Necessary and Sufficient Conditions for Reachability on a Simplex ⋆

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Abstract

In this paper we solve the general problem of designing a feedback controller to reach a set of facets of an n-dimensional simplex in finite time, for a system evolving with linear affine dynamics. Necessary and sufficient conditions are presented in the form of bilinear inequalities on the vertices of the simplex. By exploiting the structure of the problem, the bilinear inequalities are converted to a series of linear programming problems.

Key words: hybrid systems; reachability; simplex; piecewise-linear affine system.

1 Introduction

In this paper we present reachability results for *linear affine systems* defined on a polytope or simplex. We consider the synthesis of controllers that achieve a particular reachability specification on a simplex and we develop necessary and sufficient conditions for the existence of such a controller. The work of this paper is motivated by previous results on reachability and controllability on polytopes presented by Habets and van Schuppen [9], and is related to work on invariance of polyhedral sets as presented in the survey paper by Blanchini [3, pp. 1754-1756].

Our results have implications for reachability analysis in hybrid system theory. A hybrid system combines the continuous model of several systems or subsystems with discrete transitions that occur in between or within them. In recent years, the interest in hybrid systems has grown considerably and various general results have been presented in proceedings; for example [1]. Our interest lies specifically with linear affine systems introduced by Sontag [18]. The problem of reaching a particular facet of an n-dimensional polytope has been extensively studied by Habets and van Schuppen [8,9]. The present paper is motivated by their results. However, we relax a key condition in [9], allowing to find necessary and sufficient conditions for a more general problem of steering a state of the system to a set of facets of a simplex. We show through examples that even though the conditions for reachability of a facet in [9] are infeasible, we may still be able to find a continuous state feedback which guarantees the control objective. Similar necessary and sufficient conditions as the ones presented here are found in [10,15]. The present paper goes further by presenting a general algorithm that converts the bilinear inequalities that arise to a series of linear programs.

We have also investigated a special version of the reachability problem that is studied here in [16]. The distinction between the two investigations is as follows. In [16] we combine Corollary 9 on necessary and sufficient conditions for existence of a linear affine feedback solving the reachability problem with the additional restriction that $\text{rank}(B) = n-1$. The extra assumption allows one to overcome the complexity of the bilinear inequalities appearing in Corollary 9 and replace them by at most n linear programs. In this way we can bypass the complexity of the general algorithm presented in Section 4. The combined results suggest that improvements in complexity of the general solution of Section 4 are achievable if information about the number of control inputs is taken into account. An example where the method of [16] does not solve the reachability problem when the number of inputs is not $n-1$ is presented in Section 4.

Several other interesting publications are related to our work. For instance, Lee and Arapostathis [13] investigated global controllability of piecewise-linear affine

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hypersurface systems, while Veliov and Krastanov [20] studied local controllability of a system which is linear on two half-spaces. Related work on invariant polyhedral sets of linear systems has been studied by Vassilaki and Bitsoris [19], Castelan and Hennet [4], Gutman and Cwikel [7], and Blanchini [3]. In [7] invariance of polyhedra for discrete time systems is studied, and the ideas appearing in [9] to derive an affine feedback controller by using only the values of control at the vertices of a simplex and by exploiting convexity, are proposed in [7]. In [2], the problem of finding control values at the vertices to render a polyhedron invariant is formulated as a linear programming problem. The survey paper by Blanchini [3] on set invariance in control provides many other related references. Finally our Corollary 7 is a well-known result and appears for instance in [6].

1.1 Problem Statement

First, some conventions on notation are required. The set of vertices of a polytope P will be denoted as V, and S will denote an n-dimensional simplex. A *facet* of a polytope P is an $(n-1)$-dimensional intersection of the polytope with a supporting hyperplane. Associated with each facet F_i is a normal vector h_i that by convention points out of P. Finally, we use the convention that F_i is the convex hull of $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}\}$ where $v_j \in V$.

Consider a multi-input linear affine system, defined on a simplex S:

$$\dot{x} = Ax + Bu + a \,, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $a \in \mathbb{R}^n$. A facet F_i of S is called *restricted* if no trajectory of the system exits through the facet. Let I be a given set of indices of the facets to be made restricted by proper choice of control input for the system (1). We assume at least one facet F_1 is unrestricted, i.e. $1 \notin I$. Also, for each $v \in V$, let $I_v = \{k \mid k \in I, v \in F_k\}$. We consider the following problem.

