Block Circulant Control: a Geometric Approach

A.C. Sniderman, M.E. Broucke, and G.M.T. D'Eleuterio

***Abstract*—This paper studies structured control synthesis for structured systems, with a focus on block circulant systems corresponding to the structure of a topological ring. We show that several classic control synthesis problems are amenable to a structured synthesis. Moreover, the findings suggest that structured systems *naturally* admit structured controllers.**

I. Introduction

Most complex engineered systems consist of subsystems interacting in a predefined manner. This structure often takes the form of highly recognizable repeated patterns, and these patterns manifest themselves algebraically in the system model. In this work we focus on one specific system structure: a topological ring. A ring system is comprised of a number of identical subsystems whose interactions occur in a closed loop. Moreover, the first subsystem treats the second in the same way that the second subsystem treats the third, and so on. Mathematically the ring structure manifests itself as a pattern in the system matrices of the state space model: every system matrix is *block circulant*. This paper will show that several of the classic synthesis problems of multivariable control are amenable to a structured synthesis realizing block circulant feedbacks. The proposed framework can be taken as a template for future studies of more general structures.

In contrast to most prior work, we examine structured control systems through the lens of linear geometric control. We reexamine the results of [15], taking system structure into account: given a block circulant system, can the necessary and sufficient conditions of [15] recover control laws that preserve the block circulant structure? We are not the first to study structured systems through this lens. First, [4] studied controllability and observability of block circulant systems. Using the simultaneous block diagonalization property of block circulant matrices via the Fourier matrix, the full system can be decomposed into modal subsystems; by maintaining certain conditions on those subsystems when designing control laws, the full system remains block circulant. In this way, control problems for block circulant systems can be reduced to a collection of control problems on smaller modal systems with easily satisfied constraints. Most other researchers (e.g. [1], [6], [12], [13]) have followed suit, focusing solely on this decomposition in analyzing block circulant systems. We will also use this Fourier decomposition at times when it is convenient to do so. However, rather than verifying our matrices' structure through block diagonalization and the resulting Fourier decomposition, we instead exploit the commutative properties of block circulant matrices. This allows us to forgo the Fourier decomposition in favour of the standard ones (controllable decomposition, observable decomposition, etc.) used in geometric control [2], [15].

Second, [10] formulated the geometric approach for *patterned systems* by encoding the structure in a base matrix, of which all system matrices are a polynomial. While circulant matrices are patterned, block circulant matrices are not, so a generalization of their encoding method is needed. A contribution of this work is to recognize that a suitable encoding of the block circulant pattern is via commuting relationships of block circulant matrices rather than by block diagonalization [4] or by polynomial functions of a base matrix [10]. The idea of exploiting commuting relationships has also been explored in [11]. Finally, using the device of *block circulant subspaces*, we bridge the algebraic domain (of system matrices) and the geometric domain (of the system's state space). This paper is the conference announcement of [14]; here we present motivations and intuition not available in [14], whereas [14] contains the proofs.

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II. Background

The *Kronecker product* of two matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{p\times q}$, denoted by $A \otimes B$, is the $\mathbb{C}^{mp\times nq}$ matrix obtained by replacing the $(ij)^{\text{th}}$ element of A by $a_{ij}B$, for all i, j. Let I_r denote the $r \times r$ identity matrix and $\Pi_r = \operatorname{circ}(0, 1, 0, \ldots, 0)$ the $r \times r$ fundamental permutation matrix, where $\operatorname{circ}(v)$ denotes a circulant matrix whose first row is v.

We assume that the reader is familiar with the tools of linear geometric control theory [2], [15]. We make use of two projection maps which are standard in linear system theory: the insertion and the natural projection. Let $\mathcal{X}$ be an n-dimensional vector space and let $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ be two subspaces such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$ and $\dim(\mathcal{V}) = k$. The *insertion map* $S : \mathcal{V} \to \mathcal{X}$ maps $x \in \mathcal{V}$ to the corresponding element $x \in \mathcal{X}$; that is, $Sx := x$. In coordinates, it maps the $k \times 1$ coordinate vector of x in a basis for $\mathcal{V}$ to the corresponding $n \times 1$ coordinate vector for x in a basis for $\mathcal{X}$. The *natural projection* on $\mathcal{V}$ along $\mathcal{W}$, denoted $Q : \mathcal{X} \to \mathcal{V}$, maps $x \in \mathcal{X}$ to its component in $\mathcal{V}$; that is, given the unique representation $x = v + w$ with $v \in \mathcal{V}$, $w \in \mathcal{W}$, $Qx = v$. Note that $QS = I_{\mathcal{V}}$, where $I_{\mathcal{V}}$ is the identity map on $\mathcal{V}$. Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map and $\mathcal{V} \subset \mathcal{X}$ a subspace. If $\mathcal{V}$ is A-invariant; that is, $A\mathcal{V} \subset \mathcal{V}$ then the *restriction* of A to $\mathcal{V}$, denoted $A_{\mathcal{V}}$, is the unique solution of $AS = SA_{\mathcal{V}}$. Finally, $\mathcal{X}^+(A)$ denotes the unstable subspace of A.

III. Commuting Matrices

Our framework for control of block circulant systems is built up from properties of commuting matrices. The reader is referred to Chapter VIII of [7] for more background.

