Characterizing Equilibria in Reach Control Under Affine Feedback ⋆

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Abstract: The Reach Control Problem (RCP) deals with driving the states of an affine system on a simplex to leave the simplex through a pre-determined facet. A necessary condition for the solvability of the RCP by a given feedback is that there are no closed-loop equilibria in the simplex. As a stepping stone to fully characterizing when equilibria can be removed from the simplex using feedback, this paper studies the geometric structure of open-loop equilibria. Using a triangulation in which the set of potential equilibria intersects the interior of the simplex, we prove that the equilibrium set contains at most one point, in both the single-input and multi-input case. We additionally improve on the currently available results on reach controllability to characterize when the closed-loop equilibria can be pushed off the simplex using affine feedback.

Keywords: Hybrid Nonlinear Control Systems, Switching Control, Affine Feedback, Equilibrium Set, Reachable States

1. INTRODUCTION

The Reach Control Problem (RCP) is a fundamental problem in the theory of hybrid dynamical systems. The central problem of the RCP is to design a feedback that drives the trajectories of an affine system to exit a simplex through a predetermined facet in finite time (Habets et al., 2006; Roszak and Broucke, 2006). It has been shown in Habets et al. (2006) and Roszak and Broucke (2006) that equilibria play a central role in solvability of the RCP using affine feedback. Motivated by this, a currently active line of research focuses on exploring the existence, geometry and evolution of equilibria of an affine control system (Ashford and Broucke, 2013; Helwa and Broucke, 2013; Semsar-Kazerooni and Broucke, 2014; Broucke and Ganness, 2014).

One can interpret the RCP as the application of feedback control to shift closed-loop equilibria off the simplex of interest. In an effort to characterize when the system is sufficiently actuated to do so, notions such as reach control indices and reach controllability were formulated in Broucke and Ganness (2014) and Semsar-Kazerooni and Broucke (2014), respectively. In particular, reach controllability was designed to describe how an infinitesimal control actuation serves to push the closed-loop equilibria out of the simplex. In that sense, it acts as an analogue to the standard notion of local controllability, which uses infinitesimal control actuation to move the state within the neighbourhood of the starting point.

This paper has two main contributions. In Section 4 we explore properties of the set of open-loop equilibria of an affine system on a simplex. After significantly relaxing the assumptions of Semsar-Kazerooni and Broucke (2014), in particular removing the restriction on the number of inputs, we prove that this set contains at most one point. In Section 5 we provide a generalization of the key result of Semsar-Kazerooni and Broucke (2014) on reach controllability in single-input systems, by removing the need for an additional assumption made in that paper.

Notation. Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a set. The relative interior of $\mathcal{S}$ is denoted $\text{int}(\mathcal{S})$ and the relative boundary of $\mathcal{S}$ is denoted $\partial(\mathcal{S})$. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\text{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$, and the notation $\text{aff}\{v_1, v_2, \ldots\}$ denotes the affine hull of a set of points $v_i \in \mathbb{R}^n$. By the *standard orthogonal simplex* we mean a simplex $\mathcal{S} \subseteq \mathbb{R}^n$ with vertices at $v_0 = 0$, $v_i = e_i$, $i = 1, \ldots, n$. The unit normal vectors to facets $\mathcal{F}_i$ of such a simplex are given by $h_0 = [1\ 1\ 1\ \ldots\ 1]^T$ and $h_i = -e_i$, $i = 1, \ldots, n$, where each facet is indexed by the vertex it does not contain.

2. REACH CONTROL PROBLEM

We review the reach control problem. Consider an n-dimensional simplex $\mathcal{S} = \text{co}\{v_0, \ldots, v_n\}$, the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. Let its vertex set be $V = \{v_0, \ldots, v_n\}$ and its facets, indexed by the vertices they do not contain, $\mathcal{F}_0, \ldots, \mathcal{F}_n$. Let h_j, $j = 0, \ldots, n$, be the unit normal vector to each facet $\mathcal{F}_j$ pointing outside of the simplex. Facet $\mathcal{F}_0$ is called the *exit facet*. Let $I = \{1, \ldots, n\}$ and define $I(x)$ to be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \text{co}\{v_i \mid i \in I(x)\}$.

We consider the affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathcal{S}\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\mathcal{B} = \text{Im}(B)$, the image of B. Define $\mathcal{O} = \{\ x \in$

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$\mathbb{R}^n \mid Ax+a \in \mathcal{B}\}$, $\mathcal{E} = \{\, x \in \mathbb{R}^n \mid Ax+a = 0\}$, $\mathcal{O}_{\mathcal{S}} = \mathcal{S}\cap\mathcal{O}$, and $\mathcal{E}_{\mathcal{S}} = \mathcal{S} \cap \mathcal{E}$.

Notice that $\mathcal{E}$ is the set of open-loop equilibria (when $u = 0$), whereas $Ax + Bu + a$ for $x \in \mathcal{O}$ can vanish for an appropriate choice of u, so $\mathcal{O}$ is the set of possible equilibrium points of the system. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at x_0 under control input u. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{S}$.

Problem 1. (Reach Control Problem (RCP)).
Consider system (1) defined on $\mathcal{S}$. Find a state feedback $u(x)$ such that for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\delta > 0$ such that

- (i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$,
- (ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
- (iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \delta)$.

To solve the RCP we require conditions that disallow trajectories to exit from the facets $\mathcal{F}_i$, $i \in I$. For $x \in \mathcal{S}$ define the closed, convex cone $\mathcal{C}(x) = \{\ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in I \setminus I(x)\ \}$. Figure 1 (a modified version of a figure originally published in Ornik and Broucke (2015)) illustrates the notation and the cones $\mathcal{C}(x)$ for a 2D simplex.