Problem 1 *Let S be an n-dimensional simplex with a set of restricted facets F_j, $j \in I$. For the system* (1)*, construct a linear affine feedback $u = Kx + g$, with $K \in \mathbb{R}^{m\times n}$ and $g \in \mathbb{R}^m$, such that for each initial condition $x_0 \in S$ there exist a time $t_0 \geq 0$ and an $\epsilon > 0$ such that*

(1) $\forall t \in [0, t_0]$, $x(t) \in S$,
(2) $x(t_0) \in F_k$, *for some* $k \notin I$
(3) $\forall t \in (t_0, t_0 + \epsilon)$, $x(t) \notin S$.

The problem states that we must design a linear affine u to enforce all controlled trajectories originating inside S to escape S through some F_k, which is not restricted (notice that nothing is said about the trajectory for $t > (t_0 + \epsilon)$). In contrast with [9], we do not try to restrict n facets, but rather an arbitrary number. Also, we drop the restriction that t_0 must be the first time at which the state reaches the exit facet. Thus, the current problem is a generalization of the one in [9]. The implications of the extra restriction in [9] will be highlighted in the sequel.

2 Linear Affine Systems Defined on Polytopes

In this section, we present properties of linear affine systems; in particular, the location of equilibria under state feedback and we introduce the notion of feasible directions. Also, three essential results from [9] are reviewed.

Let $\mathcal{O}$ be the set of points where the autonomous vector field $Ax + a$ lies in $\mathcal{R}(B)$, the range of B. That is,

$$\begin{aligned} \mathcal{O} &:= \{x \mid \beta \cdot (Ax + a) = 0,\ \beta \in ker(B^T)\} \\ &= \{x \mid (Ax + a) \in \mathcal{R}(B)\} \,. \end{aligned}$$

The following is a basic property about the relationship between the set $\mathcal{O}$ and the equilibria of the system (1).

Lemma 2 *Given* (1)*, suppose the control input is of feedback form $u = f(x)$. If x_0 is an equilibrium point of* (1)*, then $x_0 \in \mathcal{O}$.*

Next, we define the allowable directions of flow at a point $x \in \mathbb{R}^n$

$$R_x = \begin{cases} \{y \mid y = c(Ax + a) + Bu,\ c > 0,\ u \in \mathbb{R}^m\} & x \notin \mathcal{O} \\ \{y \mid y = Bu,\ u \in \mathbb{R}^m\} & x \in \mathcal{O} \end{cases} \tag{2}$$

The cone R_x is called the set of *feasible directions* at x.

Next we review three results from [9]. The main ideas are as follows. Given a simplex S, we would like to impose certain conditions, called *restriction conditions* and *flow conditions* at the vertices of the simplex, which guarantee that trajectories may not leave from the restricted facets, but may leave from the unrestricted facets. The restriction conditions, introduced in [9], dictate that the vector field may not point "out" of a restricted facet. Lemma 3 shows that the restriction conditions guarantee that no trajectory can exit a restricted facet. The flow conditions are a new element contributed in this paper (see also [15,10]), which provide the additional element to solve the more general reachability problem of this paper. A flow condition imposes that the vector field points in a particular direction with respect to a given vector $\xi \in \mathbb{R}^n$. Lemma 4 below says that a restriction (or flow) condition on a facet (or on a simplex) can be achieved simply by imposing the restriction or flow condition on the vertices of the simplex. Thus, the procedure is to write restriction and flow conditions at the vertices only and solve them for the control values at the vertices. Lemma 5 shows that once those control values at the vertices are obtained, one can construct a linear affine feedback control defined on the entire simplex such that the closed-loop system achieves the design objective.