Definition 3.1: Let $\mathbb{F}$ be a field. We say $A \in \mathfrak{C}_{\mathbb{F}}(U,V)$ if A, U, and V all take entries in $\mathbb{F}$ and

$$UA = AV\,. \tag{1}$$

We assume the field of real numbers when not otherwise specified, i.e. $\mathfrak{C}(U,V) := \mathfrak{C}_{\mathbb{R}}(U,V)$. If $A \in \mathfrak{C}_{\mathbb{F}}(U,U)$, we write $A \in \mathfrak{C}_{\mathbb{F}}(U)$, for short.

We will see in the sequel that the commuting properties of block circulant matrices are the main mechanism by which we encode block circulant structure, even after performing decompositions of the block circulant control system. A new requirement encountered in pole placement for structured systems is to ensure that the feedbacks obtained are real matrices. Our main tool to achieve this requirement is Theorem 1, Ch. VIII, of [7] which provides explicit information on the block diagonal structure of matrices in $\mathfrak{C}(U,V)$. By bookkeeping the complex conjugate pattern of the diagonal blocks, one can ensure that the final outcome for a state feedback is not only block circulant but also real. This bookkeeping is organized around a careful ordering of the eigenvalues of U and V, which we now explain.

Let $\sigma_{\rm d}(V)$ denote the set of distinct eigenvalues of V. Suppose $\sigma_{\rm d}(V) = \{\lambda_1,\dots,\lambda_r\}$ and $\sigma_{\rm d}(U) \subset \sigma_{\rm d}(V)$. Our convention is that the indexing of eigenvalues in $\sigma_{\rm d}(U)$ follows that of $\sigma_{\rm d}(V)$; for example, $\sigma_{\rm d}(V) = \{\lambda_1,\lambda_2,\lambda_3\}$ and $\sigma_{\rm d}(U) = \{\lambda_1,\lambda_3\}$. Suppose λ_i has algebraic multiplicity k_i and m_i in $\sigma(U)$ and $\sigma(V)$, respectively. Note that some k_i's may be zero. We use the following *preferred ordering* of eigenvalues of V and U:

$$\begin{aligned}
\sigma(V) &= \{\underbrace{\lambda_1,\dots,\lambda_1}_{m_1\ \text{times}},\dots,\underbrace{\lambda_r,\dots,\lambda_r}_{m_r\ \text{times}}\}\\
\sigma(U) &= \{\underbrace{\lambda_1,\dots,\lambda_1}_{k_1\ \text{times}},\dots,\underbrace{\lambda_r,\dots,\lambda_r}_{k_r\ \text{times}}\}\,.
\end{aligned}$$

Let $\Lambda_U := \Gamma_U^{-1} U \Gamma_U$ and $\Lambda_V := \Gamma_V^{-1} V \Gamma_V$ be the Jordan forms of U and V respectively. Our convention is to partition Γ_U and Γ_V as $\Gamma_U = \begin{bmatrix} U_1 & \cdots & U_r \end{bmatrix}$ and $\Gamma_V = \begin{bmatrix} V_1 & \cdots & V_r \end{bmatrix}$, where each U_i has k_i columns and each V_i has m_i columns (note again some U_i may have "zero width"). The generalized eigenvectors are chosen so that if $\lambda_i = \overline{\lambda}_j$, then $U_i = \overline{U}_j$. If V is real, there exists a permutation $\{\ell_1,\dots,\ell_r\}$ of $I := \{1,\dots,r\}$ such that $\lambda_{\ell_i} = \overline{\lambda}_i$ for each $i = 1,\dots,r$. (In the sequel we only use eigenvectors).

Now we give sufficient conditions, (A1)-(A2) below, for a matrix A to satisfy $UA = AV$. These conditions will be seen to be reasonable for our control results. Conditions (A3)-(A5) are simply to fix notation.

Assumption 3.2: We assume that the pair (U,V) satisfies:

- (A1) U and V are diagonalizable.
- (A2) $\sigma_{\rm d}(U) \subset \sigma_{\rm d}(V)$.
- (A3) $\sigma_{\rm d}(V) = \{\lambda_1,\dots,\lambda_r\}$ where λ_i has algebraic multiplicity $0 \le k_i \le n$ in $\sigma(U)$ and m in $\sigma(V)$ for some $n, m > 0$.
- (A4) The eigenvalues of U and V are ordered according to the preferred ordering described above.
- (A5) There exists a permutation $\{\ell_1,\dots,\ell_r\}$ of $I := \{1,\dots,r\}$ such that $\lambda_{\ell_i} = \overline{\lambda}_i$ for each $i = 1,\dots,r$.

We conclude this section with an examination of the eigenvalues of $A \in \mathfrak{C}(U)$. The main result is that the eigenvalues of A follow the same complex conjugate pattern as those of U.

Definition 3.3: Let $\sigma_{\rm d}(U) = \{\lambda_1,\dots,\lambda_r\}$ be the distinct eigenvalues of U. A spectrum $\mathscr{L}$ is called *U-patterned* if it can be ordered and partitioned as $\mathscr{L} = \mathscr{L}_1 \uplus \cdots \uplus \mathscr{L}_r$ such that $\mathscr{L}_i = \overline{\mathscr{L}}_j$ whenever $\lambda_i = \overline{\lambda}_j$.

