-1}$  which satisfies  $f(\mathcal{F}_j^\mathcal{O}) \subset \mathcal{H}_j$  for all  $j \in \{1, \dots, \kappa + 1\}$ ?

*Lemma 16:* If  $f$  satisfies the conditions of Problem 8, then it satisfies the conditions of Problem 15.

*Proof:* Assume  $f$  solves Problem 8. Let  $j \in \{1, \dots, \kappa + 1\}$  and let  $x \in \text{int}(\mathcal{F}_j^\mathcal{O})$ . We have  $f(x) \in \mathcal{C}(x)$ . By (6), then  $f(x) \in \mathcal{H}_j$ . Thus,  $f(\text{int}(\mathcal{F}_j^\mathcal{O})) \subset \mathcal{H}_j$ . As  $f$  is continuous and  $\mathcal{H}_j$  is closed, we get  $f(\mathcal{F}_j^\mathcal{O}) \subset \mathcal{H}_j$ .  $\blacksquare$

We now present a technical lemma that establishes a structure-preserving homeomorphism between a  $k$ -dimensional simplex  $\Delta^k$  and a simplex whose vertices are the barycentric centers of the facets of  $\Delta^k$ .

*Lemma 17:* Let  $\Delta^k$  be a simplex with vertices  $1, 2, \dots, k + 1$ . Let  $\mathcal{A}_1, \dots, \mathcal{A}_{k+1}$  be the facets of  $\Delta^k$ , and let their barycentric centers be  $t_1, \dots, t_{k+1}$ , respectively. Then,  $\Delta' = \text{co}\{t_1, \dots, t_{k+1}\}$  is a simplex, and eomorphism  $h : \partial \Delta^k \rightarrow \partial \Delta'$  such that  $h(\text{bst}(j)) = \text{co}\{t_i \mid i = 1, \dots, k + 1, i \neq j\}$  for all  $j \in \{1, \dots, k + 1\}$ .

*Proof:* We introduce the following notation:  $C(j_1, \dots, j_r)$  is the barycentric center of  $\text{co}\{j_1, \dots, j_r\}$ . In a special case,  $t_i = C(1, 2, \dots, i - 1, i + 1, \dots, k + 1)$ . It is trivial to computationally show that  $t_i$ 's are affinely independent. Hence,  $\Delta'$  is a simplex. Let us denote its facets by  $\mathcal{A}'_i$ .

Let  $J = \{1, 2, \dots, k + 1\}$ . We define  $h : \partial \Delta^k \rightarrow \partial \Delta'$  to be piecewise affine on each simplex of the barycentric subdivision of  $\mathcal{N}$ . In other words,

$$h(C(j_1, \dots, j_r)) = C(t_j \mid j \in J \setminus \{j_1, \dots, j_r\}), \quad (7)$$

with an affine extension on each simplex  $\text{co}\{C(j_1), C(j_1, j_2), \dots, C(j_1, \dots, j_k)\}$ .  $h$  maps each simplex in the barycentric subdivision of  $\partial \Delta^k$  to a simplex in the barycentric subdivision of  $\partial \Delta'$ .  $h$  is clearly well-defined, bijective, continuous and piecewise affine, and has a continuous and piecewise affine inverse.

Now, first suppose that  $x \in \text{bst}(j)$ . Then  $x$  is in some simplex of the barycentric subdivision of  $\partial \Delta^k$  which contains  $j$ :  $x = \alpha_1 C(j) + \alpha_2 C(j, j_2) + \dots + \alpha_k C(j, j_2, \dots, j_k)$ . By (7),  $h(x) = \alpha_1 C(t_i \mid i \in J \setminus \{j\}) + \sum_{r=2}^k \alpha_r C(t_i \mid i \in J \setminus \{j, j_2, \dots, j_r\})$ . We note that none of the sets  $J \setminus \{j, j_2, \dots, j_r\}$  contain  $j$ . Hence,  $h(x) \in \mathcal{A}'_j$ . Thus,

