Formation Control of Balloons: A Block Circulant Approach

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Abstract— **This paper focuses on control of a multiagent system of high-altitude balloons, and provides an efficient method to synthesize decentralized controllers for very large systems. Feedback laws are synthesized using a novel method which maintains the distributed structure in the balloon system: the balloons' geometry imposes a block circulant pattern in the system matrices, and the synthesis method guarantees that the feedback found will also be block circulant. The block circulant feedback has a clear method of decentralization, and this decentralized control can then be used on systems much larger than the one for which the feedback was initially designed. Simulations affirm that the response time is practically unchanged between the original system and the larger decentralized system, so the feedback is both robust and scalable.**

1. Introduction

Google's Project Loon aims to provide worldwide wireless Internet access by transmitting a signal through balloons floating in the upper atmosphere. The signal travels from balloon to balloon, which means that a global communication network can be created by spacing the balloons properly around the globe. In Google's words, a complete and reliable method to coordinate balloons will require "some complex algorithms and lots of computing power" [1]. This paper will show that formation control in a network of balloons can be formulated and solved in a direct way, based on the geometric framework of [2].

A balloon network inherently has a particular structure, which arises from the fact that the balloons are identical and should treat each other identically. This structure is encoded mathematically as a *block circulant* pattern in the matrices of the system model. The goal of this work is to demonstrate a method of block circulant feedback synthesis, and to use it for formation control of balloons.

A block circulant feedback has three main advantages over an "unpatterned" feedback. First, the block circulant pattern mirrors the inherent homogeneity and ring structure of the balloon system, thereby distributing control action in a coherent way. Second, the block circulant pattern lends itself naturally to decentralization by illuminating spatial localization in the control: small controller gains for faraway balloons can be set to zero without affecting the system response. In this way, each balloon only considers its nearest neighbours in determining its motion, rather than every balloon in the system. Third, the block circulant pattern provides scalability, in that new balloons added to the system can take this same nearest neighbours approach with the same controller gains, without affecting the overall system response. Through these three advantages, a feedback for the balloon network can be found without "lots of computing power" but, rather, in under five minutes on an average household computer.

2. Problem Statement

Consider a system of r balloons, distributed around the world at a certain latitude φ. The balloons all float in the lower stratosphere (18–25 km above sea level [3]), and they move on wind currents which are fairly consistent and predictable at those altitudes. The goal is to keep these balloons evenly distributed around the latitude line; that is, if the longitude of balloon i is given by Λ_i, and $\bar{\vartheta} = {}^{2\pi}/_{r}$ is the desired angle between balloons[1], then $\Lambda_{i_{\text{mod}(r)}+1} - \Lambda_i \to \bar{\vartheta}$ as $t \to \infty$ (for each $i = 1, \ldots, r$). With this even distribution, the balloons are assumed to be close enough to communicate with their neighbours.

3. Balloon Network Model

The balloons in this system are modelled as *superpressure balloons* [4]. The balloons have pumps and vents, which allow them to use air as ballast. The balloons ride on wind currents in the stratosphere, and can only control their longitudinal speeds indirectly by changing their altitudes to ride on different wind currents.

A. Nonlinear System Model

Each balloon ($i = 1, \ldots, r$) has four state variables: its longitudinal position s_i, its altitude h_i, its vertical velocity $\dot{h}_i$, and its internal air pressure P_i. Each balloon has one control input: the volumetric flow rate of air into or out of the balloon (via the pump or vents), Q_i. The dynamics of each state variable will briefly be described here.

Longitude Dynamics: The longitudinal position s_i of balloon i is the "arc distance" of balloon i from a predetermined reference angle $(i-1)\bar{\vartheta}$ (see Figure 1), where $\bar{\vartheta}$ is the desired angle between balloons. The reference frame for s_i and $(i-1)\bar{\vartheta}$ rotates at some to-be-determined rotational speed $\bar{\omega}$; this speed is sufficiently slow that the effects of a rotating reference frame, such as the centrifugal and Coriolis forces, can be ignored in the dynamical model.

The relationship between the balloon's longitudinal position s_i and absolute longitude Λ_i is given by

$$\Lambda_i(t) = \bar{\omega}t + (i-1)\bar{\vartheta} + \frac{s_i(t)}{(\rho_\oplus + h_i(t))\cos\varphi} \; , \qquad (1)$$

* This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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[1]In all forthcoming angular calculations, the result θ is interpreted as $(\theta + \pi)_{\text{mod}(2\pi)} - \pi$, which is in the range $-\pi \leq \theta < \pi$.

