Controllability Is Not Sufficient for Pole Placement in Patterned Systems

Adam C. Sniderman[1], Mireille E. Broucke[2], and Gabriele M. T. D'Eleuterio[1]

Abstract— **In distributed systems, a common question is how to synthesize a control law that adheres to the system's distributed interconnection structure. One promising synthesis approach involves encoding the system's interconnections as a pattern in the system matrices, and carrying this pattern through all steps of a control design. To date, though, all control results on this front have had limitations on which distributed structures are admissible. In this paper, we provide a pole placement result that is valid for more general system patterns. We also show that a system's controllability does not imply patterned pole placement, so we introduce a novel notion of controllability for patterned distributed systems.**

I. Introduction

The focus of control systems is shifting from centralization to distribution — rather than doing all decision making and information processing in one place, many systems spread these jobs among a number of parts of the system. While system architectures move in this direction, control methods lag behind. In this paper, we work towards bridging this gap by providing a pole placement method for controlling a distributed system with some "pattern" using a feedback with the same pattern. Surprisingly, pole placement with a patterned feedback law cannot be guaranteed by the standard notion of controllability (somewhat like the decentralized fixed modes introduced by [1]). We provide a stronger notion of controllability that enforces a system's pattern and preserves its distributed structure.

In most previous work on distributed control, a system's distributed structure is represented as constraints in the matrices of the system model. One common approach involves fixing some matrix entries at zero [2, 3, 4]; the role of controllability was explored, but a full framework for control has not yet been produced. A more general approach to decentralized control [5] synthesizes a separate dynamic controller for each agent, though the analysis is complex.

A recent approach to distributed control is *patterned linear systems*, in which the distributed structure and matrix constraints are encoded as a pattern in the system's matrices via an algebraic relationship. In particular, [6, 7] encode the pattern as polynomials of a "base matrix", and [8, 9] encode the pattern by commuting relationships with a base matrix. We follow the latter approach here. All previous work on patterned systems places restrictions on the allowed patterns, constraining the base matrix to be, for instance, diagonalizable or unitary. These constraints cannot be met by many common distributed structures — most notably, chains — and so these methods are as of yet unusable for control of distributed systems in a general sense. This paper is the first to consider patterned control without these constraints: *any* pattern can be carried through our pole placement method without restriction.

This paper is organized as follows. In Section II, patterned systems are introduced. In Section III, we provide a counterexample to show that controllability is not a sufficient condition for patterned pole placement. In Section IV, another example shows how patterned pole placement works. In Sections V–VI , we develop the method of patterned pole placement and the notion of "patterned controllability". These results are formalized in the appendix; some proofs are suppressed because of space limitations.

II. Patterned Systems

A *patterned matrix* is a matrix M that satisfies a commuting relationship $UM = MV$ for some matrices U and V. This commuting relationship is denoted by $M \in \mathfrak{C}(U, V)$. As a shorthand, if $U = V$, then $\mathfrak{C}(U) := \mathfrak{C}(U, U)$. Standard operations on patterned matrices preserve the patterns:

- $M, N \in \mathfrak{C}(U, V)$ and $\alpha \in \mathbb{R} \Rightarrow \alpha M + N \in \mathfrak{C}(U, V)$.
- $M \in \mathfrak{C}(U, V)$ and $N \in \mathfrak{C}(V, W) \Rightarrow MN \in \mathfrak{C}(U, W)$.

A *U-patterned system* is a linear control system of the form $\dot{x} = Ax + Bu$ where $A \in \mathfrak{C}(U)$ and $B \in \mathfrak{C}(U, V)$. U is called the *base matrix* of the system. A patterned feedback for this system is of the form $u = Kx$ with $K \in \mathfrak{C}(V, U)$. A pattern often prescribes a specific form in system matrices, which lends itself to use in distributed systems. Two examples are given here.

Example 2.1: Consider the ring system in Figure 1 (left), whose parts each have n states and m inputs, are identical, and have identical interconnections (so part "1" is coupled to "2" in the same way as "2" is coupled to "3", and so on). This manifests as a *block circulant* pattern in the system matrices:

$$A = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_3 & A_1 & A_2 \\ A_2 & A_3 & A_1 \end{bmatrix}, B = \begin{bmatrix} B_1 & B_2 & B_3 \\ B_3 & B_1 & B_2 \\ B_2 & B_3 & B_1 \end{bmatrix}; \quad \Pi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

where each $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$. Every block circulant matrix commutes with the block fundamental permutation matrix: $A(\Pi_3 \otimes I_n) = (\Pi_3 \otimes I_n)A$ and $B(\Pi_3 \otimes I_m) = (\Pi_3 \otimes I_n)B$. Therefore, this system is $(\Pi_3 \otimes I_n)$-patterned. A full treatment of control of ring systems is given in [8]. ◄

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[1] A. Sniderman and G. D'Eleuterio are with the Institute for Aerospace Studies, University of Toronto, Ontario, Canada.

[2] M. Broucke is with the Department of Electrical and Computer Engineering, University of Toronto, Ontario, Canada.

