Reachability of a set of facets for linear affine systems with n-1 inputs

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Abstract

This paper provides new necessary and sufficient conditions for an n-dimensional linear affine system with $n-1$ inputs to reach an exit facet (or set of exit facets) of a simplex. The conditions reduce the original $\mathcal{N}P$-hard necessary and sufficient conditions to a set of at most n LP problems.

Index Terms

hybrid systems; reachability; simplex; linear affine systems.

I. Introduction

This paper considers linear affine systems defined on a simplex. Piecewise-linear affine systems were first introduced by Sontag [9]. The motivation for our work originates with the recent, compelling results obtained by Habets and van Schuppen [4], which have been generalized in [8], [5]. The goal of that research was to give necessary and sufficient conditions such that all trajectories initialized in a simplex escape via a facet, using linear affine state feedback.

The Reachability Problem has been recognized as a fundamental problem of hybrid system theory [1]. One variant of the problem is to consider the synthesis of a hybrid controller to satisfy a reachability specification for a continuous time system. In this paper we address this problem for linear affine systems with $n-1$ independent inputs. In [8] we showed that the solution to the problem of reachability of a facet depends on the feasibility of bilinear inequalities. Unfortunately, in general, the feasibility of Bilinear Matrix Inequalites has been shown by Toker and Özbay [10] to be an $\mathcal{N}P$-hard problem. By exposing the properties of linear affine systems,

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we present a straightforward procedure for determining when reachability of a set of facets is possible, thus avoiding the problem of solving the bilinear inequalities directly.

Linear affine systems are of interest for two reasons. First, this class contains linear systems. Second, this class arises when one linearizes a nonlinear system about a non-equilibrium point. Our study of systems with $n-1$ inputs follows a tradition established in the development of nonlinear geometric control theory; systems with $n-1$ inputs have particular geometric structure which can lead to sharp results [7], [6]. An important assumption in our method that simplifies the representation of feasible velocity vectors at a point (Lemma 3.2) is that inputs are unbounded. The introduction of input constraints within the proposed mathematical framework is an area for future research.

In addition to [4], several interesting publications are related to our work. For instance, Lee and Arapostathis [7] investigated global controllability of piecewise-linear hypersurface systems with $n-1$ inputs, while Veliov and Krastanov [12] studied local controllability of a system which is linear on two half-spaces. There is also a body of literature on invariance problems which can be viewed as a precursor to the recent work on reachability; reachability may be viewed as dual to invariance. The survey paper by Blanchini [2] on set invariance in control provides many other related references, such as the work on polyhedral invariance of Vassilaki and Bitsoris [11] and Castelan and Hennet [3]. While we do not study invariance, it is evident that the techniques of this paper can be applied to such problems.

The paper is organized as follows. The terminology and the problem statement are presented next. Section II summarizes the main result of [8] to give necessary and sufficient conditions to achieve the control objective. Properties of linear affine systems are examined in Section III leading to the main results of the paper on checking certain flow conditions.

A. Terminology

A subset M of $\mathbb{R}^n$ is called an *affine set* if $\lambda x + (1-\lambda)y \in M$ for every $x, y \in M$ and $\lambda \in \mathbb{R}$. Given a set $M \subset \mathbb{R}^n$, there exists a unique smallest affine set containing M called the *affine hull* of M, denoted as $\mathrm{aff}(M)$. A set of $q+1$ points $\{v_1, \ldots, v_{q+1}\}$ in $\mathbb{R}^n$ is said to be *affinely independent* if $\mathrm{aff}(\{v_1, \ldots, v_{q+1}\})$ is q-dimensional. The set of vertices of a polytope P will be denoted as V. If a set of points $\{v_1, \ldots, v_{n+1}\}$ is affinely independent, its convex hull is called an *n-dimensional simplex* S_n. For an n-dimensional polytope, there exist $j \geq n+1$

unit length *normals* $h_1,\dots,h_j \in \mathbb{R}^n$ and j non-zero reals $\alpha_1,\dots,\alpha_j \in \mathbb{R}$ such that $P_n := \{x \in \mathbb{R}^n \mid h_i \cdot x \leq \alpha_i, \quad i = 1,2,...,j\}$. By convention the normal vectors h_i point out of P_n. Based on this description, we define a *facet* to be the $(n-1)$-dimensional intersection of the polytope with a supporting hyperplane. If P_n is a simplex, then we use the convention that F_i is the convex hull of $\{v_1,\dots,v_{i-1},v_{i+1},\dots,v_{n+1}\}$.