Lemma 3.4: Let $\sigma_{\rm d}(U) = \{\lambda_1,\dots,\lambda_r\}$ and let $A \in \mathfrak{C}(U)$. Then $\sigma(A)$ is U-patterned.

IV. Block Circulant Matrices

Let $A_1,\dots,A_r \in \mathbb{C}^{n\times m}$. A *block circulant* matrix is an $rn \times rm$ matrix of the form

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_r \\ A_r & A_1 & \cdots & A_{r-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}.$$

Every block circulant matrix A can be represented as

$$A = \sum_{i=1}^{r} (\Pi_r^{i-1} \otimes A_i)\,. \tag{2}$$

A well-known result from [5] is that a matrix $A \in \mathbb{C}^{rn\times rn}$ is block circulant if and only if it commutes with $\Pi_r \otimes I_n$. The result easily generalizes to nonsquare matrices.

Lemma 4.1: $A \in \mathbb{C}^{rn\times rm}$ is block circulant if and only if $A \in \mathfrak{C}_{\mathbb{C}}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$.

The vector space of $rn \times rm$ block circulant matrices will henceforth be denoted as $\mathfrak{C}_{\mathbb{C}}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$.

Definition 4.2: A spectrum $\mathscr{L}$ is called a *block circulant spectrum* if it is $(\Pi_r \otimes I_n)$-patterned.

While the class of block circulant spectra is somewhat more restrictive than the class of symmetric spectra, it is always possible to find stable block circulant spectra. This makes block circulant spectra sufficiently versatile for most pole placement-related design problems of control theory.

Lemma 4.3: Let $\mathscr{L}$ be a block circulant spectrum. There exists $A \in \mathfrak{C}(\Pi_r \otimes I_n)$ such that $\sigma(A) = \mathscr{L}$.

V. Block Circulant Subspaces

In the previous two sections we presented algebraic tools used in our framework for control of block circulant systems. In this section we introduce the main geometric construct that will allow us to link the algebraic properties of block circulant and commuting matrices with the theory of linear geometric control [2], [15].

Definition 5.1: We say that $\mathcal{V} \subset \mathcal{X}$ is a *block circulant subspace* if it is $(\Pi_r \otimes I_n)$-invariant. That is, $(\Pi_r \otimes I_n)\mathcal{V} \subset \mathcal{V}$.

Lemma 5.2: Let $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ be block circulant subspaces. Then $\mathcal{V} + \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W}$ are block circulant subspaces.

Consider $A \in \mathfrak{C}(\Pi_r \otimes I_n)$. While not every A-invariant subspace is a block circulant subspace, fortunately it is possible to identify several A-invariant subspaces, useful in a control theory context, that are also block circulant subspaces.

Lemma 5.3: Let $A \in \mathfrak{C}(\Pi_r \otimes I_n)$, $B \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$, and let $\rho(s)$ be a polynomial. Then, Im B, $\operatorname{Ker} B$, Im $\rho(A)$, and $\operatorname{Ker} \rho(A)$ are block circulant subspaces.

Lemma 5.4: Let $A \in \mathfrak{C}(\Pi_r \otimes I_n)$ and let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace. Then $A\mathcal{V}$, the image of $\mathcal{V}$ under A, and $A^{-1}\mathcal{V} = \{x \in \mathcal{X} \mid Ax \in \mathcal{V}\}$, the pre-image, are both block circulant subspaces.

Another useful property is that a block circulant subspace has a block circulant complement.

Lemma 5.5: Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace. Then $\mathcal{V}^{\perp}$ is a block circulant subspace.

The significance of Lemma 5.5 is that any block circulant subspace $\mathcal{V}$ decomposes $\mathcal{X}$ relative to $\Pi_r \otimes I_n$. This means that the restrictions $(\Pi_r \otimes I_n)_{\mathcal{V}}$ and $(\Pi_r \otimes I_n)_{\mathcal{W}}$ are both defined (where $\mathcal{W}$ is any block circulant complement of $\mathcal{V}$). This important property will allow us to preserve structure in restrictions of block circulant maps; certain commuting relationships inherited from the original block circulant structure will hold for such maps. The main result is the following block circulant version of the standard representation theorem for linear maps with respect to invariant subspaces.

Theorem 5.6 (Representation Theorem): Let $A \in \mathfrak{C}(\Pi_r \otimes I_n)$ and let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace such that $\dim(\mathcal{V}) = k < rn$ and $A\mathcal{V} \subset \mathcal{V}$. Let $\mathcal{W} \subset \mathcal{X}$ be a block circulant complement of $\mathcal{V}$. Then A has a matrix representation

$$\begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \tag{3}$$

where A_1 is a matrix representation of $A_{\mathcal{V}}$. Moreover, $A_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$ and $A_2 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{W}})$.

Theorem 5.6 takes a full block circulant matrix and pushes it down to a restriction in $\mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$. Conversely, given a matrix in $\mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$ we would like to be able to lift it back up to a real block circulant matrix. This is doable for the correct choice of complementary subspace.

Lemma 5.7 (Lifting Lemma): Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace and $\mathcal{W} \subset \mathcal{X}$ a block circulant complement. Let $S_1 : \mathcal{V} \to \mathcal{X}$ and $Q_1 : \mathcal{X} \to \mathcal{V}$ be the insertion and natural projection on $\mathcal{V}$.