Example 2.2: Consider the chain system in Figure 1 (right), where again, the parts of the system are identical and have identical interconnections. This manifests as a *block lower triangular Toeplitz* pattern in the system matrices:

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & A_1 & 0 \\ A_3 & A_2 & A_1 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 & 0 \\ B_2 & B_1 & 0 \\ B_3 & B_2 & B_1 \end{bmatrix}; \quad N_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Every block lower triangular Toeplitz matrix commutes with the block fundamental nilpotent matrix: $A(N_3 \otimes I_n) = (N_3 \otimes I_n)A$ and $B(N_3 \otimes I_m) = (N_3 \otimes I_n)B$. Therefore, this system is $(N_3 \otimes I_n)$-patterned. A full treatment of control of chain systems (in a patterned sense) has not been possible in the past — the base matrix $N_r \otimes I_n$ does not fit the restrictions on base matrices in previous work [7, 9], though *scalar* chain systems ($n = m = 1$) were fully treated by [6]. ◀

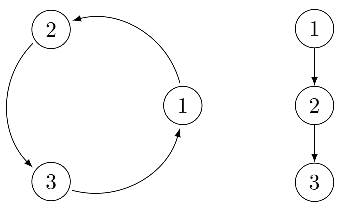

Fig. 1. Examples of Patterned Distributed Systems: Ring and Chain

III. Controllability Is Not Sufficient

The following example will show that controllability is not a strong enough condition to guarantee pole placement with a patterned feedback. Consider the base matrix U,

$$U = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -4 & 3 & -4 & 4 & 1 \\ -3 & 3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma_U = \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ -1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad (1)$$

and the U-patterned system (A, B) given by

$$A = \begin{bmatrix} 7 & -1 & 4 & -3 & 0 \\ -3 & 6 & -3 & 3 & 6 \\ 11 & -5 & 14 & -12 & -3 \\ 18 & -9 & 18 & -15 & -6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 2 \\ 2 & 0 & 3 & -3 & -1 \\ 3 & 0 & 3 & -3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $A, B \in \mathfrak{C}(U)$ (and $V = U$ for this example). The system's open-loop poles are given by $\sigma(A) = \{3, \ldots, 3\}$. To check whether these poles can be placed by a patterned feedback, the system matrices will be transformed into a particular block structured form [10, §VIII.1]: applying the transformation $\widehat{A} := \Gamma_U^{-1} A \Gamma_U$ and $\widehat{B} := \Gamma_U^{-1} \Gamma_B \Gamma_U$ (with Γ_U given in (1)) gives

$$\widehat{A} = \left[\begin{array}{ccc|cc} 3 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 3 \end{array}\right], \quad \widehat{B} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

whose blocks are all upper triangular Toeplitz (U△T)[1]. By the same transformation, any patterned feedback $K \in \mathfrak{C}(U)$ can be put in a form with the same U△T block structure:

$$\widehat{K} := \Gamma_U^{-1} K \Gamma_U = \left[\begin{array}{ccc|cc} k_1 & * & * & k_2 & * \\ 0 & k_1 & * & 0 & k_2 \\ 0 & 0 & k_1 & 0 & 0 \\ \hline 0 & k_3 & * & k_4 & * \\ 0 & 0 & k_3 & 0 & k_4 \end{array}\right].$$

By direct calculation, it can be seen that $\sigma(\widehat{A} + \widehat{B}\widehat{K}) = \{3 + k_1, 3 + k_1, 3 + k_1, 3, 3\}$, and so two eigenvalues of A — those fixed at 3 — can never be moved by patterned feedback. These immovable poles are reminiscent of the decentralized fixed modes first presented by [1].

On the other hand, the system is controllable, since its controllable subspace is the full state space $\langle A|\mathcal{B}\rangle = \mathbb{R}^5$. Thus, the poles of (A, B) can be placed into any symmetric spectrum. In sum, even though every pole can be moved by some feedback, some poles cannot be moved by any U-*patterned* feedback. To capture the idea of pole placement via patterned feedback, a stronger notion of controllability is needed.

IV. A Simple Example of Patterned Pole Placement

Before delving into the intuition and theory behind patterned pole placement, a simple example is presented here. Consider $\widehat{U}$ and $\widehat{V}$ given by the diagonal matrices

$$\widehat{U} = \left[\begin{array}{c:c|c|c} \delta^1 & & & \\ \hdashline & \delta^1 & & \\ \hline & & \delta^2 & \\ \hline & & & \delta^3 \end{array}\right], \quad \widehat{V} = \left[\begin{array}{c|c|c} \delta^1 & & \\ \hline & \delta^2 & \\ \hline & & \delta^3 \end{array}\right] \qquad (2)$$

where dashed lines separate identical entries, and solid lines separate different entries. Clearly, $\sigma(\widehat{U}) = \{\delta^1, \delta^1, \delta^2, \delta^3\}$ and $\sigma(\widehat{V}) = \{\delta^1, \delta^2, \delta^3\}$ (in that order). For a $\widehat{U}$-patterned system $(\widehat{A}, \widehat{B})$, the commuting relationships $\widehat{U}\widehat{A} = \widehat{A}\widehat{U}$ and $\widehat{U}\widehat{B} = \widehat{B}\widehat{V}$ give the form

$$\widehat{A} = \left[\begin{array}{c:c|c|c} a_1 & a_2 & & \\ \hdashline a_3 & a_4 & & \\ \hline & & a_5 & \\ \hline & & & a_6 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \end{array}}\right\} \widehat{A}^1 \\ \left.\vphantom{a}\right\} \widehat{A}^2 \\ \left.\vphantom{a}\right\} \widehat{A}^3 \end{array}, \quad \widehat{B} = \left[\begin{array}{c|c|c} b_1 & & \\ \hdashline b_3 & & \\ \hline & b_5 & \\ \hline & & b_6 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} b \\ b \end{array}}\right\} \widehat{B}^1 \\ \left.\vphantom{b}\right\} \widehat{B}^2 \\ \left.\vphantom{b}\right\} \widehat{B}^3 \end{array}. \qquad (3)$$