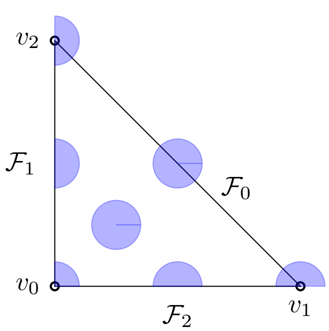

Fig. 1. Illustration of $\mathcal{C}(x)$, depicted as blue cones at several points $x \in \mathcal{S}$.

We say the *invariance conditions are solvable* if there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that,

$$Av_i + Bu_i + a \in \mathcal{C}(v_i)\,, \qquad i = 0, \ldots, n\,. \tag{2}$$

The inequalities (2) are called *invariance conditions*, and they guarantee that trajectories that exit $\mathcal{S}$ only do so through $\mathcal{F}_0$ (Habets and van Schuppen, 2004).

The following result provides the foundation for solving the RCP by affine feedback.

Theorem 2. (Habets et al., 2006; Roszak and Broucke, 2006) Given system (1) on a simplex $\mathcal{S}$ and an affine feedback $u(x) = Kx + g$, with $K \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, and $u_0 = u(v_0), \cdots, u_n = u(v_n)$, the closed-loop system is a solution to Problem 1 if and only if

- (a) The invariance conditions (2) hold,
- (b) There is no equilibrium in $\mathcal{S}$.

3. PRELIMINARIES

We note that $\mathcal{O}_{\mathcal{S}}$ is an intersection of the affine space $\mathcal{O}$ and the simplex $\mathcal{S}$. Hence, $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{o_1, \ldots, o_{\kappa+1}\}$ is a polytope with vertices $o_1, \ldots, o_{\kappa+1}$. Also, we define

$$\mathrm{cone}(\mathcal{O}_{\mathcal{S}}) = \bigcap_{k=1}^{\kappa+1} \mathcal{C}(o_k).$$

We make use of the following assumptions:

Assumption 3.

- (A1) $\mathcal{E}_{\mathcal{S}} = \mathrm{co}\{\varepsilon_1, \cdots, \varepsilon_{\kappa_0+1}\}$, a κ_0-dimensional simplex with $0 \leq \kappa_0 \leq \kappa$,
- (A2) $\mathcal{O} \cap \mathrm{int}(\mathcal{S}) \neq \emptyset$,
- (A3) $\mathcal{O} \cap \mathcal{F}_0 = \emptyset$,
- (N1) $Av_i + a \in \mathcal{C}(v_i)\,, \quad i = 0, \ldots, n$,
- (N2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}}) \neq \mathbf{0}$.

Assumptions (A2) and (A3) allow $\mathcal{O}$ to intersect the interior of simplices. Arguments motivating this choice of triangulation are found in Semsar-Kazerooni and Broucke (2014). Notice that we remove the restriction of Semsar-Kazerooni and Broucke (2014) that $\mathcal{O}_{\mathcal{S}}$ is itself a simplex. We use a different numbering for assumptions (N1) and (N2) as these arise from necessary conditions. Theorem 2 shows that a necessary condition for solvability of the RCP by an affine feedback $u = Kx + g$ is that the invariance conditions (2) hold. To achieve (N1), we may assign the affine feedback transformation $u(x) = Kx + g + v$, with v as a new exogenous input, to obtain a new affine system

$$\dot{x} = (A + BK)x + Bv + (Bg + a) =: A'x + Bv + a'\,. \tag{3}$$

Clearly, $A'v_i + a' \in \mathcal{C}(v_i)$, $i = 0, \ldots, n$. Now we abuse notation and redefine the system to be (3) — with A' replaced by A, a' replaced by a, and v replaced by u — for which (N1) is satisfied a priori. It can be shown that $\mathcal{O}$ is invariant under such feedback transformations, whereas the set $\mathcal{E}$ generally is not. It is an area of future investigation to understand the evolution of $\mathcal{E}_{\mathcal{S}}$ under repeated affine feedback transformations that preserve the invariance conditions. In this paper we assume that such an affine feedback transformation has already been performed and the new system satisfies the above assumptions.

Assumption (N2) was shown in Semsar-Kazerooni and Broucke (2014) to be a necessary condition for solvability of the RCP by continuous state feedback in the case of single-input systems. Accordingly, we will only be making use of this assumption when discussing single-input systems.

The mathematical machinery laid out in Lemmas 4-8 to derive the main arguments on characterization of equilibria is based on manipulating index sets $I(o_k)$ in order to book-keep the constraints arising from (N1). This machinery enables us to relate the combinatorial properties of index sets to geometric properties of polytopes and cones.

Lemma 4. Let $I(o_1)\cap\cdots\cap I(o_{\kappa+1}) = \{0, 1, 2, \ldots, l\}$. Then, $\mathrm{cone}(\mathcal{O}_{\mathcal{S}}) = \{y \mid h_j \cdot y \leq 0 \text{ for all } j = l + 1, \ldots, n\}$.

Lemma 5. Let $\mathcal{S} = \mathrm{co}\{v_0, \ldots, v_n\}$ be an n-dimensional simplex, and let $\mathcal{P} = \mathrm{co}\{w_1, \ldots, w_k\} \subseteq \mathcal{S}$. Let $q \leq n$ be such that

$$\bigcup_{i=1}^{k} I(w_i) = \{0, \ldots, q\}.$$

Define $\mathcal{S}' = \mathrm{co}\{v_0, \ldots, v_q\}$. Then, there exists $x \in \mathcal{P} \cap \mathrm{int}(\mathcal{S}')$.

We now give a proof of three fundamental claims which we use as building blocks for our main results. One of the key properties is the following generalization of Lemma 2 of Semsar-Kazerooni and Broucke (2014).