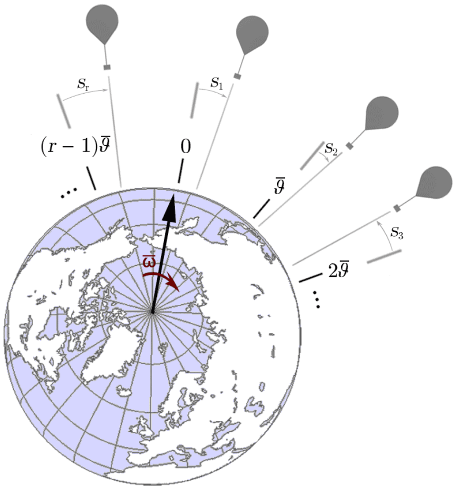

Fig. 1. Balloon Positions and Reference Frame

where $\rho_\oplus \doteq 6378$ km is the radius of the Earth, and φ is the balloon's latitude. Hence,

$$s_i(t) = (\rho_\oplus + h_i)(\cos\varphi)\left[\Lambda_i(t) - \bar{\omega}t - (i-1)\bar{\vartheta}\right] \ , \quad (2)$$

where the angle in square brackets is interpreted as in Footnote 1.

The rotational longitudinal speed $\dot{\Lambda}_i$ of balloon i is the speed of the wind current that it is riding on. The wind currents are assumed here to be a function of altitude only (while for simulations, a more realistic wind model will be used — see Section 4). Thus,

$$\dot{\Lambda}_i = \omega_{\text{wind}}(h_i) = \frac{1}{(\rho_\oplus + h_i)\cos\varphi} v_{\text{wind}}(h_i) \ , \quad (3)$$

where $\omega_{\text{wind}}(h_i)$ and $v_{\text{wind}}(h_i)$ are the rotational and translational/tangential wind speeds at altitude h_i.

Let $\bar{h} > 0$ be some to-be-determined altitude, and define $\bar{v} := (\rho_\oplus + \bar{h})(\cos\varphi)\bar{\omega}$. Observe that $\rho_\oplus \gg h_i$ and $\rho_\oplus \gg \bar{h}$, so $(\rho_\oplus + h_i)(\cos\varphi)\bar{\omega} \approx \bar{v}$. Then, taking the derivative of (2) and substituting (3) gives

$$\dot{s}_i = (v_{\text{wind}}(h_i) - \bar{v}) + \frac{s_i}{\rho_\oplus + h_i}\dot{h}_i \, . \quad (4)$$

Altitude Dynamics: The altitude dynamics of balloon i are a force balance between buoyancy, air resistance, and gravity. Let σ_i be a pump-use indicator, where $\sigma_i = 1$ if air is being pumped into balloon i, and $\sigma_i = 0$ otherwise. Let $m_{\text{tot},i}$ be the total mass of balloon i. Then,

$$m_{\text{tot},i}\ddot{h}_i + \sigma_i \dot{m}_{\text{tot},i}\dot{h}_i = F_{b,i} - F_{a,i} - F_{g,i} \, . \quad (5)$$

The total mass $m_{\text{tot},i}$ consists of two parts: the balloon structure and helium inside, which is a constant m_0; and the air inside the balloon, which is calculated using the ideal gas law. This gives

$$m_{\text{tot},i} = m_0 + \frac{P_i V_{\text{air}} M_{\text{air}}}{RT_{\text{amb}}} \ ,$$

where V_{air} is the constant volume of the balloon's air chamber, $M_{\text{air}} \doteq 29$ g/mol is the molar mass of air, $R \doteq 8.31$ J/(mol·K) is the ideal gas constant, and $T_{\text{amb}} \doteq 216$ K is the ambient air temperature outside the balloon in the lower stratosphere.

The rate of change of the mass of balloon i is related to the volumetric flow rate Q_i of air through its pump/vents using the ideal gas law:

$$\dot{m}_{\text{tot},i} = \frac{M_{\text{air}}}{RT_{\text{amb}}} P_{\text{amb}}(h_i) Q_i \ ,$$

where $P_{\text{amb}}(h_i)$ is the ambient air pressure outside the balloon, given by

$$P_{\text{amb}}(h_i) = K_P \mathrm{e}^{-\eta h_i} \quad (6)$$

with $K_P \doteq 127.76$ kPa and $\eta \doteq 0.157$ m^{-1} [5].