Notice that $(\widehat{A}, \widehat{B})$ has a block diagonal structure that splits it into three independent subsystems, denoted $(\widehat{A}^i, \widehat{B}^i)$ ($i = 1, 2, 3$, as partitioned by the solid lines). If $(\widehat{A}, \widehat{B})$ is controllable, then its poles can clearly be placed in each subsystem independently, giving a feedback of the form

$$\widehat{K} = \left[\begin{array}{c:c|c|c} k_1 & k_2 & & \\ \hline & & k_5 & \\ \hline & & & k_6 \end{array}\right] \begin{array}{l} \}\ \widehat{K}^1 \\ \}\ \widehat{K}^2 \\ \}\ \widehat{K}^3 \end{array}.$$

By direct calculation, $\widehat{K}$ is the form of the pattern $\mathfrak{C}(\widehat{V}, \widehat{U})$, and it can easily be seen that the closed-loop system has

[1]Nonsquare blocks are considered U△T if they have the form $\mathrm{row}(0, X)$ or $\mathrm{col}(X, 0)$, where X is a square upper triangular Toeplitz matrix; that is, the square upper triangular Toeplitz part can be augmented with columns of zeros on the left, or rows of zeros on the bottom.

the pattern $\widehat{A}+\widehat{B}\widehat{K} \in \mathfrak{C}(\widehat{U})$, thus preserving the system's original pattern.

Lastly, notice that the matrices $\widehat{A}^i$, $\widehat{B}^i$, and $\widehat{K}^i$ are all unpatterned (meaning that they have no nontrivial commuting relationships). So not only can the poles be placed independently in each subsystem but, if a subsystem is controllable, then its poles can also be placed arbitrarily; preserving the system's pattern is equivalent to preserving the independent subsystems.

Overall, patterned pole placement in this example is accomplished in three steps: the patterned system is split into independent subsystems, the poles are placed in each subsystem by any (unpatterned) feedbacks, and those feedbacks are then reconstructed into a patterned feedback for the overall system. More generally, as explained next, patterned systems can be split into unpatterned subsystems using the Jordan blocks of the base matrices.

V. A Method for Patterned Pole Placement

This section details the general algorithm for patterned pole placement in a patterned system, and will proceed as follows: first, we introduce important notation; second, we discuss the spectra of patterned matrices; third, we give a method to transform patterned matrices into a U$\triangle$T block form, based on the Jordan forms of their base matrices; fourth, we introduce a method for removing this U$\triangle$T block structure, resulting in unpatterned subsystems called the *reduced form*; and fifth, we perform a standard pole placement on these unpatterned subsystems.

Remark 5.1: Let $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ have eigenvalues $\sigma(U)\cup\sigma(V)$, and remove repeated eigenvalues to get the "distinct spectrum" $\sigma_{\mathrm{d}}(U)\cup\sigma_{\mathrm{d}}(V)=\{\delta^1,\dots,\delta^r\}$. Similarly, collect all the Jordan block sizes in U and V into a set, and remove repeated entries to give the distinct block sizes $\mathrm{s}_{\mathrm{d}}(U)\cup\mathrm{s}_{\mathrm{d}}(V)=\{d_1,\dots,d_\rho\}$. Then, every Jordan block of U and V is associated with some eigenvalue δ^i ($i\in\{1,\dots,r\}$) and block size $d_j\times d_j$ ($j\in\{1,\dots,\rho\}$). Define $n^i_j \geq 0$ and $m^i_j \geq 0$ to be the number of Jordan blocks corresponding to (δ^i, d_j) in U and V, respectively.

Since U and V are real, there exists a "conjugate permutation" $\{\varepsilon_1,\dots,\varepsilon_r\}$ of $\{1,\dots,r\}$ such that $\delta^{\varepsilon_i}=\bar{\delta}^i$, $n^{\varepsilon_i}_j=n^i_j$, and $m^{\varepsilon_i}_j=m^i_j$ for each i and j. ◀

Next, the attainable spectra of a patterned matrix are prescribed by its base matrix:

Definition 5.2 (U-Patterned Spectrum): Let $U \in \mathbb{R}^{n\times n}$ follow the ordering of Remark 5.5. A spectrum $\mathscr{L}$ is called *U-patterned* if it has the same cardinality as $\sigma(U)$ and can be partitioned as follows:

(E1) $\mathscr{L}=\mathscr{L}^1\uplus\cdots\uplus\mathscr{L}^r$, where $\mathscr{L}^{\varepsilon_i}=\overline{\mathscr{L}}^i$ for each i;

(E2) $\mathscr{L}^i=\mathscr{L}^i_1\uplus\cdots\uplus\mathscr{L}^i_\rho$, where $\mathscr{L}^i_j$ contains n^i_j eigenvalues occurring d_j times (for each j).

Lemma 5.3: If $A\in\mathfrak{C}(U)$, then $\sigma(A)$ is U-patterned.

The eigenvalues of $A \in \mathfrak{C}(U)$ can therefore be paired to those of U in a particular way. (E1) says that if two eigenvalues in $\sigma(U)$ are complex conjugates, then the corresponding eigenvalues in $\sigma(A)$ are also complex conjugates; and (E2) says that if two eigenvalues in $\sigma(U)$ are in the same Jordan block (and therefore equal), then the corresponding eigenvalues in $\sigma(A)$ are also equal.