Lemma 6. Let $\mathcal{S} = \text{co}\{v_0, \ldots, v_n\}$ be an n-dimensional simplex. Let $\mathcal{S}'$ be a q-dimensional face of $\mathcal{S}$, with $q \leq n$. Let $\mathcal{P} = \text{co}\{w_1, \ldots, w_{p+1}\}$ be a p-dimensional simplex. Suppose $\mathcal{P} \subseteq \mathcal{S}'$, $\mathcal{P} \cap \text{int}(\mathcal{S}') \neq \emptyset$ and $\partial(\mathcal{P}) \subseteq \partial(\mathcal{S}')$. Then each index set $I(w_k)$, $k \in \{1, \ldots, p+1\}$, has an exclusive member. That is, there exists $i_k \in I(w_k)$ such that $i_k \notin I(w_j)$ for all $j \in \{1, \ldots, p+1\} \setminus \{k\}$.

Proof. Without loss of generality, assume the vertices of $\mathcal{S}'$ are $v_0, \ldots, v_q$. By the assumption that $\mathcal{P} \cap \text{int}(\mathcal{S}') \neq \emptyset$, we have $\cup_{i=1}^{p+1} I(w_i) = \{0, \ldots, q\}$. If $p = 0$ we are done. Instead suppose w.l.o.g. $I(w_1) \subseteq \cup_{j=2}^{p+1} I(w_j)$. Thus, $\cup_{j=2}^{p+1} I(w_j) = \{0, \ldots, q\}$. Define $\mathcal{P}' = \text{co}\{w_2, \ldots, w_{p+1}\}$. Since $\mathcal{P}$ is a simplex, $\mathcal{P}'$ is a $(p-1)$-dimensional facet of $\mathcal{P}$ so $\mathcal{P}' \subseteq \partial(\mathcal{P})$. However, $\cup_{j=2}^{p+1} I(w_j) = \{0, \ldots, q\}$ implies $\mathcal{P}' \cap \text{int}(\mathcal{S}') \neq \emptyset$. This contradicts that $\partial(\mathcal{P}) \subseteq \partial(\mathcal{S}')$. ∎

The following lemma relates the dimension of $\mathcal{E}_\mathcal{S}$ with the dimension of the affine space generated by the vertices v_i present in $I(\varepsilon_k)$.

Lemma 7. Let $\mathcal{E}_\mathcal{S} = \text{co}\{\varepsilon_1, \cdots, \varepsilon_{\kappa_0+1}\}$ be a simplex with $\dim(\mathcal{E}_\mathcal{S}) = \kappa_0$. Let $q \leq n$ be such that $\cup_{k=1}^{\kappa_0} I(\varepsilon_k) = \{0, 1, \ldots, q\}$. Then, $\dim(\mathcal{E} \cap \text{aff}\{v_0, \ldots, v_q\}) = \kappa_0$.

Proof. Let $\mathcal{S}' = \text{co}\{v_0, \ldots, v_q\}$, and define $\mathcal{E}'_\mathcal{S} = \mathcal{E} \cap \mathcal{S}'$. Since $\mathcal{E}_\mathcal{S} \subseteq \mathcal{S}'$ and $\mathcal{E}_\mathcal{S} \subseteq \mathcal{E}$, we note $\mathcal{E}_\mathcal{S} \subseteq \mathcal{E}'_\mathcal{S}$. On the other hand, $\mathcal{E}'_\mathcal{S} = \mathcal{E} \cap \mathcal{S}' \subseteq \mathcal{E} \cap \mathcal{S} = \mathcal{E}_\mathcal{S}$. Hence, $\mathcal{E}'_\mathcal{S} = \mathcal{E}_\mathcal{S}$ and thus, $\dim \mathcal{E}'_\mathcal{S} = \dim \mathcal{E}_\mathcal{S} = \kappa_0$.

Now, since $\mathcal{S}' \subseteq \text{aff}(\mathcal{S}')$, $\mathcal{E}'_\mathcal{S} = \mathcal{E} \cap \mathcal{S}' = \mathcal{E} \cap \mathcal{S}' \cap \text{aff}(\mathcal{S}')$. By Lemma 5, there exists $x \in \mathcal{E}_\mathcal{S} = \mathcal{E}'_\mathcal{S}$ such that $x \in \text{int}(\mathcal{S}')$. By observing the dimension of $\mathcal{E} \cap \text{aff}(\mathcal{S}')$ locally around x, we note that $\dim(\mathcal{E} \cap \text{aff}(\mathcal{S}')) = \dim(\mathcal{E} \cap \text{aff}(\mathcal{S}') \cap \mathcal{S}') = \dim(\mathcal{E}'_\mathcal{S}) = \dim(\mathcal{E}_\mathcal{S}) = \kappa_0$. ∎

Finally, the following lemma will present the key rank argument appearing in Theorem 9 and Theorem 14. This is a generalization of Proposition 1 from Semsar-Kazerooni and Broucke (2014). The assumptions of Proposition 1 from Semsar-Kazerooni and Broucke (2014) have been relaxed, and the scope of the result has been significantly extended. In particular, it now covers a larger class of simplices on $\mathcal{S}$, instead of solely $\mathcal{O}_\mathcal{S}$ itself.

Lemma 8. Let $\mathcal{P} = \text{co}\{w_1, \ldots, w_{p+1}\} \subseteq \mathcal{S}$ be a simplex in $\mathcal{S}$ with vertices $w_1, \ldots, w_{p+1}$. We assume the following:

- (I1) $\cup_{i=1}^{p+1} I(w_i) = \{0, 1, \ldots, n\}$.
- (I2) There exists $r_0 > 0$ such that $\{0, 1, \ldots, r_0\}$ is the set of all indices that appear in more than one index set $I(w_i)$.
- (I3) Each $I(w_i)$, $i = 1, \ldots, p+1$, has at least one non-zero exclusive member.