(i) If $A_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$ then $S_1 A_1 Q_1 \in \mathfrak{C}(\Pi_r \otimes I_n)$.

(ii) If $B_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_m)$ then $S_1 B_1 \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$.

(iii) If $K_1 \in \mathfrak{C}(\Pi_r \otimes I_m, (\Pi_r \otimes I_n)_{\mathcal{V}})$ then $K_1 Q_1 \in \mathfrak{C}(\Pi_r \otimes I_m, \Pi_r \otimes I_n)$.

The ability to transition between a full space and a subspace without sacrificing commuting relationships, shown by Theorem 5.6 and Lemma 5.7, is what allows us to recover feedbacks that preserve a control system's block circulant structure using the usual decompositions of linear geometric control. We conclude this section by identifying pairs (U, V) useful in block circulant control design, that satisfy Assumption 3.2.

Lemma 5.8: Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace (including possibly $\mathcal{V} = \mathcal{X}$), and suppose (U, V) is the pair $(U, V) = ((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_m)$. Then Assumption 3.2 holds.

We have laid the foundations in the areas of commuting matrices, block circulant matrices, block circulant subspaces, and restrictions and lifts of block circulant maps. We now begin our study of block circulant control systems.

VI. Controllability

Consider the linear time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4a}$$

$$y(t) = Cx(t), \tag{4b}$$

where $x(t) \in \mathbb{R}^{rn}$ is the vector of states, $u(t) \in \mathbb{R}^{rm}$ is the vector of inputs, and $y(t) \in \mathbb{R}^{rp}$ is the vector of measurements. We denote the state space, input space and measurement space by $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$, respectively. Assume a real system with matrices $A \in \mathbb{R}^{rn \times rn}$, $B \in \mathbb{R}^{rn \times rm}$ and $C \in \mathbb{R}^{rp \times rn}$. We refer to such a system in shorthand by the triple (C, A, B) or simply by the pair (A, B) or the pair (C, A), when the third transformation is not applicable. If further $A \in \mathfrak{C}(\Pi_r \otimes I_n)$, $B \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$, and $C \in \mathfrak{C}(\Pi_r \otimes I_p, \Pi_r \otimes I_n)$, then (C, A, B) is called a *block circulant system*. The *open loop poles* of the system are the eigenvalues of A; for block circulant systems the open loop poles of the system form a block circulant spectrum.

Let $\mathcal{B} = \operatorname{Im} B$. The *controllable subspace* $\langle A|\mathcal{B}\rangle$ of the pair (A, B) is given by $\langle A|\mathcal{B}\rangle = \mathcal{B} + A\mathcal{B} + \cdots + A^{rn-1}\mathcal{B}$. Our focus on block circulant subspaces has been driven by the following observation.

Lemma 6.1: The controllable subspace is a block circulant subspace.

It is well known that the spectrum of $A + BK$ can be arbitrarily assigned to any symmetric set of poles by choice of $K : \mathcal{X} \to \mathcal{U}$ if and only if (A, B) is controllable. For a block circulant system, the question arises of what possible poles can be achieved by choice of block circulant state feedback. We address this question in the more general setting when $A \in \mathfrak{C}(U)$ and $B \in \mathfrak{C}(U, V)$ for some linear maps U and V.

Theorem 6.2 (Pole Placement): Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace and let $(U, V) = ((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_m)$. Let $A \in \mathfrak{C}(U)$ and $B \in \mathfrak{C}(U, V)$. The pair (A, B) is controllable if and only if for every U-patterned spectrum $\mathscr{L}$, there exists a map $K : \mathcal{V} \to \mathcal{U}$ with $K \in \mathfrak{C}(V, U)$ such that $\sigma(A + BK) = \mathscr{L}$.

Suppose we have a block circulant system that is not fully controllable, i.e. $\langle A|\mathcal{B}\rangle \neq \mathcal{X}$. Then there is a basis in which the controllable and uncontrollable parts of the system are displayed transparently. This is the content of our first decomposition theorem.

Theorem 6.3 (First Decomposition Theorem): Let $\mathcal{C} := \langle A|\mathcal{B}\rangle$ and suppose $\dim(\mathcal{C}) = k < rn$. Then there exists a coordinate transformation $T : \mathcal{X} \to \mathcal{X}$ such that $(\tilde{A}, \tilde{B}) := (T^{-1}AT, T^{-1}B)$ has the form

$$\tilde{A} = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{5}$$

where $A_1 = A_{\mathcal{C}} \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{C}})$, $A_2 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{C}^\perp})$, and $B_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{C}}, \Pi_r \otimes I_m)$. Moreover, the pair (A_1, B_1) is controllable.

A system, or equivalently the pair (A, B), is *stabilizable* if there exists $K : \mathcal{X} \to \mathcal{U}$ such that $\sigma(A + BK) \subset \mathbb{C}^-$. A system is stabilizable if and only if $\mathcal{X}^+(A) \subset \mathcal{C}$ [15]. For a block circulant system the question arises of whether the system can be stabilized by a block circulant state feedback. We present here a more general result to be used in later synthesis problems.

