Stabilizing Patterned Distributed Systems by State and Measurement Feedback ⋆

Adam C. Sniderman * **Mireille E. Broucke** ** **Gabriele M. T. D'Eleuterio** *

* *Institute for Aerospace Studies, University of Toronto, Canada.*
** *Dep't of Electrical and Computer Engineering, University of Toronto.*

Abstract: It is well known that a system is stabilizable if and only if its unstable part is controllable. However, constraints placed on the system's controller — such as those imposed by a distributed system's interconnection structure — can render this condition insufficient. In this paper, we provide a new necessary and sufficient condition for the stabilizability of distributed systems encoded by some *pattern* that is reflected in the system matrices. We also provide sufficient conditions for stabilizability by static measurement feedback in these systems. Our conditions are generalizations of those from the standard geometric approach.

Keywords: Stabilizability; Distributed feedback; Geometric approach; Linear systems

1. INTRODUCTION

Control system architectures have undergone a major shift in recent decades from centralized systems, in which a single unit performs the entire control action, to distributed systems, in which the control action is split among multiple interconnected subsystems. Our research is driven by the need to develop control synthesis methods to keep pace with these highly distributed architectures. Wang and Davison [1973] discovered that standard control syntheses are not viable for distributed systems — in particular, a stabilizable system is not necessarily stabilizable by *distributed* feedback. Many methods have been proposed to address this discrepancy. In this paper, we continue our presentation of a control framework for *patterned systems*, as developed by Hamilton and Broucke [2012], Sniderman et al. [2013, 2015a,b, 2017], Holmes and Broucke [2016], and Ornik et al. [2016]. A patterned system encodes a distributed system's interconnection structure via an algebraic relationship among the system matrices. Any synthesized control law must also follow that pattern. We build on our prior work [Sniderman et al., 2015a, 2017] to obtain a result on stabilization: we provide a necessary and sufficient condition for the existence of a patterned feedback law that stabilizes a patterned system. Our condition takes a form reminiscent of the standard geometric condition for unpatterned stabilizability [Wonham, 1979].

Patterned linear systems were first introduced for scalar agents by Hamilton and Broucke [2012], with the pattern encoded algebraically by polynomials of a *base matrix*. This encoding proved difficult to extend to multivariate agents (beyond one result by Massioni and Verhaegen [2009]), so we introduced a new encoding using *commuting relationships* [Sniderman et al., 2013, 2015b]. This initial work only considered distributed systems with ring interconnection structures, following a line of research begun by Brockett and Willems [1974]. We extended these results to patterned systems with diagonalizable base matrices [Sniderman et al., 2017], solving several control problems on this restricted class. Concurrently, Consolini and Tosques [2014, 2015] and Holmes and Broucke [2016] studied these problems for patterned systems with several unitary base matrices.

Our ultimate goal is to develop a control framework for multivariate agents with *any* interconnection structure amenable to a patterned encoding; as such, the prior restrictions to diagonalizable/unitary base matrices are undesirable. At present, the only result that does not require these restrictions is our Patterned Pole Placement Theorem [Sniderman et al., 2015a]. We discovered that the standard notion of controllability does not suffice to achieve pole placement by patterned feedback. This discovery led to the introduction of a patterned system's *condensed form*, whose role is to factor out the pattern, leaving an "unpatterned part" of the system on which standard controllability results apply. In this paper, we exploit the condensed form and the Pole Placement Theorem to solve the Stabilization Problem: we show that a system can be stabilized by patterned feedback if and only if its condensed form can be stabilized by any feedback. Unlike prior work on patterned stabilization, our results do not require extraneous assumptions on system topology, restrictions to univariate agents, or conditions on base matrices.

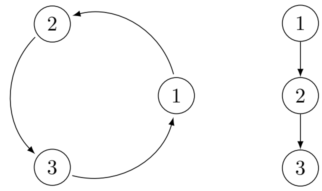

Fig. 1. Patterned distributed systems: ring and chain.

To demonstrate the utility of patterned systems, we briefly present two examples of common interconnection structures, the ring and the unidirectional chain (see

⋆ This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Figure 1). Ornik et al. [2016] showed that both these interconnection structures, when modelling with linear dynamics $\dot{x} = Ax$, can be encoded using commuting relationships $VA = AV$: the systems' state matrices (A) and base matrices (V) have the forms

$$A_{\text{ring}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_3 & A_1 & A_2 \\ A_2 & A_3 & A_1 \end{bmatrix} \quad \Leftrightarrow \quad V_{\text{ring}} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix}$$
$$A_{\text{chain}} = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & A_1 & 0 \\ A_3 & A_2 & A_1 \end{bmatrix} \quad \Leftrightarrow \quad V_{\text{chain}} = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.$$

It is notable that while V_{ring} fits the constraints of many of the above-cited works, V_{chain} does not. As such, the chain is an interconnection structure that cannot be analyzed by prior frameworks for patterned systems.

Having shown that commuting relationships are a valid encoding for distributed structures, we move on to the main problem of patterned stabilization.

2. PROBLEM FORMULATION AND MAIN RESULT

Consider a linear system $\dot{x} = Ax + Bu$ whose matrices satisfy commuting relationships $VA = AV$ and $VB = BU$ (for some matrices V and U). We call this system (V, U)*-patterned*, and the commuting relationships encode the underlying distributed interconnection structure (as demonstrated above). The standard Stabilization Problem seeks a static state feedback $u = Kx$ for which the closed-loop poles $\sigma(A+BK)$ (i.e., the eigenvalues of $A+BK$) are in the open left complex half-plane $\mathbb{C}^-$. We now repose that problem, adding the stipulation that the feedback match the system's pattern.