In (5), the gravity force is given by $F_{g,i} = m_{\text{tot},i} g$, where $g \doteq 9.75$ m/s^2 throughout the lower stratosphere. The air resistance force is given by $F_{a,i} = \beta \dot{h}_i$, where β is the damping coefficient. The buoyancy force is calculated using Archimedes' Principle and the ideal gas law, giving

$$F_{b,i} = \frac{P_{\text{amb}}(h_i) V M_{\text{air}}}{RT_{\text{amb}}} g \ ,$$

where V is the total volume of the balloon. Define the constants $\kappa := {}^{V_{\text{air}} M_{\text{air}}}/_{RT_{\text{amb}}}$ and $\nu := {}^{V}/_{V_{\text{air}}}$. Then, substituting all the above equations into (5) gives

$$\ddot{h} = \frac{\kappa\nu g - \left(\beta + \sigma_i \frac{\kappa}{V_{\text{air}}} P_{\text{amb}}(h_i) Q_i\right)\dot{h}_i}{m_0 + \kappa P_i} - g \, . \quad (7)$$

Pressure Dynamics: The pressure dynamics of balloon i are determined by the volumetric flow rate Q_i of air into and out of it. This air flow is controlled by a pump and vents. Thus,

$$\dot{P}_i = \frac{P_{\text{amb}}(h_i)}{V_{\text{air}}} Q_i \, . \quad (8)$$

State Model: Let $x_i := \text{col}(s_i, h_i, \dot{h}_i, P_i)$ and $u_i := Q_i$ be the state and input vectors of balloon i, respectively. Combining (4), (7), and (8) gives the dynamical equation

$$\dot{x}_i = \text{col}(\dot{s}_i, \dot{h}_i, \ddot{h}_i, \dot{P}_i) =: f_{\text{B}}(x_i, u_i)$$

for each balloon i. Stacking the balloons' state equations gives the dynamics of the overall system:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} f_{\text{B}}(x_1, u_1) \\ \vdots \\ f_{\text{B}}(x_r, u_r) \end{bmatrix} . \quad (9)$$

If $u_i = 0$, then balloon i will oscillate about (and eventually stabilize to) the altitude at which it is neutrally buoyant.

Control Objective: The control objective is for the balloons to be a prescribed distance apart: $\Lambda_{i_{\text{mod}(r)}+1} - \Lambda_i \to \bar{\vartheta}$ as $t \to \infty$. Using (1), and noting that $\rho_\oplus + h_i \approx \rho_\oplus$, this objective can be stated using output variables $z_i := s_{i_{\text{mod}(r)}+1} - s_i$, with the goal $z_i \to 0$ as $t \to \infty$ for each i.

B. Linearized System Model

An operating point for balloon i (at which $\dot{x}_i = 0$ and $z_i = 0$) is given by $(\bar{x}_i, \bar{u}_i) = (\bar{x}_\mathrm{B}, \bar{u}_\mathrm{B})$ for every $i = 1, \ldots, r$, where $\bar{x}_\mathrm{B} = \mathrm{col}(0, \bar{h}, 0, \bar{P})$ and $\bar{u}_\mathrm{B} = 0$. Furthermore, $\bar{h}$ and $\bar{P}$ are related by

$$\bar{P} = \nu P_\mathrm{amb}(\bar{h}) - \frac{m_0}{\kappa}\,. \tag{10}$$

Define linearized state and input variables for each balloon as the deviations from the operating point (e.g., $\Delta h_i = h_i - \bar{h}$). The rates of change of these deviations can be approximated near the operating point by linearizing the state equations (4), (7), and (8):

$$\begin{bmatrix}\Delta\dot{s}_i\\ \Delta\dot{h}_i\\ \Delta\ddot{h}_i\\ \Delta\dot{P}_i\end{bmatrix} = \overbrace{\begin{bmatrix}0 & \alpha & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -k_{hh} & -k_{h\dot{h}} & -k_{hP}\\ 0 & 0 & 0 & 0\end{bmatrix}}^{A_\mathrm{B}}\begin{bmatrix}\Delta s_i\\ \Delta h_i\\ \Delta\dot{h}_i\\ \Delta P_i\end{bmatrix} + \overbrace{\begin{bmatrix}0\\ 0\\ 0\\ k_{PQ}\end{bmatrix}}^{B_\mathrm{B}}\Delta Q_i \tag{11}$$

where, defining $\bar{m} := m_0 + \kappa\bar{P}$, the constants are given by $\alpha := ({dv_\mathrm{wind}}/{dh_i})|_{\bar{h}}$, $k_{hh} = {\nu g}/{\bar{m}} \cdot ({\partial P_\mathrm{amb}}/{\partial h_i})|_{\bar{h}}$, $k_{h\dot{h}} = {\beta}/{\bar{m}}$, $k_{hP} = {\nu\kappa^2 g}/{\bar{m}^2} \cdot P_\mathrm{amb}(\bar{h})$, and $k_{PQ} = {P_\mathrm{amb}(\bar{h})}/{V_\mathrm{air}}$. Stacking these equations gives a dynamical model for the overall system:

$$\begin{bmatrix}\Delta\dot{x}_1\\ \vdots\\ \Delta\dot{x}_r\end{bmatrix} = \overbrace{\begin{bmatrix}A_\mathrm{B} & & \\ & \ddots & \\ & & A_\mathrm{B}\end{bmatrix}}^{A}\begin{bmatrix}\Delta x_1\\ \vdots\\ \Delta x_r\end{bmatrix} + \overbrace{\begin{bmatrix}B_\mathrm{B} & & \\ & \ddots & \\ & & B_\mathrm{B}\end{bmatrix}}^{B}\begin{bmatrix}\Delta u_1\\ \vdots\\ \Delta u_r\end{bmatrix} \tag{12}$$

which has the LTI state space form $\Delta\dot{x} = A\Delta x + B\Delta u$.

Measurement: All the states of balloon i can be measured easily by sensors onboard the balloon's payload [3]. These "raw" sensor readings are reduced to two measurements per balloon; one is the balloon's internal air pressure, and the other is a linear combination of the remaining three states. The measurements are given by

$$\Delta y_i = \overbrace{\begin{bmatrix}c_1 & c_2 & c_3 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}^{C_\mathrm{B}}\Delta x_i\ , \tag{13}$$

where $c_1 \doteq 0.47$, $c_2 \doteq -9.86\cdot 10^{-5}$, and $c_3 \doteq -0.53$. These values were determined numerically based on the conditions for solvability of the geometric control problem, which will be discussed below. From this measurement for each balloon, the system measurement is given by

$$\begin{bmatrix}\Delta y_1\\ \vdots\\ \Delta y_r\end{bmatrix} = \overbrace{\begin{bmatrix}C_\mathrm{B} & & \\ & \ddots & \\ & & C_\mathrm{B}\end{bmatrix}}^{C}\begin{bmatrix}\Delta x_1\\ \vdots\\ \Delta x_r\end{bmatrix} \tag{14}$$

which has the form $\Delta y = C\Delta x$.

Linearized Control Objective: The nonlinear outputs z_i can immediately be linearized as $\Delta z_i = \Delta s_{i_{\mathrm{mod}(r)}+1} - \Delta s_i$, giving the linearized output equation

$$\begin{bmatrix}\Delta z_1\\ \vdots\\ \Delta z_{r-1}\\ \Delta z_r\end{bmatrix} = \overbrace{\begin{bmatrix}-\varsigma & \varsigma & & \\ & \ddots & \ddots & \\ & & -\varsigma & \varsigma\\ \varsigma & & & -\varsigma\end{bmatrix}}^{D}\begin{bmatrix}\Delta x_1\\ \vdots\\ \Delta x_{r-1}\\ \Delta x_r\end{bmatrix} \tag{15}$$

where $\varsigma := \mathrm{row}(1, 0, 0, 0)$, which has the form $\Delta z = D\Delta x$. The control objective for the linearized system is for $\Delta z \to 0$ as $t \to \infty$, which will space the balloons evenly.

4. Wind Model

To test the balloon model developed above, a model of the stratospheric wind currents is also needed. The wind model used here is based on data obtained from the NCEP/NCAR[2] Reanalysis Project [6], and the wind is assumed to travel east/west along a single latitude line. This simplification is not grossly inaccurate — looking at actual stratospheric winds (such as the visualization in [7]), it can be seen that most wind currents do follow a general east/west path.

A "slice" of the wind model at latitude 40°N is displayed in the left side of Figure 2. The isoclines show that the wind speed varies as a function of both longitude and altitude. In contrast, the balloon model (3) assumed that wind speed varies as a function of altitude only, as displayed in the right side of Figure 2. The actual longitudinal variations can affect the balloons' positions significantly. Fortunately, the designed feedback controller will provide the needed robustness to these unmodelled effects — see Section 6.

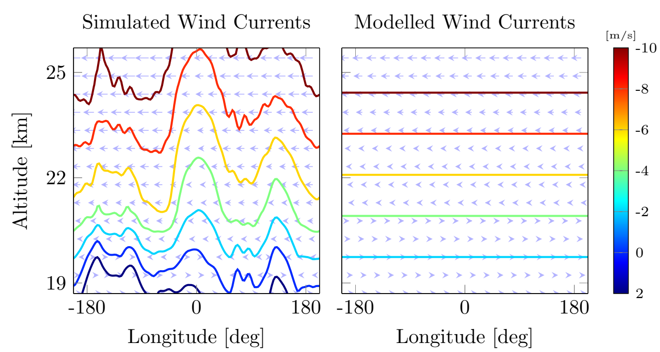

Fig. 2. Wind Speeds [m/s] at 40°N for Varying Longitudes and Altitudes

5. Controller Design

The control objective for the linearized balloon system, as stated in Section 3-B, can be converted into the *Restricted Regulator Problem* (RRP) from geometric control [8, §6.1]. The RRP was solved for a system of 50 balloons with the parameters $\bar{h} \doteq 21$ km, $V \doteq 1415$ m^3, $\bar{P} \doteq 7.43$ kPa, $V_\mathrm{air} \doteq 471$ m^3, $\bar{v} \doteq -4.3$ m/s, $m_0 \doteq 50$ kg, $\alpha \doteq -0.0017$ s^{-1}, and $\beta = 0.02$ kg/s. The steps in solving the RRP are described in Section 5-A, and the specific considerations for preserving the balloon system's distributed structure are examined in Section 5-B.