Theorem 6.4 (Stabilizability): Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace. Suppose $A \in \mathfrak{C}(\Pi_r \otimes I_n)$ and $A\mathcal{V} \subset \mathcal{V}$. Let $A_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$ be the restriction of A to $\mathcal{V}$ and let $B_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_m)$. There exists a state feedback $K_1 : \mathcal{V} \to \mathcal{U}$, $K_1 \in \mathfrak{C}(\Pi_r \otimes I_m, (\Pi_r \otimes I_n)_{\mathcal{V}})$, such that $\sigma(A_1 + B_1K_1) \subset \mathbb{C}^-$ if and only if

$$\mathcal{X}^+(A_1) \subset \mathcal{C}_1$$

where $\mathcal{C}_1$ is the controllable subspace of the pair (A_1, B_1).

VII. Observability

Consider again the block circulant system given in (4). The *unobservable subspace* $\mathcal{N}$ of the pair (C, A) is given by $\mathcal{N} = \bigcap_{i=1}^{rn} \operatorname{Ker} CA^{i-1}$.

Lemma 7.1: The unobservable subspace is a block circulant subspace.

Using duality and Theorem 6.2, we have the following result about observability of block circulant systems.

Theorem 7.2: Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace and suppose Assumption 3.2 holds for $(U, V) = ((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_p)$. Let $A \in \mathfrak{C}(U)$ and $C \in \mathfrak{C}(V, U)$. The pair (C, A) is observable if and only if for every U-patterned spectrum $\mathscr{L}$, there exists a map $K : \mathcal{Y} \to \mathcal{V}$ with $K \in \mathfrak{C}(U, V)$ such that $\sigma(A + KC) = \mathscr{L}$.

Suppose we have a block circulant system that is not fully observable, i.e. $\mathcal{N} \neq 0$. There is a basis in which the unobservable and observable parts of the system are displayed transparently. This is the content of our second decomposition theorem.

Theorem 7.3 (Second Decomposition Theorem): Suppose $\dim(\mathcal{N}) = k \neq 0$. There exists a coordinate transformation $T : \mathcal{X} \to \mathcal{X}$ such that $(\tilde{C}, \tilde{A}) := (CT, T^{-1}AT)$ has the form

$$\tilde{A} = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} 0 & C_2 \end{bmatrix}, \tag{6}$$

where $A_1 = A_{\mathcal{N}} \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{N}})$, $A_2 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{N}^\perp})$, and $C_2 \in \mathfrak{C}(\Pi_r \otimes I_p, (\Pi_r \otimes I_n)_{\mathcal{N}^\perp})$. Moreover, the pair (C_2, A_2) is observable.

Detectability is the dual concept of stabilizability. We say (C, A) is *detectable* if there exists K such that $\sigma(A + KC) \subset \mathbb{C}^-$. Thus, (C, A) is detectable if and only if $(A^\mathsf{T}, C^\mathsf{T})$ is stabilizable. From Theorem 6.4 we immediately obtain the following.

Theorem 7.4 (Detectability): Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace. Suppose $A \in \mathfrak{C}(\Pi_r \otimes I_n)$ and $A\mathcal{V} \subset \mathcal{V}$. Let $A_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$ be the restriction of A to $\mathcal{V}$ and let $C_1 \in \mathfrak{C}(\Pi_r \otimes I_p, (\Pi_r \otimes I_n)_{\mathcal{V}})$. There exists a state feedback $K_1 : \mathcal{Y} \to \mathcal{V}$, $K_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_p)$, such that $\sigma(A_1 + K_1C_1) \subset \mathbb{C}^-$ if and only if $\mathcal{N}_1 \subset \mathcal{X}^-(A_1)$.

VIII. Control Synthesis

Using the system properties obtained in the previous two sections, we now present solutions to several classic control synthesis problems adapted to block circulant systems.

A. Output Stabilization

Consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Dx(t), \end{aligned}$$

where $x(t) \in \mathbb{R}^{rn}$, $u(t) \in \mathbb{R}^{rm}$, and $z(t) \in \mathbb{R}^{rq}$. The *Output Stabilization Problem* (OSP) is to find a state feedback $u(t) = Kx(t)$ such that $z(t) \to 0$ as $t \to \infty$. The problem can be restated in more geometric terms as finding a state feedback $K : \mathcal{X} \to \mathcal{U}$ that makes the unstable subspace unobservable at the output $z(t)$. Equivalently, $\mathcal{X}^+(A + BK) \subset \operatorname{Ker} D$. The solution to the OSP requires the notion of controlled invariant subspaces. A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be *controlled invariant* if there exists a map $F : \mathcal{X} \to \mathcal{U}$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$. In this case F is called a *friend* of $\mathcal{V}$. Let $\mathscr{I}(\mathcal{X})$ denote the set of controlled invariant subspaces in $\mathcal{X}$. Similarly, for any $\mathcal{V} \subset \mathcal{X}$, let $\mathscr{I}(\mathcal{V})$ denote the set of all controlled invariant subspaces in $\mathcal{V}$. It is well-known that the OSP is solvable if and only if $\mathcal{X}^+(A) \subset \mathcal{C} + \mathcal{V}^\star$, where $\mathcal{C} = \langle A|\mathcal{B}\rangle$ and $\mathcal{V}^\star := \sup \mathscr{I}(\operatorname{Ker} D)$, the supremal controlled-invariant subspace contained in $\operatorname{Ker} D$. Consider now the OSP for block circulant systems.