$$h(\text{bst}(j)) \subset \mathcal{A}'_j. \quad (8)$$

Conversely, suppose  $x \in \mathcal{A}'_j$ . Then  $x = \alpha_1 C(t_{j_1}) + \alpha_2 C(t_{j_1}, t_{j_2}) + \dots + \alpha_k C(t_{j_1}, \dots, t_{j_k})$ , where none of  $j_i$ 's equal  $j$ . By (7),  $h^{-1}(x) = \sum_{r=1}^k \alpha_{k+1-r} C(i \mid i \neq j_1, \dots, j_r)$ . Since  $j \neq j_1, \dots, j_k$ , we know  $h^{-1}(x) \in \text{bst}(j)$ , i.e.,

$$h^{-1}(\mathcal{A}'_j) \subset \text{bst}(j). \quad (9)$$

By combining (8) and (9) we get  $h(\text{bst}(j)) = \mathcal{A}'_j$ .  $\blacksquare$

In the proof of our main result, we will also need the following technical statement. It builds on the homotopy equivalence of space  $\mathcal{X}$  and its cover  $\mathcal{N}$  from Theorem 5

by showing that each cover element  $\mathcal{X}_j \subset \mathcal{X}$  corresponds to a barycentric star of a vertex of  $\mathcal{N}$ .

*Lemma 18 (Proof of Theorem 3.3, [13]):* Let  $\mathcal{N}$  be the nerve of a closed finite regular cover  $\{\mathcal{X}_j \mid j \in J\}$  with respect to  $\mathcal{X}$ . Let us denote vertices of  $\mathcal{N}$  by  $1, \dots, j$ . Then there exist continuous functions  $f' : \mathcal{X} \rightarrow \mathcal{N}$  and  $g' : \mathcal{N} \rightarrow \mathcal{X}$  such that  $f'(\mathcal{X}_j) \subset \text{bst}(j)$  and  $g'(\text{bst}(j)) \subset \mathcal{X}_j$ .

We now reach the main result of our paper.

*Theorem 19:* Suppose Assumption 9 holds, and let  $\mathcal{H} = \mathbb{S}^{m-1}$ . Then, the answer to Problem 8 is negative.

*Proof:* Assume otherwise. Let  $f : \mathcal{O}_S \rightarrow \mathbb{S}^{m-1} = \mathcal{H}$  be a solution to Problem 8. By Lemma 16,  $f$  is then a solution to Problem 15.

By Lemma 10,  $\kappa = \dim(\mathcal{O}_S) = m$ , i.e., the vertices of  $\mathcal{O}_S$  are  $o_1, \dots, o_{m+1}$ . By the conditions of Problem 15,

$$f(o_i) \in \mathcal{H}_j, \quad i, j = 1, \dots, m+1, \quad i \neq j. \quad (10)$$

Let  $\mathcal{N}$  be the nerve of  $\{\mathcal{H}_j \mid \mathcal{H}_j \neq \mathbb{S}^{m-1}\}$ . We note that  $\{\mathcal{H}_j \mid \mathcal{H}_j \neq \mathbb{S}^{m-1}\}$  is a closed finite cover of  $\mathcal{H}$ . Additionally, by (A3) all  $\mathcal{H}_j$  which are not whole spheres are homeomorphic to closed balls. It can easily be verified that a closed ball is contractible and locally contractible, so the cover  $\{\mathcal{H}_j \mid \mathcal{H}_j \neq \mathbb{S}^{m-1}\}$  is also regular by Proposition 2. Hence, invoking Theorem 5,  $\mathcal{N} \simeq \mathcal{H} = \mathbb{S}^{m-1}$ .