Now, we discuss a particular transformation on patterned matrices. Let (A,B) be a U-patterned system, so $A\in\mathfrak{C}(U)$ and $B\in\mathfrak{C}(U,V)$. Define the Jordan forms $\widehat{U}:=\Gamma_U^{-1}U\Gamma_U$ and $\widehat{V}:=\Gamma_V^{-1}V\Gamma_V$, with Jordan blocks for identical eigenvalues listed consecutively. Apply the transformations $\widehat{A}:=\Gamma_U^{-1}A\Gamma_U$ and $\widehat{B}:=\Gamma_U^{-1}B\Gamma_V$; then, the transformed matrices have the block diagonal form [10]

$$\widehat{A}=\mathrm{diag}(\widehat{A}^1,\dots,\widehat{A}^r)\,,\quad \widehat{B}=\mathrm{diag}(\widehat{B}^1,\dots,\widehat{B}^r) \qquad (4)$$

where the size of each submatrix is determined by the algebraic multiplicity of δ^i in U and V. Clearly, (A,B) is split into r independent subsystems $(\widehat{A}^i,\widehat{B}^i)$; by [10], the $\widehat{A}^i$ and $\widehat{B}^i$ are made up of U$\triangle$T blocks, whose sizes correspond to the Jordan block sizes in $\widehat{U}$ and $\widehat{V}$. It is also important to note that the $(\widehat{A}^i,\widehat{B}^i)$ subsystems are purely mathematical, and do not necessarily correspond to agents in the physical distributed system.

Example 5.4: Consider U and V with Jordan forms $\widehat{U}:=\Gamma_U^{-1}U\Gamma_U$ and $\widehat{V}:=\Gamma_V^{-1}V\Gamma_V$ given by

$$\widehat{U}=\begin{bmatrix}\delta^1 & 1 &&&&&\\ & \delta^1 &&&&&\\ && \delta^1 & 1 &&&\\ &&& \delta^1 &&&\\ &&&& \delta^1 &&\\ &&&&& \delta^2 & 1\\ &&&&&& \delta^2\end{bmatrix},\ \widehat{V}=\begin{bmatrix}\delta^1 & 1 &&&\\ & \delta^1 &&&\\ && \delta^1 &&\\ &&& \delta^2 & 1\\ &&&& \delta^2\end{bmatrix}$$

For any U-patterned system (A,B), the transformed matrices $\widehat{A}:=\Gamma_U^{-1}A\Gamma_U$ and $\widehat{B}:=\Gamma_U^{-1}B\Gamma_V$ have the form

$$\widehat{A}=\begin{bmatrix}a_1 & * & a_2 & * & a_3 &&\\ 0 & a_1 & 0 & a_2 & 0 &&\\ a_4 & * & a_5 & * & a_6 &&\\ 0 & a_4 & 0 & a_5 & 0 &&\\ 0 & a_7 & 0 & a_8 & a_9 &&\\ &&&&& a_0 & *\\ &&&&& 0 & a_0\end{bmatrix},\ \widehat{B}=\begin{bmatrix}b_1 & * & b_3 &&\\ 0 & b_1 & 0 &&\\ b_4 & * & b_6 &&\\ 0 & b_4 & 0 &&\\ 0 & b_7 & b_9 &&\\ &&& b_0 & *\\ &&& 0 & b_0\end{bmatrix}$$

where the size of each U$\triangle$T block is determined by the corresponding Jordan block sizes in U and V. This splits (A,B) into two subsystems $(\widehat{A}^1,\widehat{B}^1)$ (top-left) and $(\widehat{A}^2,\widehat{B}^2)$ (bottom-right); these subsystems are still patterned, having a U$\triangle$T block structure. ◀

The next step is crucial to the pole placement method. In (3), the $(\widehat{A}^i,\widehat{B}^i)$ subsystems were unpatterned, so their poles could be placed arbitrarily without affecting the system's pattern. In contrast, the subsystems in Example 5.4 (and in general) are made up of U$\triangle$T blocks, and are therefore *not* unpatterned. This U$\triangle$T block structure will be removed to obtain unpatterned subsystems on which standard pole placement results can be applied. These unpatterned subsystems are generated by removing duplicate information from $(\widehat{A},\widehat{B})$; in particular, each set of equal eigenvalues from constraint (E2) (from Definition 5.2) is reduced to a single eigenvalue. To that end, the matrix entries that should be kept are those that contribute to determining the eigenvalues; it can be shown (see the proof of Lemma 5.8 in the appendix) that these entries lie on the diagonal of the matrix's square U△T blocks. To extract these entries in a precise way, it will help to put them into known positions by further ordering the Jordan blocks of U and V (from Remark 5.1) by the following steps:

1. Order the Jordan blocks of U and V so that their corresponding eigenvalues follow the order $\{\delta^1, \ldots, \delta^r\}$.
2. For each group of Jordan blocks corresponding to the same eigenvalue, further order the blocks so that their sizes follow the order $\{d_1, \ldots, d_\rho\}$. (In this paper, we always order by decreasing size.)

More formally, the Jordan block ordering is as follows.