B. Problem Statement

Consider a multi-input affine system given by

$$\dot{x} = Ax + Bu + a \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n-1}$, and $a \in \mathbb{R}^n$. We assume $rank(B) = n-1$. If the system (1) is defined on a simplex S_n and no trajectory originating in S_n is allowed to escape through a facet F_j of S_n, then we say that the facet is *restricted*. To that end, define the set $I = \{i \mid F_i \text{ is to be restricted}, \ i \neq 1\}$. Also, for all $v \in V$, let $I_v = \{k \mid k \in I, v \in F_k\}$.

Problem 1.1: Let S_n be an n-dimensional simplex with a set of restricted facets $\{F_j, j \in I\}$. For the system (1), construct a continuous feedback $u = f(x)$, $f : S_n \to \mathbb{R}^m$ such that for each initial condition $x_0 \in S_n$ there exist a time $t_0 \geq 0$ and an $\epsilon > 0$ such that

1) $\forall t \in [0, t_0]$, $x(t) \in S_n$,
2) $x(t_0) \in F_k$, for some $k \notin I$
3) $\forall t \in (t_0, t_0 + \epsilon)$, $x(t) \notin S_n$.

The problem states that we must design u to enforce all controlled trajectories originating inside S_n to leave S_n through some F_k which is not restricted.

II. Necessary and Sufficient Conditions

In this section we present necessary and sufficient conditions for the solution of Problem 1.1 by linear affine feedback. The first result pertains to the construction of a linear affine controller that achieves preselected control values at the vertices of a simplex.

Lemma 2.1 ([4]): Consider two sets of points $\{v_1,\dots,v_{n+1} \mid v_j \in \mathbb{R}^n\}$ and $\{u_1,\dots,u_{n+1} \mid u_j \in \mathbb{R}^m\}$. Suppose the v_j's are affinely independent. Then there exist unique matrices $K \in \mathbb{R}^{n\times n}$ and $g \in \mathbb{R}^m$ such that for each $j \in \{1,\dots,n+1\}$, $u_j = Kv_j + g$.

Since a simplex is defined as the convex hull of $n+1$ affinely independent points, from the above lemma we conclude that if for each vertex $v_j \in V$ a control value u_j is assigned according to some invariance and other conditions, then it is possible to construct a linear affine feedback $u = Kx + g$ that achieves those control values at the vertices.

Theorem 2.1 ([8]): Consider the system $\dot{x} = Ax + Bu + a$, with $x \in S_n$. There exists a feedback $u = Kx + g$, with $K \in \mathbb{R}^{m \times n}$ and $g \in \mathbb{R}^m$, which solves Problem 1.1 if and only if there exists a set of inputs $u_1, \ldots, u_{n+1} \in \mathbb{R}^m$ and a vector ξ such that

1). Invariance Conditions: $h_i \cdot (Av_j + Bu_j + a) \leq 0,\ j \in \{1, 2, ..., n+1\},\ i \in I_{v_j}$

2). Flow Conditions: $\xi \cdot (Av_j + Bu_j + a) < 0,\ j \in \{1, 2, ..., n+1\}$.

Theorem 2.1 tells us that we only need to satisfy several inequalities at the given vertices of the simplex. From these inequalities we obtain the control values $\{u_1, \ldots, u_{n+1}\}$ at the vertices and thence, from Lemma 2.1, we are able to construct the linear affine feedback $u = Kx + g$. However, the inequalities of the Flow conditions are bilinear. In the next two sections we will exploit the properties of linear affine systems to show that we only need to check n values for ξ to see if the inequalities of Theorem 2.1 can be met.

III. Properties of Linear Affine Systems

We define the hyperplane spanned by the columns of B passing through a point $x \in \mathbb{R}^n$ as $\mathcal{B}_x$ and the normal vector to this hyperplane as β_x. The dependence of β_x on x is to denote that a convention on the direction of β will be chosen based on x. That is, β_x is one of only two vectors, depending on the direction chosen. By definition, $\beta_x \cdot B = 0$. An important role is played by the set of points, denoted $\mathcal{O}$, where the autonomous vector field $Ax + a$ lies in $\mathcal{R}(B)$, the range of B. That is,

$$\mathcal{O} := \{x \mid \beta \cdot (Ax + a) = 0,\ \beta \in \mathcal{N}(B^T)\}\,.$$

See also [7] where the same set arises in the study of controllability problems. Notice that because any n normals h_k of the facets of S_n span $\mathbb{R}^n$, in particular, β_x can be written as a linear combination of $\{h_2, \ldots, h_{n+1}\}$[1]. Now we will choose a convention on the direction of β_x. This choice is made specifically so that using only β_x one can determine at a point x

[1] Vector β_x could be chosen as a linear combination of some other n vectors in $\{h_1, \ldots, h_{n+1}\}$. This change would effect the statements of Lemmas 4.1 and 4.3 below, but the ideas would be the same.

what are the possible directions of the vector field $Ax + Bu + a$ as a function of u. Thus, β_x, with its sense of direction specified, will provide a shorthand notation to identify what we call the "feasible directions" at each point (see Section III-A and Lemma 3.2). When $x \notin \mathcal{O}$, we choose the convention that $\beta_x \cdot (Ax + a) < 0$. When $x \in \mathcal{O}$, and β_x is spanned by either a strictly positive or strictly negative combination of $\{h_2, \ldots, h_{n+1}\}$, without loss of generality, we adopt the convention that β_x is spanned by the negative combination rather than the positive[2]. In the case when $x \in \mathcal{O}$ and β_x is not spanned by either a strictly postive or strictly negative combination of $\{h_2, \ldots, h_{n+1}\}$, then either direction for β_x may be chosen.