We note that  $\mathcal{N}$ , by definition, has at most  $m+1$  vertices. Let us show that it has exactly  $m+1$  vertices, i.e., that  $\mathcal{H}_j \neq \mathbb{S}^{m-1}$  for all  $j \in \{1, \dots, m+1\}$ . By Lemma 11, there is at most one  $\mathcal{H}_j$  such that  $\mathcal{H}_j = \mathbb{S}^{m-1}$ . Assume such an  $\mathcal{H}_j$  indeed exists. Without loss of generality, let  $\mathcal{H}_{m+1} = \mathbb{S}^{m-1}$ . By (10),  $f(o_{m+1}) \in \mathcal{H}_j$  for all  $j = 1, \dots, m$ , and also  $f(o_{m+1}) \in \mathbb{S}^{m-1} = \mathcal{H}_{m+1}$ . Thus, all  $\mathcal{H}_j$ 's intersect. Hence, by (2),  $\mathcal{N}$  is a full  $m$ -dimensional simplex  $\Delta^m$ . Since  $\Delta^m \cong \mathbb{B}^m \not\cong \mathbb{S}^{m-1}$ , this is in contradiction with  $\mathcal{N} \simeq \mathbb{S}^{m-1}$ .

Thus, let the vertices of  $\mathcal{N}$  be denoted by  $1, 2, \dots, m+1$ . Now, by (2) and (10), simplicial complex  $\mathcal{N} \neq \Delta^m$  includes all faces of an  $m$ -dimensional simplex  $\Delta^m$ . Thus,

$$\mathcal{N} = \partial\Delta^m. \quad (11)$$

By Lemma 18, there exists a continuous function  $f' : \mathcal{H} \rightarrow \mathcal{N}$  such that  $f'(\mathcal{H}_j) \subset \text{bst}(j)$ . By the conditions of Problem 15, continuous map  $f' \circ f : \mathcal{O}_S \rightarrow \mathcal{N}$  satisfies

$$(f' \circ f)(\mathcal{F}_j^\mathcal{O}) \subset f'(\mathcal{H}_j) \subset \text{bst}(j), \quad j = 1, \dots, m+1. \quad (12)$$

Let  $\mathcal{N}'$  be the simplex generated by barycentric centers of facets of  $\mathcal{N}$ , and let  $\mathcal{A}'_1, \dots, \mathcal{A}'_m$  be the facets of  $\mathcal{N}'$ . By (11) and Lemma 17, there exists a homeomorphism  $h : \mathcal{N} \rightarrow \mathcal{N}'$  such that  $h(\text{bst}(j)) = \mathcal{A}'_j$ . Hence, by (12), the continuous map  $\bar{f} = h \circ f' \circ f : \mathcal{O}_S \rightarrow \mathcal{N}'$  satisfies

$$\bar{f}(\mathcal{F}_j^\mathcal{O}) \subset h(\text{bst}(j)) = \mathcal{A}'_j, \quad j = 1, \dots, m+1. \quad (13)$$

We noted earlier that  $\mathcal{O}_S \cong \Delta^m$ . By Lemma 17 and (11),  $\mathcal{N}' \cong \mathcal{N} = \partial\Delta^m$ . Thus,  $\bar{f} : \mathcal{O}_S \rightarrow \mathcal{N}'$  can be viewed as a continuous map  $\bar{f} : \Delta^m \rightarrow \partial\Delta^m$  which by (13) satisfies

$$\bar{f}(\mathcal{P}_j) \subset \mathcal{P}_j \quad (14)$$

for all facets  $\mathcal{P}_j \subset \Delta^m$ . By taking intersections of different  $\mathcal{P}_j$ 's, it is clear from (14) that  $\bar{f} : \Delta^m \rightarrow \partial\Delta^m$  not only

preserves facets of  $\Delta^m$ , but also all faces. However, by Corollary 4, such a map cannot exist. ■

The following result completely characterizes the solvability of the problem of a topological obstruction in the RCP.

*Theorem 20:* Suppose Assumption 9 holds. The answer to Problem 8 is affirmative if and only if  $\mathcal{H} \neq \mathbb{S}^{m-1}$ .

*Proof:* It was shown in [17] that, if  $\mathcal{H} \neq \mathbb{S}^{m-1}$ , the answer to Problem 8 is affirmative. In the other direction, if the answer is affirmative, condition  $\mathcal{H} \neq \mathbb{S}^{m-1}$  holds by Theorem 19. ■

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