Remark 5.5: Continuing from Remark 5.1, order the Jordan blocks of U and V as

$$(\delta^1, d_1), \ldots, (\delta^1, d_\rho), \;\cdots\; , (\delta^r, d_1), \ldots, (\delta^r, d_\rho)\,.$$

This ordering groups together all repeated eigenvalues and all identical Jordan blocks, partitioning the Jordan forms $\widehat{U}$ (and $\widehat{V}$) as

$$\begin{aligned} \widehat{U} \; &= \mathrm{diag}(\widehat{U}^1\,, \ldots\,, \widehat{U}^r)\,, \\ \widehat{U}^i &= \mathrm{diag}(\widehat{U}^i_1\,, \ldots\,, \widehat{U}^i_\rho) \end{aligned} \tag{5}$$

where each $\widehat{U}^i_j$ is composed of all the Jordan blocks with eigenvalue δ^i and block size $d_j \times d_j$. Also, since U and V are real, it follows from the conjugate permutation of Remark 5.1 that $\widehat{U}^{\varepsilon_i}_j = \overline{\widehat{U}^i_j}$ and $\widehat{V}^{\varepsilon_i}_j = \overline{\widehat{V}^i_j}$. ◀

Applying the above ordering, each $\widehat{A}^i$ and $\widehat{B}^i$ from (4) can be further partitioned by Jordan block size, as shown in the following result.

Lemma 5.6: Let U and V follow the ordering of Remark 5.5. Let $M \in \mathfrak{C}(U, V)$, and define $\widehat{M} := \Gamma_U^{-1} M \Gamma_V$. Then, $\widehat{M}$ has the form

$$\widehat{M} = \begin{bmatrix} \widehat{M}^1 & & \\ & \ddots & \\ & & \widehat{M}^r \end{bmatrix}, \quad \widehat{M}^i = \begin{bmatrix} \widehat{M}^i_{11} & \cdots & \widehat{M}^i_{1\rho} \\ \vdots & & \vdots \\ \widehat{M}^i_{\rho 1} & \cdots & \widehat{M}^i_{\rho\rho} \end{bmatrix} \tag{6}$$

where $\widehat{M}^i_{j_1 j_2}$ is made up of $n^i_{j_1} \times m^i_{j_2}$ U△T blocks of size $d_{j_1} \times d_{j_2}$, and $\widehat{M}^{\varepsilon_i}_{j_1 j_2} = \overline{\widehat{M}^i_{j_1 j_2}}$ for each $i = 1, \ldots, r$ and $j_1, j_2 = 1, \ldots, \rho$.

Looking at the block sizes in (6), it can be deduced that all square U△T blocks will occur in the $\widehat{M}^i_{jj}$ blocks down the diagonal of the partition. Thus, a patterned system's poles are fully determined by the diagonal entries of the square U△T blocks in the $\widehat{A}^i_{jj}$, which have been localized to the block diagonal of $\widehat{A}$. Having pinpointed those entries, they are isolated into a new matrix that contains the "distilled" essence of the eigenvalues of A, called the *reduced form* of A and denoted $\breve{A}$; B is treated similarly. These reduced forms are constructed from $\widehat{A}$ and $\widehat{B}$ by taking a diagonal entry from each square U△T block, thereby creating a block diagonal matrix with submatrices $\breve{A}^i_j$ and $\breve{B}^i_j$ (corresponding to the distinct eigenvalues and Jordan block sizes of U and V, indexed by i and j respectively). Formally, the reduced form is defined as follows.

Definition 5.7 (Reduced Form): Suppose U and V follow the ordering of Remark 5.5, and let $M \in \mathfrak{C}(U, V)$. Then, the *reduced form* of M is given by

$$\begin{aligned} \breve{M} \; &= \mathrm{diag}(\breve{M}^1\,, \ldots\,, \breve{M}^r)\,, \\ \breve{M}^i &= \mathrm{diag}(\breve{M}^i_1\,, \ldots\,, \breve{M}^i_\rho) \end{aligned} \tag{7}$$

where each entry of $\breve{M}^i_j \in \mathbb{C}^{n^i_j \times m^i_j}$ is the diagonal entry of the corresponding U△T block in $\widehat{M}^i_{jj}$.

Reducing each U△T block to a single entry removes the U△T structure from $\widehat{A}$ and $\widehat{B}$, and so the reduced subsystems $(\breve{A}^i_j, \breve{B}^i_j)$ are unpatterned. Also, the reduced form contains exactly the information that affects the system's poles, as seen in the next result.

Lemma 5.8: Let $A \in \mathfrak{C}(U)$ be a patterned matrix with reduced form $\breve{A} := \mathrm{diag}(\ldots, \breve{A}^i_j, \ldots)$ as in (7). Then, the eigenvalues of A are given by those of the $\breve{A}^i_j$, with multiplicities given by the corresponding block sizes d_j:

$$\sigma(A) = \biguplus_{i=1}^{r} \biguplus_{j=1}^{\rho} \Big(\underbrace{\sigma(\breve{A}^i_j) \uplus \cdots \uplus \sigma(\breve{A}^i_j)}_{d_j \text{ times}} \Big)\,.$$

Example 5.9: In Example 5.4, the Jordan blocks in $\widehat{U}$ and $\widehat{V}$ are already ordered as in Remark 5.5. From this ordering, the square U△T blocks of $\widehat{A}$ and $\widehat{B}$ are all on their diagonals (as partitioned by solid lines), and the eigenvalues of $\widehat{A}$ are fully determined by the diagonal entries of those square U△T blocks: $\sigma\left(\left[\begin{smallmatrix} a_1 & a_2 \\ a_4 & a_5 \end{smallmatrix}\right]\right)$, a_9, and a_0 (ignoring multiplicities).