Before proceeding to other properties of linear affine systems we give a lemma which is our main tool to decide when a set of linear inequalities is trivially infeasible.

Lemma 3.1: Consider a set of k vectors $\{w_1, w_2, \ldots, w_k\}$, $w_i \in \mathbb{R}^n$ and a vector $\gamma = \lambda_1 w_1 + ... + \lambda_k w_k$ such that $\lambda_i \leq 0$ for all $i = 1, \ldots, k$. Then the set of inequalities $\gamma \cdot y < 0$ and $w_i \cdot y \leq 0$, $i = 1, \ldots, k$, has no feasible solution for $y \in \mathbb{R}^n$.

Proof: Consider $\gamma \cdot y = \lambda_1 w_1 \cdot y + ... + \lambda_k w_k \cdot y$. Since $w_i \cdot y \leq 0$ and $\lambda_i \leq 0$, we have $\gamma \cdot y \geq 0$. ■

A. Feasible Directions and Invariance Conditions

In this section we describe the set of feasible velocity vectors at a point x and we examine the relationship between this set and the Invariance conditions at a vertex. In Lemma 3.2 we explore the set of feasible velocity vectors. In Lemma 3.3 we identify a condition on β_{v_1} when the Invariance conditions are infeasible or the only feasible velocity vector at v_1 is zero.

First we define the allowable directions of flow at a point $x \in \mathbb{R}^n$

$$R_x = \begin{cases} \{y \mid y = c(Ax + Bu + a),\ c > 0,\ u \in \mathbb{R}^{n-1}\} & x \notin \mathcal{O} \\ \{y \mid y = c(Ax + Bu + a),\ c \in \mathbb{R},\ u \in \mathbb{R}^{n-1}\} & x \in \mathcal{O} \end{cases} .$$

If $x \notin \mathcal{O}$ then R_x is an open half space determined by B and $Ax + a$. If $x \in \mathcal{O}$ then $R_x = \mathcal{B}_x$. A vector w that originates at a point x and lies in R_x will be called a *feasible direction*. The following lemma gives a characterization of R_x in terms of β_x.

[2]If β_x were chosen as the positive combination, one would merely make some adjustments to the statements of Lemmas 4.1 and 4.3.

Lemma 3.2:

$$R_x = \begin{cases} \{y \mid \beta_x \cdot y < 0\} & x \notin \mathcal{O} \\ \{y \mid \beta_x \cdot y = 0\} & x \in \mathcal{O} \end{cases}.$$

A useful geometric interpretation of the Invariance conditions of Theorem 2.1 can be obtained in terms of R_x. The Invariance conditions at $v \in V$ dictate that the feasible directions are confined to lie in a polyhedral set C_v given by $C_v = \bigcap_{k \in I_v} \{y \mid h_k \cdot y \leq 0\}$. Solvability of the Invariance conditions at $v \in V$ is equivalent to the condition $C_v \cap R_v \neq \emptyset$. Solvability of both the Flow and Invariance conditions at $v \in V$ is equivalent to the condition $C_v \cap R_v \cap \{y \mid \xi \cdot y < 0\} \neq \emptyset$. It may happen that for $v_i \in \mathcal{O}$, $C_{v_i} \cap R_{v_i} = \{0\}$, so that v_i is an equilibrium point. It was shown in [8] that the presence of an equilibrium of the closed-loop dynamics in the simplex is equivalent to the failure of the Flow conditions. If we have one facet F_1 unrestricted, so $I = \{2, ..., n+1\}$, it is seen in the next lemma that vertex v_1 may be an equilibrium point of the closed-loop dynamics. Moreover, the lemma shows that both situations, $C_v \cap R_v = \emptyset$ and $C_v \cap R_v = \{0\}$, can be detected using only β_v, $v \in V$.

Lemma 3.3: Consider the system (1) defined on a simplex S_n.

(1) Let $I = \{2, ..., n+1\}$ and $v_1 \in \mathcal{O}$. If $\beta_{v_1} = \lambda_2 h_2 + \cdots + \lambda_{n+1} h_{n+1}$ with $\lambda_k < 0$, then $C_{v_1} \cap R_{v_1} = \{0\}$.