Problem 8.1 (Output Stabilization Problem): Given a block circulant triple (D, A, B), find a block circulant state feedback $K : \mathcal{X} \to \mathcal{U}$ such that

$$\mathcal{X}^+(A + BK) \subset \operatorname{Ker} D\,.$$

In contrast with the standard OSP, in the block circulant OSP it is necessary to be able to construct block circulant friends for block circulant controlled invariant subspaces.

Lemma 8.1: Let (A, B) be a block circulant pair with $A \in \mathfrak{C}(\Pi_r \otimes I_n)$ and $B \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$. If $\mathcal{V} \subset \mathcal{X}$ is a block circulant subspace and $\mathcal{V} \in \mathscr{I}(\mathcal{X})$, then there exists $F \in \mathfrak{C}(\Pi_r \otimes I_m, \Pi_r \otimes I_n)$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.

The next result states that the supremal controlled invariant subspace contained in a block circulant subspace is itself a block circulant subspace.

Lemma 8.2: Let (D, A, B) be a block circulant triple with $A \in \mathfrak{C}(\Pi_r \otimes I_n)$, $B \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$, and $D \in \mathfrak{C}(\Pi_r \otimes I_q, \Pi_r \otimes I_n)$. Then $\mathcal{V}^\star = \sup \mathscr{I}(\operatorname{Ker} D)$ is a block circulant subspace.

Given a block circulant friend F of a block circulant subspace $\mathcal{V}$, there is a basis in which $A + BF$ has a useful matrix representation. This is the content of our third decomposition theorem.

Theorem 8.3 (Third Decomposition Theorem): Let $\mathcal{V} \subset \mathcal{X}$ be a block circulant subspace with $\mathcal{V} \in \mathscr{I}(\mathcal{X})$ and let $\mathcal{W} \subset \mathcal{X}$

be a block circulant complement of $\mathcal{V}$. There exists a state and feedback transformation (T, F) with $T : \mathcal{X} \to \mathcal{X}$ and $F \in \mathfrak{C}(\Pi_r \otimes I_m, \Pi_r \otimes I_n)$ such that $(\tilde{A}, \tilde{B}) := (T^{-1}(A + BF)T, T^{-1}B)$ has the form

$$\tilde{A} = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{7}$$

where $A_1 = (A + BF)_{\mathcal{V}} \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}})$, $A_2 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{W}})$, $B_1 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{V}}, \Pi_r \otimes I_m)$, and $B_2 \in \mathfrak{C}((\Pi_r \otimes I_n)_{\mathcal{W}}, \Pi_r \otimes I_m)$.

Using the previous results on controlled invariant subspaces, we proceed to the solution of the block circulant OSP. The proof relies on properties of the insertion and natural projection maps and will be illustrated by an example at the end of the paper.

Theorem 8.4: The block circulant OSP is solvable if and only if

$$\mathcal{X}^+(A) \subset \mathcal{C} + \mathcal{V}^\star . \tag{8}$$

B. Disturbance Decoupling

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{9}$$
$$z(t) = Dx(t) \tag{10}$$

where $x(t) \in \mathbb{R}^{rn}$, $u(t) \in \mathbb{R}^{rm}$, $z(t) \in \mathbb{R}^{rq}$, and $w(t) \in \mathbb{R}^{rs}$. The signal $w(t)$ is a disturbance which is assumed not to be directly measurable by the controller. We would like to find a state feedback $u = Kx$ so that the controlled output $z(t)$ is not affected by any disturbance $w(t)$. The disturbance is assumed to belong to some sufficiently rich class of signals, reflecting our lack of knowledge about the characteristics of this signal. We say the closed-loop system is *disturbance decoupled* if for each initial condition $x(0) \in \mathcal{X}$, the output $z(t)$ is the same for every $w(t)$. Let $\mathcal{E} = \operatorname{Im} E$. Turning to block circulant systems, a geometric statement of the disturbance decoupling problem (DDP) is as follows.

Problem 8.2 (Disturbance Decoupling Problem): Given the block circulant system (D, A, B, E) with $A \in \mathfrak{C}(\Pi_r \otimes I_n)$, $B \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_m)$, $D \in \mathfrak{C}(\Pi_r \otimes I_q, \Pi_r \otimes I_n)$, and $E \in \mathfrak{C}(\Pi_r \otimes I_n, \Pi_r \otimes I_s)$, find a block circulant state feedback $u = Kx$ such that

$$\langle A + BK \mid \mathcal{E} \rangle \subset \operatorname{Ker} D . \tag{11}$$

Theorem 8.5: The block circulant DDP is solvable if and only if

$$\mathcal{E} \subset \mathcal{V}^\star \tag{12}$$

where $\mathcal{V}^\star = \sup \mathscr{I}(\operatorname{Ker} D)$.