The reduced forms of A and B are given by

$$\breve{A} = \left[\begin{array}{cc|c|c} a_1 & a_2 & & \\ a_4 & a_5 & & \\ \hline & & a_9 & \\ \hline & & & a_0 \end{array}\right], \quad \breve{B} = \left[\begin{array}{c|c|c} b_1 & & \\ b_4 & & \\ \hline & b_9 & \\ \hline & & b_0 \end{array}\right]$$

and it can clearly be seen that the eigenvalues of $\breve{A}$ are exactly the same as those of A (ignoring multiplicities). Also, each block is unpatterned, since there is no particular relationship among its entries. Thus, the original patterned system (A, B) has been split into three unpatterned subsystems[2] $(\breve{A}^1_1, \breve{B}^1_1)$, $(\breve{A}^1_2, \breve{B}^1_2)$, and $(\breve{A}^2_1, \breve{B}^2_1)$ (from the solid-line partitions), and the size of each submatrix is determined by the number of times the corresponding Jordan blocks repeat. ◀

In short, placing the poles of the $\breve{A}^i_j$ using arbitrary feedbacks is equivalent to placing the poles of $\widehat{A}$ using a U△T block feedback, and also to placing the poles of A using a patterned feedback. The reduced form thereby opens the door to patterned pole placement: poles can be placed arbitrarily in each subsystem $(\breve{A}^i_j, \breve{B}^i_j)$ using any known method (e.g., [11]); then, these individual reduced feedbacks can be combined and "expanded" into a patterned feedback $\widehat{K}$ by turning each

[2]In this notation, the superscript corresponds to the eigenvalue number in $\widehat{U}$ and $\widehat{V}$: "1" is δ^1 and "2" is δ^2. The subscript corresponds to the enumerated Jordan block sizes, in order of appearance: "1" is size 2×2, and "2" is size 1×1.

entry into a correctly sized U△T block, as demonstrated below.

Example 5.10: Continuing from Example 5.9, place the poles in each reduced subsystem in the standard way, using feedbacks $\check{K}_1^1 = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, $\check{K}_2^1 = \begin{bmatrix} k_9 \end{bmatrix}$, and $\check{K}_1^2 = \begin{bmatrix} k_0 \end{bmatrix}$. Then, expand those into a feedback for $(\widehat{A}, \widehat{B})$ by the mapping

$$\check{K} = \left[\begin{array}{c:c|c|c} k_1 & k_2 & & \\ \hline & & k_9 & \\ \hline & & & k_0 \end{array}\right] \mapsto \widehat{K} = \left[\begin{array}{cc:cc|c|cc} k_1 & 0 & k_2 & 0 & 0 & & \\ 0 & k_1 & 0 & k_2 & 0 & & \\ \hline 0 & 0 & 0 & 0 & k_9 & & \\ \hline & & & & & k_0 & 0 \\ & & & & & 0 & k_0 \end{array}\right]$$

so each entry has been expanded into a U△T block, whose length and height match the Jordan block sizes in $\widehat{V}$ and $\widehat{U}$, respectively. It can be seen that $\widehat{A} + \widehat{B}\widehat{K}$ has the same U△T block structure as $\widehat{A}$, guaranteeing that the original pattern is preserved in the overall feedback $K := \Gamma_V \widehat{K} \Gamma_U^{-1} \in \mathfrak{C}(V, U)$ and the closed-loop system $A + BK \in \mathfrak{C}(U)$. Lastly, a direct calculation shows that the eigenvalue-determining entries have all been moved:

$$\begin{aligned} &a_1 \mapsto a_1 + b_1 k_1 \,, \quad a_2 \mapsto a_2 + b_1 k_2 \,, \quad a_4 \mapsto a_4 + b_4 k_1 \,, \\ &a_5 \mapsto a_5 + b_4 k_2 \,, \quad a_9 \mapsto a_9 + b_9 k_9 \,, \quad a_0 \mapsto a_0 + b_0 k_0 \,. \end{aligned}$$

Thus, arbitrary pole placement in the reduced system gives rise to patterned pole placement in the overall system. ◀

In general, expanding the reduced feedbacks $\check{K}_j^i$ into diagonal blocks as above results in a matrix $\widehat{K}$ with the same U△T block structure as $\widehat{A}$ and $\widehat{B}$. Then, applying the initial system transformation in reverse is guaranteed to give a patterned feedback $K := \Gamma_V \widehat{K} \Gamma_U^{-1} \in \mathfrak{C}(V, U)$ [10] that moves the poles of (A, B) as desired.

Algorithm 1: Overall, the method for pole placement in a patterned system is as follows:

1. Group together the base matrix's repeated eigenvalues and identical Jordan blocks, as in Remark 5.5.
2. Construct the reduced form of the system matrices following Lemma A.1. This splits the patterned system into a number of unpatterned subsystems.
3. Place the poles individually in each subsystem by finding any arbitrary feedbacks, using any method.
4. Expand the "reduced feedbacks" back into a full patterned feedback following Lemma A.2.

VI. Main Result on Patterned Controllability

How can the usual notion of controllability be extended to capture when a patterned system can be "controlled in any patterned way"? When the poles of a patterned system can be placed into any patterned spectrum using patterned feedback? In Section V, this patterned pole placement was accomplished by splitting the system into unpatterned reduced subsystems, and placing the poles independently on those. Thus, the "patterned controllability" of the full patterned system can be fully characterized by the "traditional controllability" of the reduced system.