Further we assume

- (N1) $Av_i + a \in \mathcal{C}(v_i)$ for all $i = 0, \ldots, n$.
- (E1) $h_j \cdot (Av_i + a) = 0$ for all $i = 0, \ldots, r_0$, $j = r_0+1, \ldots, n$.
- (E2) $h_j \cdot (Aw_k + a) = 0$ for all $k = 1, \ldots, p+1$, $j = r_0 + 1, \ldots, n$, $j \notin I(w_k)$.
- (E3) $h_j \cdot (Aw_k + a) = 0$ for all $k = 1, \ldots, q+1$, $q \leq p$, $j = r_0 + 1, \ldots, n$, j is an exclusive member of $I(w_k)$.

Then, $\text{rank}(A) < n - q$.

Proof. Without loss of generality, the vertices $v_0, \ldots, v_n$ can be ordered according to the non-zero exclusive members of $I(w_k)$. That is, the indices are ordered as $\{0, 1, \ldots, r_0, r_0 + 1, \ldots, r_1, \ldots, r_p + 1, \ldots, r_{p+1}\}$, where $r_{p+1} = n$. Here, $\{0, 1, \ldots, r_0\}$ appear in more than one index set $I(w_k)$. Indices $\{r_{k-1} + 1, \ldots, r_k\}$ only appear in $I(w_k)$. We assume w.l.o.g. that $\mathcal{S}$ is the standard orthogonal simplex: $v_0 = 0$, $v_i = e_i$, $h_i = -e_i$ for $i = 1, \ldots, n$.

Now we examine the consequences of (N1), (E1)-(E2) on the forms of A and a. First consider (E1). Setting $i = 0$ we have

$$h_j \cdot (Av_0 + a) = h_j \cdot a = (-e_j) \cdot a = a_j = 0, \; j = r_0 + 1, \ldots, n. \tag{4}$$

Then again from (E1):

$$h_j \cdot Av_i = [A]_{ji} = 0, \; i = 1, \ldots, r_0, \; j = r_0 + 1, \ldots, n. \tag{5}$$

Next we examine (N1) and (E2). First we use (4) to simplify (E2):

$$\begin{aligned} h_j \cdot Aw_k = 0, \; & k = 1, \ldots, p+1, \\ & j = r_0 + 1, \ldots, n, \; j \notin I(w_k). \end{aligned} \tag{6}$$

Let $w_k = \sum_{i \in I(w_k)} \alpha_i^{w_k} v_i$, with $\alpha_i^{w_k} > 0$, $\sum_{i \in I(w_k)} \alpha_i^{w_k} = 1$. Then for each $k = 1, \ldots, p+1$, (6) becomes

$$\sum_{i \in I(w_k)} \alpha_i^{w_k} h_j \cdot Av_i = 0, \; j = r_0 + 1, \ldots, n, \; j \notin I(w_k). \tag{7}$$

By (N1) and (4):

$$h_j \cdot (Av_i + a) = h_j \cdot Av_i \leq 0, \; j = r_0 + 1, \ldots, n, \; i \neq j. \tag{8}$$

Combining (7) and (8) and using $\alpha_i^{w_k} > 0$, we get

$$\begin{aligned} h_j \cdot Av_i = [A]_{ji} = 0, \; & k = 1, \ldots, p+1, \\ & i \in I(w_k), \; j \in \{r_0 + 1, \ldots, n\} \backslash I(w_k). \end{aligned} \tag{9}$$

Consider $j = r_{k-1} + 1, \ldots, r_k$, the exclusive indices of $I(w_k)$. By exclusivity, $r_{k-1} + 1, \ldots, r_k \in \{r_0 + 1, \ldots, n\} \backslash I(w_l)$ for all $l = 1, \ldots, p+1$, $l \neq k$. Also,

$$\bigcup_{l \neq k} I(w_l) = \{0, \ldots, n\} \backslash \{r_{k-1} + 1, \ldots, r_k\}.$$

Applying these observations to (9) we get

$$\begin{aligned} [A]_{ji} = 0, \; & k = 1, \ldots, p+1, \\ & i \in \{0, \ldots, n\} \backslash \{r_{k-1} + 1, \ldots, r_k\}, \\ & j \in \{r_{k-1} + 1, \ldots, r_k\}. \end{aligned} \tag{10}$$

Putting together the information in (4), (5) and (10), the forms of A and a are:

$$A = \begin{bmatrix} A_{00} & A_{01} & \cdots & A_{0,p+1} \\ & A_{11} & & \\ & & \ddots & \\ & & & A_{p+1,p+1} \end{bmatrix}, \; a = \begin{bmatrix} a_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{11}$$

These forms are obtained as follows:

- The block of zero elements below $a_0 \in \mathbb{R}^{r_0}$ in a is due to (4).
- The block of zero elements below $A_{00} \in \mathbb{R}^{r_0 \times r_0}$ in A is due to (5).
- The off-diagonal zero blocks in the same rows as $A_{11}, \ldots, A_{p+1,p+1}$ are due to (10).

- The first r_0 rows of A correspond to the values of j in (5), (10) for which we have no constraints.
- There are $p+1$ blocks of rows corresponding to sets of indices $\{r_{k-1}+1,\ldots,r_k\}$, $k=1,\ldots,p+1$. Thus, $A_{kk} \in \mathbb{R}^{(r_k-r_{k-1})\times(r_k-r_{k-1})}$.