Lemma 3 ([9,15]) *Consider the linear affine system $\dot{x} = Ax + a$, $a \in \mathbb{R}^n$ defined on a polytope P. Suppose that for facets F_i, $i = 1, ...k$ with normal vectors h_i, respectively, the following conditions hold: $h_i \cdot (Ax + a) \leq 0$ for all $x \in F_i$. Then all trajectories originating in P that leave P do so via an unrestricted facet F_j, $j \notin \{1, \ldots, k\}$.*

Lemma 4 ([9]) *Consider system (1) defined on a polytope P with vertices $V = \{v_1, \ldots, v_k\}$. Given $\xi \in \mathbb{R}^n$, $\xi \neq 0$, for all $x \in P$ there exists an input $u \in \mathbb{R}^m$ such that $\xi \cdot (Ax + Bu + a) < 0$ if and only if for all $v_j \in V$ there exists $u_j \in \mathbb{R}^n$ such that $\xi \cdot (Av_j + Bu_j + a) < 0$.*

Note that in the previous Lemma $<$ can be replaced with any of $\{\leq, >, \geq, =\}$. In particular, we use $\leq$ when dealing with restriction properties.

Lemma 5 ([9]) *Consider two sets of points $\{v_1, \ldots, v_{n+1}\}$, $v_j \in \mathbb{R}^n$ and $\{u_1, \ldots, u_{n+1}\}$, $u_j \in \mathbb{R}^m$. Suppose the v_j's are affinely independent. Then there exists a unique matrix $K \in \mathbb{R}^{m \times n}$ and a unique vector $g \in \mathbb{R}^m$ such that for each v_j, $u_j = Kv_j + g$.*

Since a simplex is the convex hull of $n+1$ affinely independent points, it is now clear that if we enforce input values $u_1, ..., u_{n+1}$ at the $n+1$ vertices of the simplex, then we can also construct a corresponding linear affine controller $u = Kx + g$.

3 Necessary and Sufficient Conditions

This section derives necessary and sufficient conditions for the general problem of reaching a set of facets of an n-dimensional simplex in finite time, for a system evolving with linear affine dynamics. The first result is central to this development.

Theorem 6 *Consider the linear affine system $\dot{x} = Ax + a$ with $x, a \in \mathbb{R}^n$, and a compact, convex set P. We have $Ax + a \neq 0$ for all $x \in P$ if and only if there exists a $\xi \in \mathbb{R}^n$ such that $\xi \cdot \dot{x} = \xi \cdot (Ax + a) < 0$ for all $x \in P$.*

PROOF.
($\Leftarrow$) Since $\xi \cdot (Ax + a) < 0$ for all $x \in P$, then clearly for all $x \in P$, $Ax + a \neq 0$.
($\Rightarrow$) Since P is compact and convex it follows that the image of P under the map $x \mapsto Ax + a$, denoted by $C_1 = AP + a$ is also compact and convex and does not contain the origin, by assumption. Thus, letting $C_2 = \{0\}$ and using the Separating Hyperplane Theorem [[14] pg.98] there exists a hyperplane H that separates C_1 and C_2 strongly. In other words, there exists $\epsilon > 0$ and some $\xi \in \mathbb{R}^n$ such that for all $x \in P$, $\xi \cdot (Ax + a) \leq -\epsilon$, or $\xi \cdot (Ax + a) < 0$. $\square$

A consequence of the above theorem is the following corollary, which ensures that all trajectories originating in a compact, convex set P containing no equilibria eventually leave the set.

Corollary 7 *Consider the system $\dot{x} = Ax + a$, with $x, a \in \mathbb{R}^n$. Let $P \subseteq \mathbb{R}^n$ be compact and convex. Suppose that for all $x \in P$, $Ax + a \neq 0$. Then, for each $x_0 \in P$, the trajectory starting at x_0 eventually leaves P, i.e. $x(t_1) \notin P$ for some $t_1 > 0$.*

The background results and the above observations lead to the first solution to Problem 1.

Theorem 8 *Consider an affine system $\dot{x} = Ax + Bu + a$, with $x \in S$ and $u \in \mathbb{R}^m$. Problem 1 is solvable if and only if there exists a linear affine control u with $u(v_1) = u_1, \ldots, u(v_{n+1}) = u_{n+1}$ such that the closed loop system has no equilibria in S and the restriction conditions*

$$h_i \cdot (Av_j + Bu_j + a) \leq 0 \quad j \in \{1, ..., n+1\}, \ i \in I_{v_j},$$

are satisfied.