A. Restricted Regulator Problem (RRP)

Consider a linear time-invariant system

$$\begin{aligned}\Delta\dot{x} &= A\Delta x + B\Delta u\\ \Delta y &= C\Delta x\,,\quad \Delta z = D\Delta x\end{aligned}$$

[2]National Centers for Environmental Prediction and National Center for Atmospheric Research

The goal of the RRP is to find a feedback $\Delta u = K\Delta x$ that stabilizes the system output Δz, where K only uses state information that can be seen in the measurement Δy. This constraint is imposed by a subspace $\mathcal{L}$ that masks the unmeasurable states. The RRP can be stated as follows:

Problem 5.1 ([8]). Given a system of the above form with state space $\mathcal{X}$, and an A-invariant subspace $\mathcal{L} \supset \operatorname{Ker} C$, find a state feedback $\Delta u = K\Delta x$ such that $\mathcal{L} \subset \operatorname{Ker} K$ and $\mathcal{X}^+(A+BK) \subset \operatorname{Ker} D$, where $\mathcal{X}^+(\cdot)$ denotes the unstable modal subspace [8, §0.11].

The condition $\operatorname{Ker} C \subset \mathcal{L} \subset \operatorname{Ker} K$ in Problem 5.1 ensures that the state feedback only uses information that shows up in the measurement. The condition $\mathcal{X}^+(A+BK) \subset \operatorname{Ker} D$ guarantees that the control objective to stabilize the output is met. Generally, $\mathcal{L}$ is taken to be $\langle A|\operatorname{Ker} C\rangle$ (the minimal A-invariant subspace containing $\operatorname{Ker} C$), and $\operatorname{Ker} D$ is replaced with $\mathcal{V}$ (the largest (A,B)-controlled invariant subspace contained in $\operatorname{Ker} D$ [8, §4.2]). For the 50-balloon system with linearized model (11)–(15), these subspaces are given by[3]

$$\mathcal{V} = \operatorname{Im}\begin{bmatrix} e_1 & \cdots & e_4 \\ \vdots_{(\times 50)} & & \vdots \\ e_1 & \cdots & e_4 \end{bmatrix}, \ \mathcal{L} \doteq \mathcal{V} + \operatorname{Im}\left(I_{50} \otimes \begin{bmatrix} 10 & 0 \\ 0 & 10^4 \\ 9 & -2 \\ 0 & 0 \end{bmatrix} \right).$$

where the e_i are the standard basis vectors in $\mathbb{R}^4$. Also, the balloons' controllable subspace is given by $\mathcal{C} = \mathbb{R}^{200}$, and its unstable subspace is given by

$$\mathcal{X}^+(A) \doteq \operatorname{Im}\left(I_{50} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -14 \\ 0 & 0 \\ 0 & 31 \end{bmatrix} \right).$$

Geometric control gives necessary and sufficient conditions for the solvability of the RRP [8, thm. 6.1] based on the above subspaces; these conditions are satisfied on the balloon system, and so the RRP is solvable. The RRP can be solved by placing poles in the part of the system that shows up in both the measurement and the output — in this case, in the subspace $\mathcal{R} := (\mathcal{L}+\mathcal{V})^\perp$. For the balloon system,

$$\mathcal{R} = \operatorname{Im}\left[\begin{array}{cc|cc|cc|cc} e_4 & p & & & & & -e_4 & -p \\ \hline -e_4 & -p & e_4 & p & & & & \\ \hline & & \ddots & & \ddots & & & \\ \hline & & & & -e_4 & -p & e_4 & p \end{array}\right]$$

where $p \doteq \operatorname{col}(-480, 10^{-2}, 535, 0)$. In general, it is necessary to find a matrix F such that $\mathcal{V}$ is $(A+BF)$-invariant; for the balloon system, since $\mathcal{V}$ is A-invariant, it suffices to take $F = 0$. Also define a transformation matrix T by concatenating columns that form a basis for $\mathcal{L}+\mathcal{V}$, followed by columns that form a basis for $\mathcal{R}$. Applying this transformation to the system matrices gives

$$T^{-1}(A+BF)T = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{16}$$

[3]All numbers in these subspaces are approximate, except for 0 and 1.

and the full system has been decomposed into two independent subsystems (A_1, B_1) and (A_2, B_2). Stabilizing the second subsystem solves the RRP; further, the original solvability conditions [8, thm. 6.1] guarantee that this is possible. Therefore, stabilize (A_2, B_2) with some feedback K_2, and define $K = \begin{bmatrix} 0 & K_2 \end{bmatrix} T^{-1} + F$. Then, the overall feedback $u = Kx$ solves the RRP.