Finally, we consider (E3). First, partition

$$w_k = (w_k^0, w_k^1, \ldots, w_k^{p+1})$$

according to $\{1,\ldots,r_0,r_0+1,\ldots,r_1,\ldots,r_p+1,\ldots,r_{p+1}\}$. Combining (11) and (E3):

$$\begin{bmatrix} A_{11} & & \\ & \ddots & \\ & & A_{q+1,q+1} \end{bmatrix} \begin{bmatrix} w_k^1 \\ \vdots \\ w_k^{q+1} \end{bmatrix} = 0,\ k=1,\ldots,q+1. \tag{12}$$

Since $\{r_{k-1}+1,\ldots,r_k\}$ is exactly the set of exclusive members of $I(w_k)$, we know $w_k^k \neq 0$. Thus, A_{kk} is singular for all $k=1,\ldots,q+1$. Hence, from (12), $\operatorname{rank}(A) < n-q$. ■

4. EQUILIBRIUM SET

We now proceed to the main contributions of this paper. The central result of this section is that $\dim(\mathcal{E}_\mathcal{S}) = 0$. A characterization of $\mathcal{E}_\mathcal{S}$ was previously explored in Helwa and Broucke (2014), where it was also shown that $\mathcal{E}_\mathcal{S}$ is a point. However, Helwa and Broucke (2014) dealt only with single-input systems, whereas we deal with multi-input systems. It was shown in Semsar-Kazerooni and Broucke (2014) that the equilibria in single-input systems can lie only on the boundary of $\mathcal{S}$. As the set of possible equilibria in controllable single-input systems is a line segment, the result of Helwa and Broucke (2014) was not unreasonable to expect. Our multi-input result, in contrast, is more surprising, and represents a significant improvement. Moreover, the fact that $\mathcal{E}_\mathcal{S}$ is a point was used in Helwa and Broucke (2014) to apply multi-affine feedback to solve the RCP for single-input systems in the case when affine feedback fails. Our result may provide an avenue to apply multi-affine feedback in the multi-affine case. This may ultimately serve to answer the question of an appropriate class of feedback for solvability of the RCP. We remark that there are also other differences from the result of Helwa and Broucke (2014). In particular, we will not be assuming that $\mathcal{O}_\mathcal{S}$ is a simplex and we do not require assumptions (A2) and (N2).

Theorem 9. Consider the system (1) defined on a simplex $\mathcal{S}$. Suppose assumptions (A1), (A3) and (N1) hold. If $\mathcal{E}_\mathcal{S} \neq \emptyset$, then $\dim(\mathcal{E}_\mathcal{S}) = 0$.

Proof. Suppose $\dim(\mathcal{E}_\mathcal{S}) = \kappa_0 > 0$. Without loss of generality, let $q \le n$ be such that $\cup_{k=1}^{\kappa_0+1} I(\varepsilon_k) = \{0,1,\ldots,q\}$. Let $\mathcal{S}' = \operatorname{co}\{v_0,\ldots,v_p\}$.

Let $\mathcal{O}'_\mathcal{S} = \mathcal{O} \cap \mathcal{S}'$ and $\mathcal{E}'_\mathcal{S} = \mathcal{E} \cap \mathcal{S}'$. By a variant of Lemma 1 in Semsar-Kazerooni and Broucke (2014), $\partial\mathcal{E}_\mathcal{S} \subseteq \partial\mathcal{S}'$.

By Lemma 5, $\mathcal{E}_\mathcal{S} \cap \operatorname{int}(\mathcal{S}') \neq \emptyset$, so Lemma 6 applies with $\mathcal{P} = \mathcal{E}_\mathcal{S}$. By (A3), 0 is not an exclusive member of any $I(\varepsilon_k)$, $k=1,\ldots,\kappa_0+1$. Thus, by Lemma 6, the vertices of $\mathcal{S}$ can be ordered according to non-zero exclusive members of $I(\varepsilon_k)$. That is, the indices are ordered as $\{0,1,\ldots,r_0,r_0+1,\ldots,r_1,\ldots,r_{\kappa_0}+1,\ldots,r_{\kappa_0+1},q+1,\ldots,n\}$, with $r_0 < r_1 < \ldots < r_{\kappa_0+1} = q$. Here $\{0,1,\ldots,r_0\}$ are the indices appearing in more than one index set $I(\varepsilon_k)$, $k=1,\ldots,\kappa_0+1$. Indices $\{r_{k-1}+1,\ldots,r_k\}$ only appear in $I(\varepsilon_k)$.

Consider any vertex $\varepsilon_k \in \mathcal{E}_\mathcal{S}$, $k=1,\ldots,\kappa_0+1$. We have $A\varepsilon_k + a = 0$, and thus $h_j \cdot (A\varepsilon_k + a) = 0$, $j \in I$. Let $\varepsilon_k = \sum_{i \in I(\varepsilon_k)} \alpha_i^{\varepsilon_k} v_i$ with $\alpha_i^{\varepsilon_k} > 0$ and $\sum_{i \in I(\varepsilon_k)} \alpha_i^{\varepsilon_k} = 1$. Then

$$\sum_{i \in I(\varepsilon_k)} \alpha_i^{\varepsilon_k} h_j \cdot (Av_i + a) = 0\,, \quad j \in I\,. \tag{13}$$

By (N1),

$$h_j \cdot (Av_i + a) \le 0\,, \quad i \in I(\varepsilon_k), \quad j \in I \setminus I(\varepsilon_k)\,. \tag{14}$$

Combining (13), (14), and the fact that $\alpha_i^{\varepsilon_k} > 0$ for $i \in I(\varepsilon_k)$ we get

$$\begin{aligned} h_j \cdot (Av_i + a) = 0,\ & k=1,\ldots,\kappa_0+1 \\ & i \in I(\varepsilon_k),\ j \in I \setminus I(\varepsilon_k). \end{aligned} \tag{15}$$

Consider $j = r_{k-1}+1,\ldots,r_k$. By exclusivity, $r_{k-1}+1,\ldots,r_k \in I \setminus I(\varepsilon_l)$ for all $l=1,\ldots,\kappa_0+1$, $l \neq k$. Also, $\cup_{l\neq k} I(\varepsilon_l) = \{0,\ldots,q\} \setminus \{r_{k-1}+1,\ldots,r_k\}$. Applying these observations to (15) we get

$$\begin{aligned} h_j \cdot (Av_i + a) = 0\,, & k=1,\ldots,\kappa_0+1\,, \\ & i \in \{0,\ldots,q\} \setminus \{r_{k-1}+1,\ldots,r_k\}\,, \\ & j \in \{r_{k-1}+1,\ldots,r_k\}\,. \end{aligned} \tag{16}$$

Let us now invoke Lemma 8 for the simplex $\mathcal{E}_\mathcal{S}$. We note that Lemma 8 requires assumption (I1). This does not necessarily apply directly. However, we are interested only in solutions $x \in \mathcal{E} \cap \operatorname{aff}(\mathcal{S}')$. Thus, instead of looking at the whole simplex $\mathcal{S}$, we will be observing only $\mathcal{S}' = \operatorname{co}\{v_0,\ldots,v_q\}$, which was chosen exactly in a way that (I1) applies on it. (I2) and (I3) are satisfied.