(2) Let $I \subset \{2, ..., n+1\}$ and $v \notin \mathcal{O}$. If $\beta_v = \lambda_2 h_2 + \cdots + \lambda_{n+1} h_{n+1}$ with $\lambda_k \leq 0$ for $k \in I_v$ and $\lambda_k = 0$ for $k \notin I_v$, then $C_v \cap R_v = \emptyset$.

Proof: (1) Consider $y \in C_{v_1} \cap R_{v_1}$. According to the Invariance conditions at v_1, $h_i \cdot y \leq 0$ for $i \in I$. Also, by Lemma 3.2 at $v_1 \in \mathcal{O}$, $\beta_{v_1} \cdot y = 0$ for all $y \in R_{v_1}$. Combined with the assumption on β_{v_1} it is easily verified that the only solution is $y = 0$.

(2) Let $y \in C_v \cap R_v$. Consider $\beta_v \cdot y = \sum_{i \in I_v} \lambda_i h_i \cdot y$. According to the Invariance conditions at v, $h_i \cdot y \leq 0$, for all $i \in I_v$, so $\beta_v \cdot y \geq 0$. But this contradicts Lemma 3.2 that at $v \notin \mathcal{O}$ and for all $y \in R_v$, $\beta_v \cdot y < 0$. ■

IV. Feasibility of Flow & Invariance Conditions

This section contains the main results of the paper. We give checkable conditions for the existence of ξ such that the Flow and Invariance conditions of Theorem 2.1 can be simultaneously satisfied. Our strategy is as follows. We replace the bilinear Flow conditions with a set of linear inequalities by fixing the value of ξ, i.e. the Flow conditions are only in terms of u_j. We show that if there is no ξ belonging to a particular finite set that yields a feasible solution of the Flow and Invariance conditions, then the reachability problem is not solvable. Because the Invariance conditions are necessary to solve the problem, we must assume that they hold in order to investigate solvability with our particular choices of ξ. For this reason we impose the following assumption.

Assumption 4.1: Throughout this section we assume that $I = \{2, ..., n+1\}$, unless otherwise specified. Also, we assume that the Invariance conditions are satisfiable. That is, $C_v \cap R_v \neq \emptyset$. In light of Lemma 3.3, this implies that if $v \notin \mathcal{O}$, then β_v is not a negative combination of $\{h_j \mid j \in I_v\}$.

First, we present a lemma that gives conditions when Problem 1.1 is always solvable. The setup is as follows. Consider $i \in \{1, \ldots, n+1\}$ and let $\beta_{v_i} = \lambda_2 h_2 + \ldots + \lambda_{n+1} h_{n+1}$. Suppose that we set $\xi = h_{n+1}$. Then with ξ fixed, the Flow and Invariance conditions translate to solving a set of linear inequalities at each vertex. More precisely at vertex v_i we must find a solution y to the constraints:

$$\beta_{v_i} \cdot y \leq_{\mathcal{O}} 0 \tag{2a}$$

$$h_j \cdot y \leq 0 \qquad j = 2, \ldots, i-1, i+1, \ldots, n \tag{2b}$$

$$h_{n+1} \cdot y < 0\,. \tag{2c}$$

The symbol $\leq_{\mathcal{O}}$ is "$=$" if $v_i \in \mathcal{O}$ and "$<$" if $v_i \notin \mathcal{O}$. The first equation follows from Lemma 3.2. The second equation comprises the Invariance conditions, while the third equation is the Flow condition. The above conditions are equivalent to $C_{v_i} \cap R_{v_i} \cap \{y \mid h_{n+1} \cdot y < 0\} \neq \emptyset$. Note that vector y is understood to be the vector field evaluated at v_i. If y satisfies constraints (2), then by the proof of Lemma 3.2 a control u_i can always be found to achieve a vector parallel to y at v_i. Finally, note that if $i = 1$ or $i = n+1$, then we have $h_j \cdot y \leq 0$ for all $j = 2, \ldots, n$.

Lemma 4.1: Suppose that, after reordering vertices $V \setminus \{v_1\}$ if necessary, there exists $v^* \in V \setminus \{v_1\}$, with $\beta_{v^*} = \lambda_2^* h_2 + \ldots + \lambda_{n+1}^* h_{n+1}$, such that one of the following conditions holds:

(a) $\lambda_{n+1}^* = 0$.

(b) $\lambda_{n+1}^* \lambda_n^* < 0$.

(c) $\lambda_j^* > 0$ for all $j = 2, \ldots, n+1$ and $v_1 \notin \mathcal{O}$.

Then the Flow and Invariance conditions of Theorem 2.1 with $\xi = h_{n+1}$ can be simultaneously satisfied.