The balloon system requires some considerations beyond the RRP: the feedback should not only solve the RRP, but should also correspond to the system's distributed structure. The RRP does not account for that structure, only providing "any" feedback that works for the system model.

B. Block Circulant Restricted Regulator Problem (BC RRP)

The matrices of the linearized balloon system (11)–(15) all have a *block circulant* pattern [9]. This pattern arises inherently from the distributed and homogeneous interconnection structure of the balloons — they are identical and treat each other identically. This distributed structure should be maintained by a controller for the system, as discussed in the introduction, and so any feedback synthesized for the system should also be block circulant. In particular, for the balloon system, the RRP should be solved by a block circulant feedback; this is called the Block Circulant RRP (BC RRP). The BC RRP is solvable whenever the usual RRP is solvable — that is, if *any* feedback exists, then a *block circulant* feedback exists [2, thm. 8.8].

The key to solving the BC RRP is a commuting property inherent in all block circulant matrices: a matrix $M \in \mathbb{R}^{rn\times rm}$ is block circulant if and only if $(\Pi_r \otimes I_n)M = M(\Pi_r \otimes I_m)$, where "$\otimes$" denotes the Kronecker product and $\Pi_r := \left[\begin{smallmatrix} 0 & I_{r-1} \\ 1 & 0 \end{smallmatrix}\right]$ is the fundamental permutation matrix [9, §5.6]. In particular, for the balloon system, $(\Pi_{50} \otimes I_4)A = A(\Pi_{50} \otimes I_4)$ and $(\Pi_{50} \otimes I_4)B = B(\Pi_{50} \otimes I)$. By exploiting these relationships, and also choosing $\mathcal{L}$ and $\mathcal{V}$ to be $(\Pi_r \otimes I_n)$-invariant (which holds for the balloon system's subspaces given above), the solvability of the BC RRP can be determined by the same conditions as the general RRP, and can be solved by the same system decomposition as outlined in Section 5-A [2]. Preserving the block circulant pattern in the solution requires three additional considerations:

1. The chosen matrix F (such that $\mathcal{V}$ is $(A+BF)$-invariant) must be block circulant;
2. The chosen complementary subspace $\mathcal{R}$ to $\mathcal{L}+\mathcal{V}$ must be $(\Pi_r \otimes I_n)$-invariant;
3. The chosen feedback law K_2 for the decomposed subsystem must satisfy $(\Pi_r \otimes I_m)K_2 = K_2(\Pi_r \otimes I_n)_{\mathcal{R}}$, where $(\Pi_r \otimes I_n)_{\mathcal{R}}$ is the restriction of $\Pi_r \otimes I_n$ to $\mathcal{R}$ [8, §0.4].

As long as the RRP is solvable, such choices can always be made, as proven in [2]. In particular, for the balloon system, $F = 0$ is block circulant, $\mathcal{R}$ is $(\Pi_{50} \otimes I_4)$-invariant, and K_2 can be chosen to satisfy $(\Pi_{50} \otimes I)K_2 = K_2(\Pi_{50} \otimes I_4)_{\mathcal{R}}$. These choices guarantee that the overall feedback K will be block circulant [2, thm. 8.8], thus solving the BC RRP.

6. CONTROL OF THE BALLOON NETWORK

Following the control design method of Section 5, a block circulant feedback $\Delta u = K\Delta x$ was found for the 50-balloon system by implementing a MATLAB framework developed for block circulant control (and using the Geometric Approach Toolbox [10] to perform subspace operations). This feedback was then decentralized and scaled up to 600 balloons. Synthesizing a controller for the larger system directly — i.e., finding a 600×2400 feedback matrix — would have been a very daunting computational task; by exploiting the block circulant pattern, this task was made markedly simpler and considerably quicker.

A. Linear Feedback

The 50-balloon feedback K is visualized in Figure 3, which shows the magnitude of each balloon's "influence" on others: the block in the i^{th} row and j^{th} column represents the part of balloon i's control u_i that depends on balloon j's states x_j, with darker blocks implying larger dependencies.