(N1) is satisfied by the assumptions of the theorem, and it still holds for the reduced system on $\mathcal{S}'$. (E1) is satisfied by (16), (E2) is satisfied since $A\varepsilon_k + a = 0$ for all $k = 1,\ldots,\kappa_0+1$, and (E3) is also satisfied for that reason, with q from Lemma 8 equalling κ_0 in this theorem.

Hence, by Lemma 8 $\operatorname{rank}(\tilde{A}) < q - \kappa_0$, where $\tilde{A}$ is the matrix A with rows and columns $q+1,\ldots,n$ removed. Hence, equation

$$\tilde{A}\tilde{x} + \tilde{a} = 0 \tag{17}$$

has at least κ_0+1 linearly independent solutions, where $\tilde{x}$ and $\tilde{a}$ differ from x and a, respectively, by having their rows $q+1,\ldots,n$ removed. As each $x \in \mathcal{E} \cap \operatorname{aff}(\mathcal{S}')$ corresponds to exactly one solution $\tilde{x}$ of (17), $\dim(\mathcal{E} \cap \operatorname{aff}(\mathcal{S}')) \ge \kappa_0+1$. This is in contradiction with Lemma 7. ■

We note that one of our triangulation assumptions, (A2), was not strictly necessary for the above theorem, although its use would have made our invocation of Lemma 8 more elegant, as assumption (I1) in Lemma 7 would have been automatically satisfied. However, assumption (A3) is necessary.

From Theorem 9, we know that, under assumptions (A1), (A3) and (N1), $\mathcal{E}_\mathcal{S}$ is a single point. As mentioned, these assumptions represent a significant relaxation of the assumptions made in Helwa and Broucke (2014) and Semsar-Kazerooni and Broucke (2014). However, assumption (A1) still contains the imposition that $\mathcal{E}_\mathcal{S}$ is a simplex. In a particular case of controllable single-input systems, which is the main subject of inquiry of Semsar-Kazerooni and Broucke (2014), this assumption can be removed as well. We invoke the following well-known result:

Lemma 10. If (A, B) is controllable, then $\mathcal{O}$ is an affine subspace with $\dim(\mathcal{O}) = m$.

Combined with assumption (A2), we can use Lemma 10 to prove the following result for single-input systems.

Proposition 11. Suppose assumptions (A2), (A3) and (N1) hold. Suppose $m = 1$ and (A, B) is controllable. Then, $\mathcal{E}_{\mathcal{S}}$ is either empty or a single point.

Proof. From Lemma 10, we obtain $\dim(\mathcal{O}) = 1$. From (A2), $\dim(\mathcal{O}_{\mathcal{S}}) = \dim(\mathcal{O}) = 1$. Since $\mathcal{O}_{\mathcal{S}}$ is a polytope, the only option is that $\mathcal{O}_{\mathcal{S}}$ is a line segment. Now, if $\mathcal{E}_{\mathcal{S}}$ is not empty, we have $\mathcal{E}_{\mathcal{S}} \subseteq \mathcal{O}_{\mathcal{S}}$. Since $\mathcal{E}_{\mathcal{S}}$ is a polytope as well, this implies that $\mathcal{E}_{\mathcal{S}}$ is either a single point or a segment. In both cases, this implies it is a simplex. Hence, (A1) is satisfied as well. Now, by Theorem 9, we get that $\mathcal{E}_{\mathcal{S}} = \emptyset$ or $\mathcal{E}_{\mathcal{S}}$ is a single point. ■

5. REACH CONTROLLABILITY

The notion of reach controllability has been defined for single-input systems in Semsar-Kazerooni and Broucke (2014). It provides a way to describe the ability of infinitesimal control actuation in system (1) to move the equilibria located on the boundary out of the simplex $\mathcal{S}$. This is similar to the notion of local controllability, which also uses infinitesimal control actuation to reach points in the local neighbourhood. However, in the case of reach controllability it is not the states of the system that we are directly interesting in moving. Instead, the desire is to move the equilibrium set $\{x \in \mathbb{R}^n \mid Ax + Bu(x) + a = 0\}$ out of the simplex.

The definition we provide here is a slight generalization of the definition in Semsar-Kazerooni and Broucke (2014). Our definition allows for the vertices of $\mathcal{E}_{\mathcal{S}}$ not to be contained in the vertex set of $\mathcal{O}_{\mathcal{S}}$. We will see such a situation appearing in Example 13.

Definition 12. Suppose (N1) holds, and there exists $0 \neq b \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}})$. We say the triple (A, B, a) is *reach controllable* if either $\mathcal{E}_{\mathcal{S}} = \emptyset$, or $\mathcal{E}_{\mathcal{S}} = \mathrm{co}\{\varepsilon_1, ..., \varepsilon_{\kappa_0+1}\}$ with $0 \leq \kappa_0 < \kappa$, and for each $\varepsilon_k \in I_{\mathcal{E}_{\mathcal{S}}}$ there exists $i \in I(\varepsilon_k)$ and $u_i > 0$ such that $Av_i + bu_i + a \in \mathcal{C}(v_i)$.