PROOF.
($\Rightarrow$) It is obvious that the closed loop system will not have an equilibrium in S if Problem 1 is solved. For the proof of necessity of the restriction conditions, see [9].

($\Leftarrow$) By assumption, for the set of vertices $\{v_1, \ldots, v_{n+1}\}$ there exists a corresponding set of inputs $\{u_1, \ldots, u_{n+1}\}$. Invoking Lemma 5, there exists a linear affine control $u = Kx + g$, which guarantees that the desired input values are achieved at each vertex. Now we must show that the resultant input $u = Kx + g$ solves Problem 1. First, by substituting for u, we obtain $\dot{x} = (A + BK)x + (Bg + a) = \tilde{A}x + \tilde{a}$, and from the assumption that for all $x \in S$, $\tilde{A}x + \tilde{a} \neq 0$, Corollary 7 guarantees that all trajectories of this system will eventually leave S. That is, for each initial condition $x_0 \in S$, there exists a time $t_1 > 0$ such that the trajectory starting at x_0 satisfies $x(t_1) \notin S$. Now it is evident that this implies there exists $t_0 < t_1$ and $\epsilon > 0$ such that $x(t_0) \in \partial S$ and $x(t) \notin S$ for all $t \in (t_0, t_0 + \epsilon)$. Finally, using Lemma 4 and Lemma 3, the trajectory cannot leave via the restricted facets. This concludes the proof. $\square$

Theorem 8 gives conditions for the solvability of Problem 1. The restriction conditions agree with the results in [9], while the equilibrium condition introduces the missing link for solving the general problem of reaching a desired facet in finite time. However, if we were to construct an algorithm based on the conditions of Theorem 8, we would only be able to satisfy the restriction conditions and hope that the equilibrium condition would hold with the chosen controller. Instead, from Theorem 6 we can replace the condition of no equilibria in a compact, convex set by the existence of a $\xi \in \mathbb{R}^n$ such that for all

$x \in P$, $\xi \cdot (Ax + a) < 0$. This observation leads to the main result of this section.

Corollary 9 *Consider the system $\dot{x} = Ax + Bu + a$, with $x \in S$. Problem 1 is solvable if and only if there exists a set of inputs $u_1, \ldots, u_{n+1} \in \mathbb{R}^m$ and a vector ξ such that the following hold:*

1). Restriction Conditions: $h_i \cdot (Av_j + Bu_j + a) \leq 0$
$j \in \{1, \ldots, n+1\},\ i \in I_{v_j}$
2). Flow Conditions: $\xi \cdot (Av_j + Bu_j + a) < 0$
$j \in \{1, \ldots, n+1\}$.

Corollary 9 tells us that we only need to check several inequalities at the given vertices of the simplex. Moreover, if we know what the value of ξ is, then the problem reduces to solving a set of linear inequalities. In fact, the sufficient conditions presented in [9] are a specific case of Corollary 9, with ξ set to $-h_1$, and $I = \{2, ..., n+1\}$. The example presented next will illustrate a situation where setting ξ to h_1 will not solve the general problem of leaving via a particular facet or set of facets, but a different ξ will. The inequalities above are still problematic, as both ξ and u_i are unknown, leading to a set of bilinear inequalities. This problem will be addressed in Section 4.

4 Algorithm

In [16] it was shown that for systems with $n-1$ inputs, ξ could be selected from the set $\{h_2, ..., h_{n+1}\}$ in order to obtain linear inequalities for the flow conditions. To motivate the need for a general algorithm to find ξ, we first present an example in which $\xi \notin \{h_2, ..., h_{n+1}\}$, in contrast with the results of [16], and moreover, $S \cap \mathcal{O} \neq \emptyset$. The latter condition is relevant, for if $S \cap \mathcal{O} = \emptyset$, then by Lemma 2 and Theorem 8, the problem reduces to checking only the restriction conditions.

Example 10 *Consider the system*

$$\dot{x} = \begin{bmatrix} -1.5 & -0.5 & -0.9 \\ 0 & 0 & 0 \\ 0.5 & -0.5 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

defined on a simplex S with vertices $v_1 = (0,0,0)$, $v_2 = (1,-1,0)$, $v_3 = (1,1,0)$ and $v_4 = (0,0,1)$. Let $I = \{2,3,4\}$.