C. Measurement Feedback

We study the problem of finding a static measurement feedback $u = K'y$ such that $x(t) \to 0$ as $t \to \infty$. Following our geometric approach, the problem is transformed to find an A-invariant subspace $\mathcal{L} \subset \mathcal{X}$ and a state feedback $u = Kx$ such that

$$\operatorname{Ker} C \subset \mathcal{L} \subset \operatorname{Ker} K .$$

The requirement $\mathcal{L} \subset \operatorname{Ker} K$ gives the interpretation to $\mathcal{L}$ as a "masking subspace" that characterizes what state information cannot be used in state feedback, or equivalently what state information is masked out by K. Intuitively, the larger the dimension of $\mathcal{L}$, the less state information that can appear in the feedback. The requirement $\operatorname{Ker} C \subset \mathcal{L}$ imposes that only the measurements y can be used in the feedback. Now if $\operatorname{Ker} C$ were A-invariant, then the best choice for $\mathcal{L}$ would be $\mathcal{L} = \operatorname{Ker} C = \mathcal{N}$. Generally, $\operatorname{Ker} C$ is not A-invariant, and the next best choice is the smallest A-invariant subspace containing $\operatorname{Ker} C$, namely $\langle A | \operatorname{Ker} C \rangle$. Thus we arrive at the stabilization by measurement feedback problem (SMFP) for block circulant systems.

Problem 8.3 (SMFP): Given a block circulant triple (C, A, B), find a block circulant state feedback $u = Kx$ such that

$$\operatorname{Ker} C \subset \operatorname{Ker} K \tag{13}$$
$$\sigma(A + BK) \subset \mathbb{C}^- . \tag{14}$$

Lemma 8.6: The subspace $\langle A | \operatorname{Ker} C \rangle$ is a block circulant subspace.

Theorem 8.7: The block circulant SMFP is solvable if

$$\mathcal{X}^+(A) \subset \langle A | \mathcal{B} \rangle , \tag{15}$$
$$\mathcal{X}^+(A) \cap \langle A \mid \operatorname{Ker} C \rangle = 0 . \tag{16}$$

D. Output Stabilization by Measurement Feedback

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{17}$$
$$y(t) = Cx(t) \tag{18}$$
$$z(t) = Dx(t) \tag{19}$$

where $x(t) \in \mathbb{R}^{rn}$, $u(t) \in \mathbb{R}^{rm}$, $y(t) \in \mathbb{R}^{rp}$, and $z(t) \in \mathbb{R}^{rq}$. The *output stabilization by measurement feedback problem* is to find a measurement feedback $u = Ky$ such that $z(t) \to 0$ as $t \to \infty$. If (C, A, B) is controllable and observable, then the problem can be solved by observer-based feedback. In the more general case, ideas originating from the solutions of OSP and SMFP must be used.

Recall from SMFP that the set of states which cannot be used in a measurement feedback can be characterized by some A-invariant subspace $\mathcal{L}$ for which the state feedback $u = Kx$ must satisfy $\mathcal{L} \subset \operatorname{Ker} K$. Ideally, one would choose $\mathcal{L}$ to be the unobservable subspace of (C, A). However, this choice may not always be feasible, given the constraint that $z(t) \to 0$. If a valid $\mathcal{L}$ can be found, then the problem can be converted to the more tractable Restricted Regulator Problem (RRP) [15], stated here for block circulant systems.

Problem 8.4 (Restricted Regulator Problem (RRP)): Given a block circulant subspace $\mathcal{L}$ such that $A\mathcal{L} \subset \mathcal{L}$, find $K \in \mathfrak{C}(\Pi_r \otimes I_m, \Pi_r \otimes I_n)$ such that

$$\mathcal{L} \subset \operatorname{Ker} K , \tag{20}$$
$$\mathcal{X}^+(A + BK) \subset \operatorname{Ker} D . \tag{21}$$

Theorem 8.8: The block circulant RRP is solvable if and only if there exists a block circulant subspace $\mathcal{V} \in \mathscr{I}(\operatorname{Ker} D)$ such that

$$A(\mathcal{L} \cap \mathcal{V}) \subset \mathcal{L} \cap \mathcal{V} \tag{22}$$

$$\mathcal{X}^+(A) \cap \mathcal{L} \subset \mathcal{L} \cap \mathcal{V} \tag{23}$$

$$\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{V}\,. \tag{24}$$

IX. Example

There are many physical systems which lend themselves to our block circulant control methodology. We presented a number of these applications earlier, and our future work will specifically explore power systems and formation control. Here, we present a purely pedagogical example of the block circulant output stabilization problem.

For notational simplicity, define $U := \Pi_4 \otimes I_2$ and $V := \Pi_4 \otimes I_1$. Consider the block circulant system (D, A, B), where $A \in \mathfrak{C}(U)$, $B \in \mathfrak{C}(U, V)$, and $D \in \mathfrak{C}(V, U)$ are given by

$$A = \left[\begin{array}{rr|rr|rr|rr} -1 & 5 & 3 & 1 & -4 & -3 & 3 & 1 \\ 0 & -7 & 0 & -1 & 0 & -3 & 0 & -1 \\ \hline 3 & 1 & -1 & 5 & 3 & 1 & -4 & -3 \\ 0 & -1 & 0 & -7 & 0 & -1 & 0 & -3 \\ \hline -4 & -3 & 3 & 1 & -1 & 5 & 3 & 1 \\ 0 & -3 & 0 & -1 & 0 & -7 & 0 & -1 \\ \hline 3 & 1 & -4 & -3 & 3 & 1 & -1 & 5 \\ 0 & -1 & 0 & -3 & 0 & -1 & 0 & -7 \end{array}\right], \quad B = \left[\begin{array}{r|r|r|r} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right],$$

$$D = \left[\begin{array}{rr|rr|rr|rr} -5 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ \hline -1 & 0 & -5 & 0 & -1 & 0 & 1 & 0 \\ \hline 1 & 0 & -1 & 0 & -5 & 0 & -1 & 0 \\ \hline -1 & 0 & 1 & 0 & -1 & 0 & -5 & 0 \end{array}\right].$$