Definition 6.1 (Patterned Controllability): A U-patterned system (A, B) is called *U-patterned controllable* if the corresponding reduced subsystems $(\check{A}_j^i, \check{B}_j^i)$ are all controllable (in the usual sense).

From this definition, moving poles in (A, B) in a "patterned way" is equivalent to moving poles in each $(\check{A}_j^i, \check{B}_j^i)$ in any arbitrary way. Therefore, just as controllability captures the possibility of pole placement, patterned controllability captures the possibility of patterned pole placement.

Theorem 6.2 (Patterned Pole Placement): Let $A \in \mathfrak{C}(U)$ and $B \in \mathfrak{C}(U, V)$. The U-patterned system (A, B) is U-patterned controllable if and only if for every U-patterned spectrum $\mathscr{L}$, there exists a patterned feedback $K \in \mathfrak{C}(V, U)$ such that $\sigma(A + BK) = \mathscr{L}$.

Thus, we have provided an expanded notion of controllability and pole placement in patterned systems, which opens the door to a more general theory for control of distributed systems.

Appendix
Formalization and Selected Proofs

In Section V, patterned pole placement was achieved by first putting a patterned system in its reduced form, in which the poles could be placed arbitrarily. This reduced form comes about in two main steps. First, a Jordan transformation of the base matrices induces a U△T block form in a patterned matrix [10, §VIII.1], with certain blocks guaranteed to be zero. The nonzero blocks can be grouped together by following the ordering of Remark 5.5, making the matrix block diagonal (as in Lemma 5.6). Second, from this block diagonal form, the matrix can be put into its reduced form (as defined in Definition 5.7) by keeping a diagonal entry from each square U△T block of $\widehat{M}$. An explicit equation for determining a patterned matrix's reduced form is given here.

Lemma A.1: Let $M \in \mathfrak{C}(U, V)$, where U and V follow the ordering of Remark 5.5. Define the Jordan forms $\widehat{U} := \Gamma_U^{-1} U \Gamma_U$ and $\widehat{V} := \Gamma_V^{-1} V \Gamma_V$ as in (5), where Γ_U^{-1} and Γ_V are ordered and partitioned such that

$$\begin{aligned} \Gamma_U^{-1} &= \operatorname{col}\left(\mathrm{T}_1^1, \ldots, \mathrm{T}_\rho^1, \ \cdots \ , \mathrm{T}_1^r, \ldots, \mathrm{T}_\rho^r\right) , \\ \Gamma_V &= \operatorname{row}\left(\Gamma_1^1, \ldots, \Gamma_\rho^1, \ \cdots \ , \Gamma_1^r, \ldots, \Gamma_\rho^r\right) . \end{aligned} \tag{8}$$

Here, Γ_U and Γ_V are constructed from generalized eigenvectors (in Jordan chains) of U and V, where Γ_j^i has $d_j n_j^i$ columns and T_j^i has $d_j n_j^i$ rows. Furthermore, choose these matrices to satisfy the conjugate permutation, so that $\Gamma_j^{\varepsilon_i} = \overline{\Gamma}_j^i$ and $\mathrm{T}_j^{\varepsilon_i} = \overline{\mathrm{T}}_j^i$ for each i and j. Also define $e_k := \operatorname{col}(1, 0, \ldots, 0) \in \mathbb{R}^k$. Then, the reduced form of M is given by $\check{M} = \operatorname{diag}\left(\check{M}_1^1, \ldots, \check{M}_\rho^1, \ \cdots \ , \check{M}_1^r, \ldots, \check{M}_\rho^r\right)$ (as in (7)), where

$$\check{M}_j^i = \left(I_{n_j^i} \otimes e_{d_j}^{\mathrm{T}}\right)\left(\mathrm{T}_j^i M \Gamma_j^i\right)\left(I_{m_j^i} \otimes e_{d_j}\right) \in \mathbb{C}^{n_j^i \times m_j^i} . \tag{9}$$

It can be deduced from (9) that the $\check{M}_j^i$ are unique up to choice of generalized eigenvectors in Γ_U and Γ_V. Conversely to Lemma A.1, a reduced matrix can be expanded into a full patterned matrix by turning each entry into a full U△T block.

Lemma A.2: Suppose $\breve{M} := \mathrm{diag}(\ldots, \breve{M}^i_j, \ldots)$ is in reduced form with respect to the pattern $\mathfrak{C}(U,V)$ (following Remark 5.5), and define $\widehat{M}^i := \mathrm{diag}(\breve{M}^i_1 \otimes I_{d_1},\ \ldots\ ,\breve{M}^i_\rho \otimes I_{d_\rho})$. Then, $M := \Gamma_U\ \mathrm{diag}(\widehat{M}^1, \ldots, \widehat{M}^r)\ \Gamma_V^{-1} \in \mathfrak{C}(U,V)$.

Different patterned matrices might have the same reduced form, and the "lifting" procedure in Lemma A.2 is certainly not the only way to produce a patterned matrix from reduced form. Nevertheless, with these results for transforming a matrix to and from its reduced form, the main results of the paper can be proven.