Notice, with the help of software provided by [21], we obtain

$$S \cap \mathcal{O} = \{v \mid v = \alpha(\frac{1}{2}, \frac{1}{2}, 0) + (1-\alpha)(\frac{11}{21}, \frac{-9}{21}, \frac{10}{21}),\ \alpha \in [0,1]\},$$

which is clearly non-empty.

Next, let's show that $\xi \in \{h_2, h_3, h_4\}$ will not satisfy the flow and restriction conditions simultaneously. First, if we set $\xi = h_2$ then at vertex v_2 the restriction and flow conditions are

$$\mathit{Restriction} = \begin{cases} h_3 \cdot (Av_2 + Bu_2 + a) \leq 0 \Rightarrow u_2 \geq 0 \\ h_4 \cdot (Av_2 + Bu_2 + a) \leq 0 \Rightarrow 0 \leq 2 \end{cases}$$
$$\mathit{Flow} \quad = \quad h_2 \cdot (Av_2 + Bu_2 + a) \leq 0 \quad \Rightarrow \quad u_2 < 0$$

and cannot be satisfied, i.e. $0 \leq u_2 < 0$. Next, if we set $\xi = h_3$, then by the same procedure, at vertex v_3 we obtain $1 < u_3 \leq -1$. Finally, by setting $\xi = h_4$, once again the flow condition does not hold at vertex v_3, i.e. we obtain the inequality $0 < 0$. Thus, $\xi \notin \{h_2, h_3, h_4\}$. It can also be verified that the conditions in [9], when $\xi = -h_1$, do not hold for this example.

The necessary and sufficient conditions of Corollary 9 require the solution of bilinear inequalities. The solution to bilinear inequalities is in general NP hard; thus a more computationally tractable solution is sought. In [16] we showed that $\xi \in \{h_2, ..., h_{n+1}\}$, thereby transforming the solution of the bilinear inequalities to the solution of at most n LP problems. Unfortunately, this does not hold in general, as seen in Example 10. In this section we present an algorithm to determine ξ for a linear affine system with m inputs. The analysis begins by interpreting the flow and restriction conditions geometrically using R_x, the set of feasible directions.

The restriction conditions at $v \in V$ dictate that the feasible directions are confined to lie in a polyhedral cone C_v given by

$$C_v := \bigcap_{k \in I_v} \{y \mid h_k \cdot y \leq 0\}.$$

Solvability of the restriction conditions at $v \in V$ is equivalent to the condition $C_v \cap R_v \neq \emptyset$. Solvability of both the flow and restriction conditions at $v \in V$ is equivalent to the condition $C_v \cap R_v \cap \{y \mid \xi \cdot y < 0\} \neq \emptyset$. Thus, a necessary condition for the solvability of the flow and restriction conditions at $v \in V$ is that $C_v \cap R_v \neq \{0\}$ and $C_v \cap R_v \neq \emptyset$. Notice that if $v \in \mathcal{O}$, then $C_v \cap R_v \neq \emptyset$ by definition of R_v, since $0 \in C_v \cap R_v$.

Consider the closed polyhedral cone $C_v \cap \overline{R}_v$, where $\overline{R}_v$ denotes the closure of R_v. This cone is finitely generated by a set of extremal vectors, denoted E_v. The following algorithm finds a legitimate ξ which satisfies the flow and restriction conditions for a system with an arbitrary number of inputs. If the algorithm fails to construct a ξ, then none exists and Problem 1 is infeasible.

Algorithm 1. We are given a linear affine system $\dot{x} = Ax + Bu + a$ defined on an n-dimensional simplex S and a set of restricted facets with indices in I.