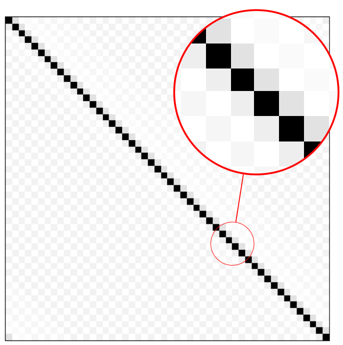

Fig. 3. BC RRP Feedback Matrix for 50 Balloons (darker blocks have larger magnitude)

The block circulant pattern in the feedback is inherent in the diagonal "bands": each diagonal consists of blocks of the same shade, meaning that each balloon i considers balloon $(i+k)_{\text{mod}(50)}$ in the same way. These diagonal bands — and by extension, the block circulant pattern — encourage a feedback which can readily be decentralized: by keeping the darkest matrix blocks and zeroing all others, small controller gains can be removed (similar to [11]). In Figure 3, the darkest blocks are on and to the right of the main diagonal. Zeroing all other blocks gives a decentralized control law in which each balloon's motion only depends on its own states and those of the balloon next to it (rather than all of them), while preserving the same closed-loop system behaviour. This decentralized feedback is given by

$$u_i \cdot 10^6 \doteq \begin{bmatrix}-0.94 & (1.96\cdot10^{-4}) & 1.05 & -16.83\end{bmatrix} x_i + \begin{bmatrix}-0.11 & (0.23\cdot10^{-4}) & 0.12 & -0.85\end{bmatrix} x_{i_{\text{mod}(50)}+1}\,.$$

This decentralized feedback was scaled from 50 to 600 balloons by replicating the above two blocks — each balloon in the larger system still uses only its own states and those of the balloon next to it, with the same controller gains as in the smaller system. This scaling method drastically reduces the calculation time by circumventing the computational complexity of pole placement on high-dimensional matrices. Moreover, this method is possible because of the block circulant pattern; without identical controller gains, it does not appear clear how to implement the feedback on a larger system.

B. Unmodelled Wind Effects

For proper balloon spacing to be maintained over time, the balloons must all be travelling at the same longitudinal speed — that is, they must all be on same-speed wind currents. The wind currents were assumed in (3) to be a function of altitude only; thus, in the system model, balloons at the same altitude all travel at the same speed. In contrast, the actual wind currents are a function of both altitude and longitude (as seen on the left side of Figure 2), so balloons at the same altitude do not always travel at the same speed; rather, balloons *on the same isocline* (in the figure) travel at the same speed. In particular, if the balloons are all at altitude $\bar{h}$, they will not all travel at speed $\bar{v}$. Therefore, for at least some balloons, $\dot{s}_i \neq 0$ at $\bar{h}$, so $\bar{x}_{\text{B}}$ is no longer a valid operating point. Instead, balloon i must take $\bar{h}_i$ as the altitude at which the wind speed *at its current longitude* is $\bar{v}$. Then, the $\bar{h}_i$ will all lie on the $\bar{v}$ isocline. It is important to note that the $\bar{h}_i$ are no longer the same for all i.

In the sequel, the balloons do not have immediate access to actual wind data. In lieu of such data, $\bar{h}_i$ cannot be perfectly aligned with the $\bar{v}$ isocline. Rather, balloon i approximates $\bar{h}_i$ by estimating the altitude of the $\bar{v}$ isocline at its current longitude Λ_i as $\bar{h}_i = h_i - \alpha^{-1}(\dot{s}_i - \bar{v})$, where α is the linearized wind gradient from (11). Furthermore, as Λ_i changes (as the balloon traverses the Earth), the local altitude of $\bar{v}$ at Λ_i also changes according to Figure 2, and so $\bar{h}_i$ will be a worse approximation of the $\bar{v}$ isocline over time. Therefore, $\bar{h}_i$ must be recalculated periodically in order to keep it near the $\bar{v}$ isocline. Lastly, whenever $\bar{h}_i$ is recalculated, so should $\bar{P}_i$ be recalculated according to (10).

C. Simulation Results

The balloon network was simulated using the feedback described above, in the wind currents described in Section 4. The nonlinear control law is given by $u = \bar{u} + K(x - \bar{x})$, where $\bar{x}$ uses the most recently calculated values for the $\bar{h}_i$ and $\bar{P}_i$, as in Section 6-B. Figure 4 shows the states and inputs for 50 of the 600 balloons[4]. The "steps" seen in the control inputs are the operating point recalculations (as described in Section 6-B), which were performed every 30 simulated minutes. Over time, the system approaches a steady state: each balloon's altitude and internal air pressure stabilize, and the control input tends to 0.