Proof: First, we observe that since vectors $\{\beta_v\}_{v\in V}$ are parallel, if (a) holds at v^*, then it holds at all $v \in V$; if (b) holds at v^*, then it holds at all $v \in V$; and if (c) holds at v^*, then for all $v \in V$, either $\lambda_j < 0$ for all $j \in \{2,\dots,n+1\}$ or $\lambda_j > 0$ for all $j \in \{2,\dots,n+1\}$. We consider the constraints (2) for $i \in \{1,\dots,n+1\}$ under each of the three cases. Recall that $\beta_{v_i} = \lambda_2 h_2 + \cdots \lambda_{n+1} h_{n+1}$.

(a) Assume $\lambda^*_{n+1} = 0$. Let $H = [h_2 \ \dots \ h_{n+1}]^T$, a non-singular $n \times n$ matrix. Assume w.l.o.g that $\lambda^*_2,\dots,\lambda^*_r \neq 0$, and $\lambda^*_{r+1} = \cdots = \lambda^*_{n+1} = 0$, $r \le n$. Let $\epsilon_{n+1} > 0$ be an arbitrary constant and define an index j and constant ϵ_j as follows. If $i \in \{2,\dots,r\}$ set $j := i$ and $\epsilon_j := \lambda_i \neq 0$. If $i \in \{1, r+1,\dots,n+1\}$ and $v_i \notin \mathcal{O}$, by Assumption 4.1 there must exist $j' \in \{2,\dots,r\}$ such that $\lambda_{j'} > 0$, so set $j := j'$ and $\epsilon_j := \lambda_{j'}$. Now define the vector $\epsilon = [\epsilon_2 \cdots \epsilon_{n+1}]^T$ as follows:

$$\epsilon = \begin{cases} [0 \dots 0 \ -\epsilon_{n+1}]^T & \text{if} \quad v_i \in \mathcal{O}, \\ [0 \dots 0 \ -\alpha_j \ 0 \dots - \epsilon_{n+1}]^T & \text{if} \quad v_i \notin \mathcal{O}, \end{cases}$$

The constraints (2) are feasible at $v_i \in V$ if the following (more stringent) constraints are feasible:

$$[\lambda_2 \ \lambda_3 \dots \lambda_r \ 0 \ \dots \ 0] H y \ \le_{\mathcal{O}} \ 0 \qquad (3)$$

$$H y \ = \ \epsilon \qquad (4)$$

Note that (3) is exactly (2a). Equation (4) combines (2b)-(2c) with three primary differences. First, (2c) is replaced by $h_{n+1} \cdot y = -\epsilon_{n+1}$. Second, (4) includes an extra constraint on $h_i \cdot y$ which does not appear in (2b). In particular, when $i \in \{2,\dots,r\}$ and $v_i \notin \mathcal{O}$, the extra constraint is $h_i \cdot y \le -\epsilon_i$ (note that the sign of ϵ_i is irrelevant). Instead, when $v_i \in \mathcal{O}$, or when $i \in \{1, r+1,\dots,n+1\}$ and $v_i \notin \mathcal{O}$, the extra constraint is $h_i \cdot y \le 0$. The third difference occurs when $i \in \{1, r+1,\dots,n+1\}$ and $v_i \notin \mathcal{O}$. Then (4) has the more stringent constraint $h_j \cdot y \le -\epsilon_j$ where $\epsilon_j > 0$. In sum, a nonzero solution of (3)-(4) yields a solution of (2).

Now we solve (4) to obtain $y = H^{-1}\epsilon$. Then we can verify that (3) also holds since

$$[\lambda_2 \ \lambda_3 \dots \lambda_r \ 0 \ \dots \ 0]\epsilon = \begin{cases} 0 & \text{if} \quad v_i \in \mathcal{O}, \\ -\epsilon_j^2 < 0 & \text{if} \quad v_i \notin \mathcal{O}. \end{cases}$$

(b) Assume $\lambda^*_{n+1}\lambda^*_n < 0$. Let $H = [h_2 \; \dots \; h_{n+1}]^T$, select some $\epsilon_{n+1} > 0$, and define $\epsilon_n > 0$ as follows. If $v_i \in \mathcal{O}$, set $\epsilon_n = -\frac{\epsilon_{n+1}\lambda_{n+1}}{\lambda_n} > 0$. If $v_i \notin \mathcal{O}$, set $\epsilon_n > -\frac{\epsilon_{n+1}\lambda_{n+1}}{\lambda_n} > 0$. Finally, let $\epsilon = [0 \dots 0 \;\; -\epsilon_n \;\; -\epsilon_{n+1}]^T$. Then the constraints (2) are feasible if the following constraints are feasible:

$$[\lambda_2 \; \lambda_3 \dots \lambda_{n+1}]Hy \;\; \leq_{\mathcal{O}} \;\; 0 \tag{5}$$

$$Hy \;\; = \;\; \epsilon \,. \tag{6}$$

Solving the second equation, we have $y = H^{-1}\epsilon$. Considering the first constraint, we have $[\lambda_2 \; \lambda_3 \dots \lambda_{n+1}]\epsilon = -\epsilon_n\lambda_n - \epsilon_{n+1}\lambda_{n+1}$. Then the first constraint is satisfied.