(1) If $S \cap \mathcal{O} \neq \emptyset$, proceed to the next step. Otherwise, check the restriction conditions only. If they do not hold, stop, as there is no solution to Problem 1. Otherwise, go to the last step.
(2) At each $v \in V$, construct the cone $C_v \cap \overline{R}_v$ and find the set of extremal vectors E_v. If $E_v = \emptyset$, or if $v \notin \mathcal{O}$ and for all $e \in E_v$, $e \in \mathcal{R}(B)$, then stop, as there is no solution.
(3) Find a set $\tilde{E}$ of extremal vectors composed of one vector from each E_v, $v \in V$, such that no vector in $\tilde{E}$ can be written as a negative combination of any other vectors in $\tilde{E}$. If no set $\tilde{E}$ exists, then stop, as there is no solution.
(4) Solve the feasibility LP problem:

$$e_{v_i} \cdot \xi < 0\,, \qquad e_{v_i} \in \tilde{E} \cap E_{v_i}\,, i = 1, \ldots, n+1\,. \quad (3)$$

(5) Find the u_i's for each vertex v_i, by solving the restriction and flow conditions of Corollary 9, with ξ obtained from the previous step.
(6) Construct $u = Kx + g$ using Lemma 5.

Remark 11 *For the polyhedral operations needed for the algorithm one can use software provided by [11] or [21], among others. Notice that in Step (5), if e_{v_i} is a feasible direction then u_i can be found using* (2)*. That is, we can solve for u_i directly by considering*

$$e_{v_i} = \begin{cases} \begin{bmatrix} (Av_i + a) & B \end{bmatrix} \begin{bmatrix} c \\ u_i \end{bmatrix} & v_i \notin \mathcal{O} \\ Bu_i & v_i \in \mathcal{O} \end{cases}$$

Theorem 12 *Problem 1 is solvable if and only if Algorithm 1 terminates successfully.*

The proof involves the following points.

(i) If $S \cap \mathcal{O} = \emptyset$, and the restriction conditions can be satisfied with some $\{u_1, ..., u_{n+1}\}$, then Problem 1 is solvable. This point follows from Lemma 2 and Theorem 8.
(ii) If $E_v = \emptyset$, or if $v \notin \mathcal{O}$ and for all $e \in E_v$, $e \in \mathcal{R}(B)$, then Problem 1 is not solvable. This point follows from the following arguments. First, if $E_v = \emptyset$, then $C_v \cap R_v \subset C_v \cap \overline{R}_v = \emptyset$. Second, suppose $v \notin \mathcal{O}$ and for all $e \in E_v$, $e \in \mathcal{R}(B)$. However, with $v \notin \mathcal{O}$, for all $u \in \mathbb{R}^m$, $Av + Bu + a \notin \mathcal{R}(B)$. Thus $C_v \cap R_v = \emptyset$.
(iii) If there is a set $\tilde{E}$ of extremal vectors such that no vector in $\tilde{E}$ can be written as a negative combination of any other vectors in $\tilde{E}$, then the linear program (3) has a solution.
(iv) If there is no set $\tilde{E}$ with the aforementioned properties, then there is no $\xi \in \mathbb{R}^n$ satisfying the restriction and flow conditions.
(v) The use of $\overline{R}_v$ rather than R_v is sound. That is, if the LP problem (3) is solvable using vectors in $C_v \cap \overline{R}_v$, then it is also solvable with feasible directions in $C_v \cap R_v$.

Point (iii) is addressed by Lemma 13, which is an immediate consequence of the Infeasibility Theorem ([5], p.52), so its proof is omitted. In fact, the lemma shows that Step 3 of the algorithm can be skipped and one could go directly to constructing a $\tilde{E}$ and solving the LP problem of Step 4. Point (iv) is addressed by Lemma 14 and point (v) is addressed by Lemma 15.

Lemma 13 *Given a set of vectors $W := \{w_1, ..., w_k\}$, $w_i \in \mathbb{R}^n$, there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot w < 0$, for all $w \in W$ if and only if no $w \in W$ can be written as $w = \sum_{i=1}^{k} \alpha_i w_i$ with $\alpha_i \leq 0$.*

Lemma 14 *Let $\{y_v \in R_v \cap C_v \mid v \in V\}$ be a set of feasible directions, with one at each $v \in V$. If there exists $\xi \in \mathbb{R}^n$ such that $\xi \cdot y_v < 0$ for all $v \in V$, then there exists a set of extremal vectors $\{e_v \in E_v \mid v \in V\}$, one at each $v \in V$, such that $\xi \cdot e_v < 0$, for all $v \in V$.*