An example of a system which satisfies Assumption 3, but is not reach controllable was provided as Example 3 in Semsar-Kazerooni and Broucke (2014). We omit it here for lack of space. Along with our Assumption 3, results in Semsar-Kazerooni and Broucke (2014) make use of the following additional assumption:

(A0) $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{o_1, \cdots, o_{\kappa+1}\}$ is a κ-dimensional simplex with $1 \leq \kappa < n$.

Under assumptions (A0)-(A3) and (N1)-(N2), it has been shown in Semsar-Kazerooni and Broucke (2014) that the vertices of $\mathcal{E}_{\mathcal{S}}$ in the single-input case are indeed the vertices of $\mathcal{O}_{\mathcal{S}}$. However, this does not hold if we just remove the assumption (N2). This is shown in Example 13.

Example 13. Let $m = 1$ and let

$$A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}, B = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $\mathcal{S}$ be the standard orthogonal simplex. We note that $\mathcal{C}(v_0) = \{x \in \mathbb{R}^3 \mid x_j \geq 0, j = 1, 2, 3\}$, while $\mathcal{C}(v_i) = \{x \in \mathbb{R}^3 \mid x_j \geq 0, j \neq i\}$.

It can be verified by direct calculations that (N1) holds, and we can additionally easily calculate $\mathcal{O} = \{x \mid Ax + a \in \mathcal{B}\}$: it equals $\mathrm{aff}\{v_0/2 + v_1/2, v_0/2 + v_2/4 + v_3/4\}$. Hence, $\mathcal{O}_{\mathcal{S}} = \mathcal{O} \cap \mathcal{S} = \mathrm{co}\{v_0/2 + v_1/2, v_0/2 + v_2/4 + v_3/4\}$. Thus, $\mathcal{O}_{\mathcal{S}}$ clearly satisfies (A0), (A2) and (A3).

Finally, $\mathcal{E}$ is given by $\{x \mid Ax + a = 0\}$, and in this particular case, we can easily calculate $\mathcal{E} = \mathcal{E}_{\mathcal{S}} = \{v_0/2 + v_1/4 + v_2/8 + v_3/8\}$. Clearly, $\mathcal{E}_{\mathcal{S}}$ then satisfies (A1). So, this system satisfies (A0)-(A3) and (N1). However, $\mathcal{E}_{\mathcal{S}}$ is not a vertex of $\mathcal{O}_{\mathcal{S}}$. The resulting picture is given in Figure 13.

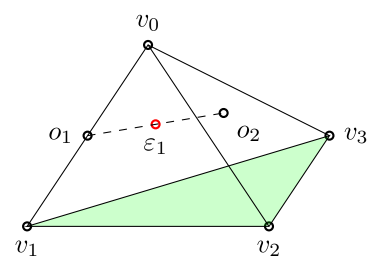

Fig. 2. Illustration of Example 13. $\mathcal{O}_{\mathcal{S}}$ is depicted by a dashed line, and the exit facet $\mathcal{F}_0$ is painted green.

In the remainder of this paper, we will be assuming (N2). Thus, by Semsar-Kazerooni and Broucke (2014), we may assume without loss of generality that $\mathcal{E}_{\mathcal{S}} = \{o_1, \ldots, o_{\kappa_0+1}\}$, and, by Theorem 9, we know that $\kappa_0 = 0$.

The main result of Semsar-Kazerooni and Broucke (2014), which connects reach controllability with solvability of the RCP, contains the assumption that $Ao_{\kappa+1} + a \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}})$. Along with being unintuitive, this assumption seems to significantly constrain the potential values of $Ao_{\kappa+1} + a$, allowing only those values which lie in the ray of $\mathcal{B}$ that points through $\mathrm{cone}(\mathcal{O}_{\mathcal{S}})$. We now show that this assumption is in fact unnecessary. Our proof relies on the use of assumption (N1) to derive the zero structure of matrices A and a, and invoke Lemma 8.

Theorem 14. Suppose (A0) and Assumption 3 hold. Suppose $m = 1$, and $\mathcal{E}_{\mathcal{S}} = \{o_1, \ldots, o_{\kappa_0+1}\}$. Then, if $\kappa > \kappa_0$, it is impossible that $Ao_k + a \notin \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}})$ for all $k = \kappa_0 + 2, \ldots, \kappa + 1$.

Proof. The conditions of Theorem 9 hold, so $\kappa_0 = 0$, $\mathcal{E}_{\mathcal{S}} = \{o_1\}$ and $Ao_k + a \neq 0$, $k = 2, \ldots, \kappa + 1$.

Suppose by way of contradiction that $Ao_k + a \notin \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}})$ for all $k = 2, \ldots, \kappa + 1$. Since $0 \neq Ao_k + a \in \mathcal{B}$ for $k = 2, \ldots, \kappa + 1$ and $m = 1$, we have

$$-(Ao_k + a) \in \mathcal{B} \cap \mathrm{cone}(\mathcal{O}_{\mathcal{S}}),\ k = 2, \ldots, \kappa + 1. \tag{18}$$

Assume without loss of generality that $\mathcal{S}$ is a standard orthogonal simplex, i.e., $v_0 = 0$, $v_i = e_i$ and $h_i = -e_i$ for all $i \in I$. Also, assume there exists $l \geq 0$ such that $I(o_1) \cap \cdots \cap I(o_{\kappa+1}) = \{0, 1, 2, \ldots, l\}$. By Lemma 4, $\text{cone}(\mathcal{O}_\mathcal{S}) = \{y \mid h_j \cdot y \leq 0 \text{ for all } j = l+1, \ldots, n\}$. Combining this with (18), we obtain

$$h_j \cdot (Ao_k + a) \geq 0,\ k = 2, \ldots, \kappa+1,\ j = l+1, \ldots, n. \quad (19)$$

Since we know $I \backslash I(o_k) \subseteq \{l+1, \ldots, n\}$ for all $k = 2, \ldots, \kappa+1$, (19) gives

$$h_j \cdot (Ao_k + a) \geq 0,\ k = 2, \ldots, \kappa+1,\ j \in I \backslash I(o_k). \quad (20)$$