(c) $\lambda^*_j > 0$ for all $j = 2, \dots, n+1$. W.l.o.g we can assume $v* = v_{n+1}$. Notice that $v_{n+1} \notin \mathcal{O}$ since if it were, we would have the convention that $\beta_{v_{n+1}}$ is a strictly negative combination of $\{h_2, \dots, h_{n+1}\}$.

(c1) $i = 1$ or $i = n+1$. If $i = 1$, by assumption $v_1 \notin \mathcal{O}$, so by Assumption 4.1, $\lambda_j > 0$, for all $j = 2, \dots, n+1$. If $i = n+1$, then the same is true for v_{n+1}. Let $H = [h_2 \; \dots \; h_{n+1}]^T$ and $\epsilon = [0 \dots 0 \; -\lambda_{n+1}]^T$. Then constraints (2) are feasible if constraints (5)-(6) are feasible. In particular, set $y = H^{-1}\epsilon$ to satisfy the second constraint. Then for (5), we have $[\lambda_2 \dots \lambda_{n+1}]\epsilon = -\lambda^2_{n+1} < 0$.

(c2) $i \in \{2, \dots, n\}$. Let $\overline{H} = [\beta_{v_i} \; h_2 \dots h_{i-1} \; h_{i+1} \dots h_{n+1}]^T$, a non-singular matrix as β_{v_i} is a strictly positive or strictly negative combination of $h_2, \dots, h_{n+1}$. Let $\epsilon = [-\epsilon_{v_i} \; 0 \dots \; 0 \; -\epsilon_{n+1}]^T$, with $\epsilon_{n+1} > 0$, and $\epsilon_{v_i} = 0$ if $v_i \in \mathcal{O}$, or $\epsilon_{v_i} > 0$ if $v_i \notin \mathcal{O}$. Then the constraints (2) are satisfied if the constraint $\overline{H}y = \epsilon$ is satisfied. The latter can be solved to obtain $y = \overline{H}^{-1}\epsilon$.

■

The next goal is to capture all the cases where Problem 1.1 is not solvable. Before doing so, a supporting lemma is needed.

Lemma 4.2: Consider n linearly independent vectors $\{w_1, \dots, w_n\}$ and a vector w_{n+1} that is a negative linear combination of the preceding n vectors. Then any vector $\gamma \in \mathbb{R}^n$ can be written as a negative combination of some n or less vectors among $\{w_1, \dots, w_n\}$.

Proof: First, since $w_{n+1} = -\alpha_1 w_1 - \alpha_2 w_2 - \dots - \alpha_n w_n$, with $\alpha_j > 0$ for all $j \in \{1, \dots, n\}$, then any w_i, $i \in \{1, 2, \dots, n\}$ can be written as a negative combination of the other vectors. That is,

$$w_i = \frac{-\alpha_1 w_1 \cdots - \alpha_{i-1} w_{i-1} - \alpha_{i+1} w_{i+1} \cdots - \alpha_n w_n - w_{n+1}}{\alpha_i}. \quad (7)$$

Also, since any n vectors among $\{w_1, \ldots, w_{n+1}\}$ are linearly independent, we can express γ as a linear combination of any n vectors. Suppose, w.l.o.g. that, after renumbering indices, if necessary, $\gamma = \lambda_2 w_2 + \cdots + \lambda_{n+1} w_{n+1}$, where either all $\lambda_j < 0$ or there is $j \in \{2, \ldots, n+1\}$ such that $\lambda_j > 0$, and $\lambda_k < 0$ for all $k < j$. Then we can substitute the expression (7) for w_j to obtain $\gamma = \lambda_2 w_2 + \cdots + \lambda_{j-1} w_{j-1} + \lambda_j (\frac{-\alpha_1 w_1 - \cdots - \alpha_{j-1} w_{j-1} - \alpha_{j+1} w_{j+1} - \cdots - \alpha_n w_n - w_{n+1}}{\alpha_j}) + \lambda_{j+1} w_{j+1} + \cdots + \lambda_{n+1} w_{n+1}$. This is a linear combination of n vectors $w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_{n+1}$ and the coefficients for w_k up to $k = j-1$ are negative. Iterating on this process, at each successive step a new linear combination of vectors among $\{w_1, \ldots, w_{n+1}\}$ appears, but importantly, one additional positive coefficient is replaced by negative coefficients. Thus, the process terminates in a finite number of steps when we obtain a negative combination of at most n vectors. ■

Lemma 4.1 gives the normal vector ξ that satisfies the Flow conditions of Theorem 2.1. The next Lemma provides conditions when no ξ can be found.