Proof of Lemma 5.3 (U-Patterned Spectra): Clearly, $\sigma(M)$ and $\sigma(U)$ have the same cardinality since $UM = MU$. Define $\widehat{M}$ as in Lemma 5.6, and take $\mathscr{L}^i := \sigma(\widehat{M}^i)$ for $i = 1, \ldots, r$. Then, $\sigma(M) = \mathscr{L}^1 \uplus \cdots \uplus \mathscr{L}^r$. Since $\widehat{M}^{\varepsilon_i} = \overline{\widehat{M}^i}$ by Lemma 5.6, it follows that $\mathscr{L}^{\varepsilon_i} = \overline{\mathscr{L}}^i$, giving (E1). Next, define $\breve{M}$ as in Lemma A.1, and take $\mathscr{L}^i_j = \sigma(\breve{M}^i_j) \uplus \cdots \uplus \sigma(\breve{M}^i_j)$ (repeated d_j times). Clearly, $\mathscr{L}^i_j$ contains n^i_j eigenvalues occurring d_j times each. Also, $\mathscr{L}^i = \mathscr{L}^i_1 \uplus \cdots \uplus \mathscr{L}^i_\rho$ by Lemma 5.8, giving (E2). ■

Proof Sketch of Lemma 5.8 (Spectra from Reduced Form): Follow the ordering of Remark 5.5, and define $\widehat{A} := \Gamma_U^{-1} A \Gamma_U = \mathrm{diag}(\widehat{A}^1, \ldots, \widehat{A}^r)$ as in (6). Clearly, $\sigma(A) = \sigma(\widehat{A}) = \biguplus_{i=1}^r \sigma(\widehat{A}^i)$. Also, it can be shown by induction that $\sigma(\widehat{A}^i) = \biguplus_{j=1}^\rho \sigma(\widehat{A}^i_{jj})$.

It remains to show that $\sigma(\widehat{A}^i_{jj})$ is given by the eigenvalues of $\breve{A}^i_j$, repeated d_j times each. If $\breve{A}^i_j$ has a generalized eigenvector v, then by direct calculation, $\breve{A}^i_j \otimes I_{d_j}$ has d_j generalized eigenvectors $v \otimes e_k$ corresponding to the same eigenvalue (where the e_k are the standard basis vectors of $\mathbb{R}^k$). Thus, $\sigma(\breve{A}^i_j \otimes I_{d_j})$ must be given by $\sigma(\breve{A}^i_j)$ repeated d_j times. Lastly, it can be shown that since $\widehat{A}^i_{jj}$ and $\breve{A}^i_j \otimes I_{d_j}$ have the same entries along the diagonal of each square U△T block, they also have the same eigenvalues. ■

Proof of Theorem 6.2 (Patterned Pole Placement): Partition the U-patterned spectrum $\mathscr{L}$ as in Definition 5.2, so $\mathscr{L} = \left(\mathscr{L}^1_1 \uplus \cdots \uplus \mathscr{L}^1_\rho\right) \uplus \cdots \uplus \left(\mathscr{L}^r_1 \uplus \cdots \uplus \mathscr{L}^r_\rho\right)$. For each $i = 1, \ldots, r$, and $j = 1, \ldots, \rho$, $\mathscr{L}^i_j$ contains n^i_j eigenvalues occurring d_j times each. Define $\breve{\mathscr{L}}^i_j$ as the set containing one of each of the n^i_j eigenvalues.

(Only If) Suppose (A,B) is U-patterned controllable. By definition, $(\breve{A}^i_j, \breve{B}^i_j)$ is controllable for each i and j, where $\breve{A}^i_j \in \mathbb{C}^{n^i_j \times n^i_j}$ and $\breve{B}^i_j \in \mathbb{C}^{n^i_j \times m^i_j}$. Therefore, there exists $\breve{K}^i_j \in \mathbb{C}^{m^i_j \times n^i_j}$ such that $\sigma(\breve{A}^i_j + \breve{B}^i_j \breve{K}^i_j) = \breve{\mathscr{L}}^i_j$. These individual reduced feedbacks can be combined into an overall patterned feedback $K := \Gamma_U\ \mathrm{diag}(\ldots, \breve{K}^i_j \otimes I_{d_j}, \ldots)\ \Gamma_V^{-1} \in \mathfrak{C}(V,U)$ by Lemma A.2. Thus, by Lemma 5.8,

$$\sigma(A + BK) = \biguplus_{i=1}^{r} \biguplus_{j=1}^{\rho} \Big(\underbrace{\breve{\mathscr{L}}^i_j \uplus \cdots \uplus \breve{\mathscr{L}}^i_j}_{(d_j \text{ times})} \Big) = \mathscr{L}\,.$$

(If) Suppose that for any U-patterned spectrum $\mathscr{L}$, there exists a patterned feedback $K \in \mathfrak{C}(V,U)$ such that $\sigma(A + BK) = \mathscr{L}$. Put the system into reduced form as in Lemma A.1, so $A + BK$ is split into reduced submatrices $\breve{A}^i_j + \breve{B}^i_j \breve{K}^i_j$, with $\sigma(\breve{A}^i_j + \breve{B}^i_j \breve{K}^i_j) = \breve{\mathscr{L}}^i_j$. Furthermore, from Definition 5.2, it can be seen that $\breve{\mathscr{L}}^i_j$ can be made to have any desired set of n^i_j eigenvalues by choice of $\mathscr{L}$. Therefore, the poles of $(\breve{A}^i_j, \breve{B}^i_j)$ can be placed arbitrarily, and so $(\breve{A}^i_j, \breve{B}^i_j)$ is controllable for each i and j. By definition, (A,B) is U-patterned controllable. ■

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