From (N1) we also know

$$h_j \cdot (Av_i + a) \leq 0,\ i = 0, \ldots, n,\ j \in I \backslash \{i\}.$$

This implies

$$h_j \cdot (Av_i + a) \leq 0,\ k = 1, \ldots, \kappa+1, \quad i \in I(o_k),\ j \in I \backslash I(o_k). \quad (21)$$

Combining (20), (21) and convexity, we obtain

$$h_j \cdot (Ao_k + a) = h_j \cdot (A \sum_{i \in I(o_k)} \alpha_i^{o_k} v_i + a) = = \sum_{i \in I(o_k)} h_j \cdot (Av_i + a) = 0,\ k = 1, \ldots, \kappa+1,\ j \in I \backslash I(o_k). \quad (22)$$

Note that we include $k = 1$ here because $Ao_1 + a = 0$.

Then, since $\alpha_i^{o_k} > 0$, reapplying (21) we get

$$h_j \cdot (Av_i + a) = 0,\ k = 1, \ldots, \kappa+1, \quad i \in I(o_k),\ j \in I \backslash I(o_k). \quad (23)$$

Now, assume that $\{0, 1, \ldots, r_0\}$ is the set of all indices that appear in more than one index set $I(o_k)$. Hence, for any $j = r_0+1, \ldots, n$, j is an exclusive member of some $I(o_{k'})$, $k' = 1, \ldots, \kappa+1$. For any $i = 0, \ldots, r_0$, i is not an exclusive member of $I(o_{k'})$. Thus, there exists $k \in \{1, \ldots, \kappa+1\}$ such that $i \in I(o_k)$ and $j \in I \backslash I(o_k)$. From (23) we get

$$h_j \cdot (Av_i + a) = 0,\ i = 0, \ldots, r_0,\ j = r_0+1, \ldots, n. \quad (24)$$

We first invoke Lemma 6. By Lemma 1 in Semsar-Kazerooni and Broucke (2014), $\partial \mathcal{O}_\mathcal{S} \subseteq \partial \mathcal{S}$. Thus, the assumptions for Lemma 6 are satisfied. Hence, each index set $I(o_k)$, $k = 1, \ldots, \kappa+1$, has an exclusive member.

We now invoke Lemma 8 for $\mathcal{P} = \mathcal{O}_\mathcal{S}$. (I1) is satisfied by assumption (A2). (I2) is also satisfied by definition of r_0. (I3) is satisfied by our invocation of Lemma 6. Assumption (N1) holds. (E1) holds by (24). (E2) is satisfied by (22).

We now distinguish between two cases. First, suppose $\kappa + 1 = 2$. (E3) is certainly satisfied for at least $q = 0$, as $Ao_1 + a = 0$. Hence, by Lemma 8 $\text{rank}(A) \leq n - 1 = n - \kappa$. This implies $\dim(\mathcal{E}) \geq \kappa$. However, by (A2) $\dim(\mathcal{O}) = \dim(\mathcal{O}_\mathcal{S}) = \kappa$, and $\mathcal{E} \subseteq \mathcal{O}$. As both $\mathcal{E}$ and $\mathcal{O}$ are affine spaces, this implies $\mathcal{E} = \mathcal{O}$, i.e., $\mathcal{O}_\mathcal{S} = \mathcal{E}_\mathcal{S}$. This means $\kappa = \kappa_0$, and we reach a contradiction.

Suppose now that $\kappa + 1 > 2$. Then, without loss of generality we may assume

$$I(o_2) \cap \cdots \cap I(o_{\kappa+1}) = \{0, 1, 2, \ldots, l'\} \quad (25)$$

for some $l' \geq l$.

As $\mathcal{B} \ni Ao_k + a \neq 0$ for $k = 2, \ldots, \kappa+1$, we know $Ao_k + a = \lambda_k b$ for some $\lambda_k \neq 0$, $k = 2, \ldots, \kappa+1$. By (22) we get

$$h_j \cdot b = 0,\ k = 2, \ldots, \kappa+1,\ j \in I \backslash I(o_i). \quad (26)$$

By (25) and (26), $h_j \cdot b = 0$, $j = l'+1, \ldots, n$.

Hence, $h_j \cdot (Ao_k + a) = 0$, for all $k = 2, \ldots, \kappa+1$, $j = l'+1, \ldots, n$.

By (25) and $\kappa + 1 > 2$, all exclusive members of $I(o_k)$, $k = 2, \ldots, \kappa+1$ are contained in $\{l'+1, \ldots, n\}$ Thus, $h_j \cdot (Ao_k + a) = 0$ for all $k = 1, \ldots, \kappa+1$ and j which are exclusive members of $I(o_k)$. We again included $k = 1$ above as $Ao_1 + a = 0$.

We can now take $q = \kappa$ in (E3) of Lemma 8. By Lemma 8, $\text{rank}(A) \leq n - \kappa - 1$. This implies $\kappa = \dim(\mathcal{O}) \geq \dim(\mathcal{E}) \geq \kappa + 1$. ■

A generalization of Theorem 4 of Semsar-Kazerooni and Broucke (2014) follows immediately.

Theorem 15. Let $m = 1$, and let (A0) and Assumption 3 hold. System (1) is RCP solvable by affine feedback if and only if (A, B, a) is reach controllable.

Proof. The theorem was proved in Semsar-Kazerooni and Broucke (2014) under the assumption that $Ao_{\kappa+1} + a \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$. Clearly, by renaming vertices this is equivalent to stating $Ao_s + a \in \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$ for any $s > \kappa_0 + 1$. Assume otherwise: then, $Ao_s + a \notin \mathcal{B} \cap \text{cone}(\mathcal{O}_\mathcal{S})$ for all $s > \kappa_0 + 1$. However, this was shown to be impossible by the previous theorem. We are done. ■

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