Lemma 4.3: Suppose that for all $v \in V \backslash \{v_1\}$, $\beta_v = \lambda_2^* h_2 + \ldots + \lambda_{n+1}^* h_{n+1}$ is such that $\lambda_i < 0$ for all $i \in \{2, \ldots, n+1\}$. Then there does not exist $\xi \in \mathbb{R}^n$ such that at each $v \in V$, the Invariance and Flow conditions simultaneously have a solution.

Proof: By contradiction, suppose there exists $\xi \in \mathbb{R}^n$ such that the Invariance conditions and $\xi \cdot y < 0$ are simultaneously solvable at each $v \in V$. Let $h_{n+2} := \beta_{v_i}$. Consider the following three conditions at $v_i \in V$: conditions (2b), $\xi \cdot y < 0$ and $h_{n+2} \cdot y \leq 0$. If these conditions cannot be solved, then neither can the Flow and Invariance conditions. To that end, we write ξ as a linear combination of at most n vectors among $\{h_2, \ldots, h_{n+2}\}$. Since, by assumption for all $i \in \{2, \ldots, n+1\}$, h_{n+2} is a negative combination of the preceding n vectors, we can invoke Lemma 4.2 to rewrite ξ as $\xi = -\alpha_2 h_2 - ... - \alpha_{k-1} h_{k-1} - \alpha_{k+1} h_{k+1} - ... - \alpha_{n+2} h_{n+2}$. Let $H := [h_2 \ldots h_{k-1} \; h_{k+1} \ldots h_{n+2}]^T$. Suppose $k \in \{2, \ldots, n+1\}$. Then the constraints at v_k are $Hy \leq 0$ and $\xi \cdot y < 0$. By Lemma 3.1 these constraints have no solution. Similarly, if $k = n+2$, we obtain a set of constraints at v_1 which have no solution. This provides the contradiction. ■

The following is the main result of the paper.

Theorem 4.1: Problem 1.1 is solvable if and only if there exists $\xi \in \{h_2, \ldots, h_{n+1}\}$ such that for all $v_i \in V$, $C_{v_i} \cap R_{v_i} \cap \{y \mid \xi \cdot y < 0\} \neq \emptyset$.

Proof: ($\Leftarrow$) Trivial. ($\Rightarrow$) If the Flow and Invariance conditions are simultaneously solvable for some $\xi \in \mathbb{R}^n$, then by Lemma 4.3, there is some $i \in \{2, \ldots, n+1\}$ such that β_{v_i} is not a strictly negative combination of $\{h_2, \ldots, h_{n+1}\}$. Then one of the cases of Lemma 4.1 is applicable. If case (c) applies and $v_1 \in \mathcal{O}$, then by the convention on direction of β_{v_1} and by Lemma 3.3, the Flow conditions fail, a contradiction. Instead, it must be that $v_1 \notin \mathcal{O}$. In that case, applying Lemma 4.1 and after renumbering indices if necessary, $\xi = h_{n+1}$ satisfies all constraints at each vertex. ■

Note that in the previous theorem, ξ may not be unique. So far we have concentrated on showing when it is possible to satisfy Problem 1.1 with n restricted facets. The next theorem shows that if fewer than n facets are restricted and the Invariance conditions can be satisfied, then there always exists $\xi \in \{h_2, \ldots, h_{n+1}\}$ such that the Flow and Invariance conditions can be satisfied and Problem 1.1 solved.

Theorem 4.2: If the cardinality of I is less than n, then under Assumption 4.1, Problem 1.1 is solvable and there exists $\xi \in \{h_2, \ldots, h_{n+1}\}$ such that for all $v_i \in V$, $C_{v_i} \cap R_{v_i} \cap \{y \mid \xi \cdot y < 0\} \neq \emptyset$.

Proof: We will show there exists $\xi \in \{h_2, \ldots, h_{n+1}\}$ such that for all $v \in V$, $C_v \cap R_v \cap \{y \mid \xi \cdot y < 0\} \neq \emptyset$, from which it follows that Problem 1.1 is solvable. Assume w.l.o.g $I = \{2, ..., k \mid 2 \leq k \leq n\}$ or $\emptyset$ (i.e at least F_1 and F_{n+1} are unrestricted). Denote $\beta_v = \lambda_2 h_2 + \cdots + \lambda_{n+1} h_{n+1}$. We can assume $\lambda_j \neq 0$ for all $j \in \{2, \ldots, n+1\}$; otherwise we can proceed as in Lemma 4.1, case (a). Let $\xi = h_2$. If we can satisfy the following inequalities at $v \in V$:

$$\begin{aligned} \beta_v \cdot y &\leq_{\mathcal{O}} 0 & \\ h_2 \cdot y &\leq -\epsilon_2 & \\ h_j \cdot y &\leq 0 & j = 3, \ldots, n \\ h_{n+1} \cdot y &= \epsilon_{n+1} & \end{aligned}$$

where $\epsilon_2 > 0$ and the sign of ϵ_{n+1} is irrelevant, then $C_v \cap R_v \cap \{y \mid h_2 \cdot y < 0\} \neq \emptyset$. Let $H = [h_2 \; h_3 \; ... \; h_{n+1}]^T$ and $\epsilon = [-\epsilon_2 \; 0 \; ... \; 0 \; \epsilon_{n+1}]^T$. Then the last three inequalities are satisfied by $y = H^{-1}\epsilon$. Checking the first inequality, we have $\beta_v \cdot y = [\lambda_2 \; ... \; \lambda_{n+1}]Hy = -\epsilon_2\lambda_2 + \lambda_{n+1}\epsilon_{n+1} \leq_{\mathcal{O}} 0$, so solving for ϵ_{n+1} yields the result. ■

The following algorithm finds an affine feedback controller that solves Problem 1.1.

Algorithm 4.1: Given a linear affine system $\dot{x} = Ax + Bu + a$ defined on an n-dimensional simplex S_n with vertices $v_i \in V$, and the set I.

1) Check if the Invariance and Flow conditions of Theorem 2.1 can be satisfied for some $\xi \in \{h_2, ..., h_{n+1}\}$, and solve the LP problem for the control values $\{u_1, \dots, u_{n+1}\}$ at the vertices. If for each ξ no solution of the LP problem exists, then the reachability problem is unsolvable.
2) Using the control values $\{u_1, \dots, u_{n+1}\}$ obtained in the previous step, solve

$$\begin{bmatrix} v_1^T & 1 \\ v_2^T & 1 \\ \vdots & \\ v_{n+1}^T & 1 \end{bmatrix} \begin{bmatrix} K^T \\ g^T \end{bmatrix} = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_{n+1}^T \end{bmatrix}. \tag{8}$$

for K and g and construct the linear affine controller $u = Kx + g$. See [4] for the origin of this equation.

Remark 4.1: If $I = \{2, ..., n+1\}$, then from Assumption 4.1 and Lemma 4.3 the expression $\beta_{v_j} = \lambda_2 h_2 + ... + \lambda_{n+1} h_{n+1}$ at each v_j tells us whether Problem 1.1 is feasible. If it is feasible, then Lemma 4.1 tells us how to pick the value of ξ for step one of the algorithm above. In particular, we can select $\xi = h_j$ if the corresponding β_{v_j} has coefficients satisfying either (a) $\lambda_j = 0$, (b) $\lambda_j \lambda_k < 0$ for some k, or (c) $\lambda_k > 0$ for all $k \in I$. Instead, if the cardinality of I is less than n, I is not empty, and Assumption 4.1 is satisfied, we can let $\xi = h_i$ for any $i \in I$ (this follows from the proof of Theorem 4.2). If I is the empty set then any h_i will work for ξ.

REFERENCES

[1] R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science,* Volume 2993, Springer-Verlag, Philadelphia, 2004

[2] F. Blanchini. Set invariance in control (Survey paper). *Automatica*, 35 (1999) 1747-1767.

[3] E.B. Castelan and J.C. Hennet. On Invariant Polyhedra of Continuous-Time Linear Systems. *IEEE Transactions on Automatic Control*, 38 (1993), 1680-1685.

[4] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica* 40:21-35 2004.

[5] L.C.G.J.M. Habets and J.H. van Schuppen. Control to facet problems for affine systems on simplices and polytopes - With applications to control of hybrid systems. *IEEE Conference on Decision and Control.*Seville Spain, December 2005.

[6] L.R. Hunt. N-dimensional controllability with $n-1$ controls. *IEEE Transactions on Automatic Control*, 27 (1982), 113-117.

[7] K.K. Lee and A. Arapostathis. On the controllability of piecewise-linear hypersurface systems. *Systems & Control Letters*, 9 (1987) 89-96.

[8] B. Roszak and M.E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *IEEE Conference on Decision and Control*, December 2005.

[9] E.D. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Trans. Automatic Control* 26:346-358, 1981.

[10] O. Toker and H. Özbay. On the $\mathcal{N}P$-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback. *Proceedings of the 1995 American Control Conference*, Seattle, Washington, 1995.

[11] M. Vassilaki and G. Bitsoris. Constrained regulation of linear continuous-time dynamical systems. *Systems & Control Letters*, 13 (1989), 247-252.

[12] V.M. Veliov and M. I. Krastanov. Controllability of piecewise linear systems. *Systems & Control Letters*, 7 (1986), 335-341.