PROOF. Consider $v \in V$. By the Fundamental Theorem of Linear Inequalities ([17], p. 86) we can write $y_v \in C_v \cap R_v \subset C_v \cap \overline{R}_v$ as

$$y_v = \alpha_1 e_v^1 + \cdots + \alpha_k e_v^k\,, \qquad e_v^j \in E_v\,, \alpha_j \geq 0\,.$$

Note that there exists l such that $\alpha_l > 0$ since $y_v \neq 0$. Thus we have

$$\xi \cdot \left(\alpha_1 e_v^1 + ... + \alpha_k e_v^k\right) < 0.$$

This implies there exists l such that $\xi \cdot e_v^l < 0$. □

Lemma 15 *Consider $\xi \in \mathbb{R}^n$ and a set $\{e_v \in E_v \mid v \in V\}$ of extremal vectors. Suppose that for all $v \in V$, $\xi \cdot e_v < 0$. Then there exists a set of feasible directions $\{y_v \mid y_v \in C_v \cap R_v\}$ such that $\xi \cdot y_v < 0$ for all $v \in V$.*

PROOF. If for all $v \in V$, $e_v \in C_v \cap R_v$, then there is nothing to prove. Suppose instead for some $v \in V$, $e_v \notin C_v \cap R_v$ and $e_v \in C_v \cap \overline{R}_v$. There exists $\epsilon > 0$ such that $\xi \cdot e_v < -\epsilon$. We will show there exists $y \in C_v \cap R_v$ such that $\xi \cdot y < 0$. From point (ii) we know that there exists some $e_v' \in E_v \cap R_v$. Then by convexity $y = \lambda e_v' + (1-\lambda) e_v \in C_v \cap R_v$, for $\lambda \in (0, 1]$. Let $\lambda < \frac{\epsilon}{2}$. W.l.o.g assume $\|\xi\| = \|e_v\| = \|e_v'\| = 1$. It now follows:

$$\begin{aligned}(\lambda e_v' + (1-\lambda) e_v) \cdot \xi &= (\lambda(e_v' - e_v)) \cdot \xi + e_v \cdot \xi \\ &\leq |\ (\lambda(e_v' - e_v)) \cdot \xi\ | - \epsilon \\ &\leq \lambda \|e_v' - e_v\| \, \|\xi\| - \epsilon \\ &= 2\lambda - \epsilon < 0\,.\end{aligned}$$

□

Now we are ready to complete the motivating example presented in the beginning of the section.

Example 16 *We continue Example 10. First, by definition we have*

$$\mathcal{O} = \left\{ x \mid \begin{bmatrix} -1.5 & -0.5 & -0.9 \\ 0.5 & -0.5 & -1 \end{bmatrix} x = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

which results in $S \cap \mathcal{O} \neq \emptyset$, as previously shown. The next step of the algorithm is to construct the E_{v_i}'s. We obtain

$$\begin{aligned} E_{v_1} &= \{[1\ -1\ 0]^T,\ [1\ 1\ 0]^T\} \\ E_{v_2} &= \{[0\ 1\ 0]^T,\ [0\ 0\ 1]^T\} \\ E_{v_3} &= \{[-1\ -1\ 0]^T,\ [0\ -1\ 0]^T\} \\ E_{v_4} &= \{[1\ 1\ 10]^T,\ [1\ -1\ -10]^T\}\,. \end{aligned}$$

Now letting $\tilde{E} = \{[1\ -1\ 0]^T,\ [0\ 0\ 1]^T,\ [0\ -1\ 0]^T,\ [1\ -1\ -10]^T\}$ it can easily be verified that $\xi = [-1\ 1\ -0.2]^T$ will satisfy the restriction and flow conditions simultaneously. For instance, checking $\xi \cdot \tilde{e} < 0$, for all $\tilde{e} \in \tilde{E}$, concludes that indeed ξ exists. Once we have found the needed ξ we can proceed to solve the linear inequalities of Corollary 9 to obtain the needed inputs u_i for each vertex v_i from which a linear affine controller can be constructed satisfying the restrictions of Problem 1.

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