The unstable subspace of A is given by $\mathcal{X}^+(A) := \operatorname{Ker}(A - 8I) = \operatorname{span}\{a_1, a_2\}$ where $a_1 = (1, 0, 0, 0, -1, 0, 0, 0)$ and $a_2 = (0, 0, 1, 0, 0, 0, -1, 0)$. We also compute the controllable subspace $\langle A|\mathcal{B}\rangle = \operatorname{span}\{c_1, \cdots, c_6\}$, where $c_1 := e_1$, $c_2 := e_3$, $c_3 := e_5$, $c_4 := e_7$, $c_5 := (0, 1, 0, 0, 0, -1, 0, 0)$, and $c_6 := (0, 0, 0, 1, 0, 0, 0, -1)$. Following Lemma 6.1, $\langle A|\mathcal{B}\rangle$ is a block circulant subspace. It can immediately be seen that $a_1 = c_1 - c_3$ and $a_2 = c_2 - c_4$. Thus, $\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle$.

Following Theorem 4.3 of [15], the supremal controlled invariant subspace contained in $\operatorname{Ker} D$ can be shown to be $\mathcal{V}^\star = \operatorname{span}\{v_1, v_2\}$, where $v_1 = (0, 1, 0, 0, 0, 1, 0, 0)$ and $v_2 = (0, 0, 0, 1, 0, 0, 0, 1)$; it can also be verified that $\mathcal{V}^\star$ is a block circulant subspace. Because $\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle$, we immediately have that $\mathcal{X}^+(A) \subset \mathcal{V}^\star + \langle A|\mathcal{B}\rangle$. By Theorem 8.4, the block circulant output stabilization problem is solvable.

Following Lemma 8.1, we find a feedback transformation $u = Fx + v$, where $\mathcal{V}^\star$ is $(A + BF)$-invariant for the matrix

$$F = \frac{1}{2}\left[\begin{array}{rr|rr|rr|rr} 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \end{array}\right] \in \mathfrak{C}(V, U)\,.$$

This gives the transformed system $\dot{x} = (A + BF)x + Bv$.

Following Theorem 8.3, we define a coordinate transformation based on the state space decomposition $\mathcal{X} = \mathcal{V}^\star \oplus (\mathcal{V}^\star)^\perp$. In this particular example, $(\mathcal{V}^\star)^\perp = \langle A|\mathcal{B}\rangle$, so we choose the coordinate transformation $T = \begin{bmatrix} v_1 & v_2 & c_1 & \cdots & c_6 \end{bmatrix}$, giving

$$T^{-1}(A + BF)T = \begin{bmatrix} \widetilde{A}_1 & 0 \\ 0 & \widetilde{A}_2 \end{bmatrix}, \; T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; DT = \begin{bmatrix} 0 & D_2 \end{bmatrix}$$

where, in particular, $\widetilde{A}_2 \in \mathfrak{C}(U_{(\mathcal{V}^\star)^\perp})$ and $B_2 \in \mathfrak{C}(U_{(\mathcal{V}^\star)^\perp}, V)$. Therefore, to output-stabilize (D, A, B), it is sufficient to stabilize $(\widetilde{A}_2, B_2)$. One such stabilizing state feedback is

$$K_2' = \frac{1}{2}\begin{bmatrix} -25 & 0 & 25 & 0 & -32 & 0 \\ 0 & -25 & 0 & 25 & 0 & -32 \\ 25 & 0 & -25 & 0 & 32 & 0 \\ 0 & 25 & 0 & -25 & 0 & 32 \end{bmatrix} \in \mathfrak{C}(V, U_{(\mathcal{V}^\star)^\perp})$$

Using Lemma 5.7, K_2' can be lifted back into a matrix K' (in the full space $\mathcal{X}$) which output-stabilizes the feedback-transformed system $(D, A + BF, B)$. The overall block circulant output stabilizing feedback for the original system (D, A, B) is $K := F + K'$, given by

$$K = \frac{1}{2}\left[\begin{array}{rr|rr|rr|rr} -25 & -17 & 0 & -1 & 25 & 15 & 0 & -1 \\ \hline 0 & -1 & -25 & -17 & 0 & -1 & 25 & 15 \\ \hline 25 & 15 & 0 & -1 & -25 & -17 & 0 & -1 \\ \hline 0 & -1 & 25 & 15 & 0 & -1 & -25 & -17 \end{array}\right] \in \mathfrak{C}(V, U)\,.$$

X. Conclusion

In this paper we showed that control systems that possess a certain structure — that of subsystems connected in a ring — can be controlled in a way that respects and maintains that structure. The ring structure is algebraically encoded in commuting properties of the system's block circulant matrices. An important outcome is that the conditions for solvability of each of the design problems are almost completely decoupled from the block circulant structure. Indeed, we did not have to modify any linear geometric control conditions in order to guarantee the possibility of recovering a block circulant control law. Our work suggests that structured systems *naturally* admit structured controllers.

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