Figure 5 shows the distances between adjacent balloons, for 150 of the 600. This plot confirms that at the system's steady state, the control objective is met: the balloons are all about 50 km apart, which spaces them evenly at 40°N.

[4]The upper- and lower-bounding state responses are included in Figure 4. The plot of $\dot{h}_i$ only shows four balloons, because of the high oscillations.

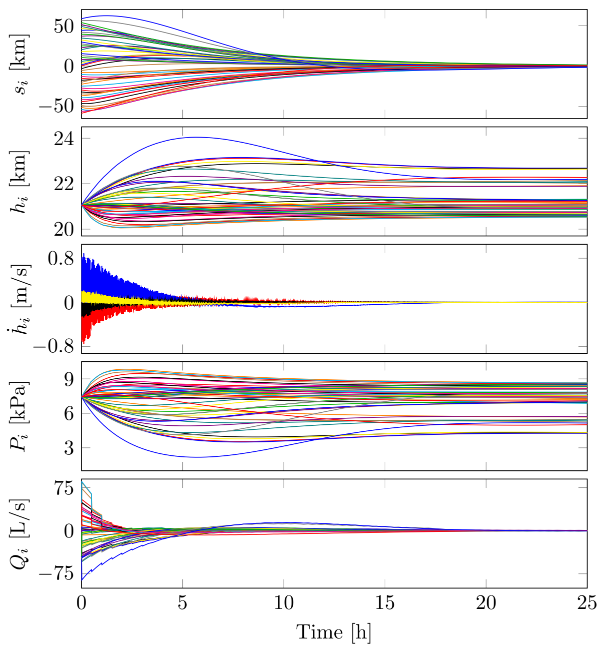

Fig. 4. BC RRP States/Inputs (50 of 600 shown)

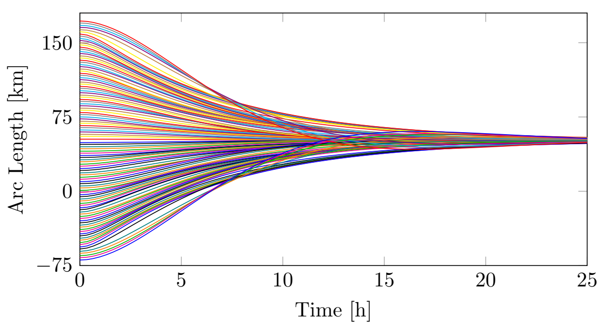

Fig. 5. BC RRP Arc Length between Balloons (from i to $i_{\text{mod}(600)} + 1$, 150 of 600 shown)

Figure 6 compares the output response of the original 50-balloon system to the decentralized and scaled 600-balloon system, showing the outputs z_i for four selected balloons. Most important, it can be seen that the decentralization and scaling process barely affects the overall response of the balloons; while the balloons in the smaller system tend to have slightly shorter settling times, the differences are small when considering that the larger system has 12 times as many balloons, whereas the feedback remains unmodified. The similarity between the system responses shows that the synthesized feedback is both robust and scalable.

In reality, the system behaviour could be affected by physical limits; for example, the balloon's air chamber can only support a certain pressure range, and the pump can only operate up to a maximum flow rate. These operating ranges could cause the balloons to saturate at the upper and lower bounds of pressure and flow rate, thereby affecting the system's response time. However, the response itself would not change as a result of these unmodelled limits; the balloons would still space themselves evenly.

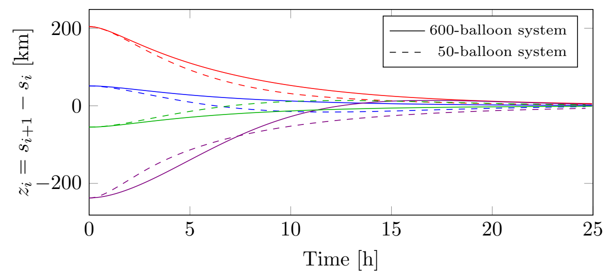

Fig. 6. BC RRP Output Before and After Decentralization and Scaling

7. Concluding Remarks

Formation control of balloons has been formulated and solved in a very direct way. Our results show that maintaining the system's inherent block circulant pattern in the feedback accomplishes three significant outcomes: the balloons can be coordinated through complicated wind currents; the feedback can be decentralized, enabling the balloons to space themselves properly with very limited information; and the feedback can be scaled up to control at least an order of magnitude more balloons than the synthesis involved.

Our findings extend well beyond their application to high-altitude balloons. Synthesizing feedback laws for large multi-agent systems is often difficult and time-consuming, but these results show that it need not be: by preserving a system's inherent pattern, feedbacks can be decentralized and scaled efficiently and intuitively. Our simulations demonstrate that this technique can be robust enough to control many more agents without affecting the overall system behaviour.

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