Problem 1. (Patterned Stabilization Problem). Given a (V, U)-patterned system (A, B), find a patterned state feedback K ($UK = KV$) such that $\sigma(A + BK) \subset \mathbb{C}^-$.

The standard Stabilization Problem is solvable if and only if the system's instabilities are all controllable; in geometric terms, $\mathcal{X}^+(A) \subset \mathcal{C}$, where $\mathcal{X}^+(A)$ and $\mathcal{C}$ are the unstable subspace and controllable subspace, respectively (defined in §3). This condition is not sufficient for solvability of Problem 1. Instead, we convert the system (A, B) into its *condensed form* $(\breve{A}, \breve{B})$, a reduced-order model that will be defined in (9), which removes certain redundancy imposed by the pattern. Using this construction, the standard stabilization condition can once again be checked.

Theorem 2. A patterned system (A, B) is stabilizable by patterned feedback if and only if

$$\breve{\mathcal{X}}^+(\breve{A}) \subset \breve{\mathcal{C}} \tag{1}$$

where $\breve{\mathcal{X}}^+(\breve{A})$ and $\breve{\mathcal{C}}$ are the unstable subspace and controllable subspace of the condensed system $(\breve{A}, \breve{B})$.

When the Patterned Stabilization Problem is solvable, it can be solved algorithmically. First, transform the system into a particular form based on the Jordan form of the base matrices (Theorem 6). Use that structured form to find the system's condensed form (Definition 8), and determine an induced pattern (Lemma 10). Next, decompose the condensed system into its controllable and uncontrollable parts; each part has its own pattern (Theorem 18). Place the poles of the controllable part as desired (Corollary 16), and then expand that pole-placing feedback to the full system (Lemma 11). The resulting feedback will be patterned, and will stabilize the system. These steps will be explained in depth through §§4–5, and then Theorem 2 will be formally proven in §6. We will discuss an extension to measurement feedback in §7.

3. MATHEMATICAL PRELIMINARIES

The following notation is used. A square matrix M has eigenvalues in the spectrum $\sigma(M)$. The Kronecker product is denoted $A \otimes B$. I_n is the $n \times n$ identity matrix.

An upper triangular Toeplitz (U△T) matrix has the form

$$X_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ & \ddots & \ddots & \vdots \\ \vdots & & x_1 & x_2 \\ 0 & \cdots & & x_1 \end{bmatrix}. \tag{2}$$

A nonsquare $n \times m$ matrix X is U△T if it has the form $X = \text{row}(0, X_n)$ (for $n < m$) or $X = \text{col}(X_m, 0)$ (for $n > m$); that is, the square part can be augmented with columns of zeros on the left, or rows of zeros on the bottom.

We assume the reader is familiar with linear geometric control theory, [e.g., Wonham, 1979]. Let $\mathcal{X}$ be an n-dimensional vector space with k-dimensional subspace $\mathcal{S} \subset \mathcal{X}$. Let $\mathcal{S}^c \subset \mathcal{X}$ be a complementary subspace, so $\mathcal{X} = \mathcal{S} \oplus \mathcal{S}^c$. We make use of two standard projection maps. First, the *insertion map* $S : \mathcal{S} \to \mathcal{X}$ maps $x \in \mathcal{S}$ to the corresponding element $x \in \mathcal{X}$; in coordinates, S maps $k \times 1$ coordinate vectors to the corresponding $n \times 1$ coordinate vectors. Second, the *natural projection* $Q : \mathcal{X} \to \mathcal{S}$ (along $\mathcal{S}^c$) maps $x \in \mathcal{X}$ to its component in $\mathcal{S}$; that is, given the unique representation $x = s + r$ with $s \in \mathcal{S}$ and $r \in \mathcal{S}^c$, $Qx = s$. Note that $QS = I_k$.

Let $M : \mathcal{X} \to \mathcal{X}$ be a linear map. If $\mathcal{S} \subset \mathcal{X}$ is M-invariant ($M\mathcal{S} \subset \mathcal{S}$), then the *restriction* of M to $\mathcal{S}$ is the unique solution $M_{\mathcal{S}}$ of $MS = SM_{\mathcal{S}}$. Next, define projection maps S_1, Q_1 on $\mathcal{S}$ and S_2, Q_2 on $\mathcal{S}^c$ (which is not necessarily M-invariant). The columns of $T := [S_1 \ S_2]$ form a basis for $\mathcal{X}$ adapted to $\mathcal{S}$, and $T^{-1} = \text{col}(Q_1, Q_2)$. Thus,

$$T^{-1}MT = \begin{bmatrix} Q_1MS_1 & Q_1MS_2 \\ Q_2MS_1 & Q_2MS_2 \end{bmatrix} =: \begin{bmatrix} M_1 & * \\ 0 & M_2 \end{bmatrix}. \tag{3}$$

Lemma 3. Let $\mathcal{S} \subset \mathcal{X}$ be an M-invariant subspace with complement $\mathcal{S}^c$, and projection maps as above.

(i) Let $M_1 := Q_1MS_1$. Then, $MS_1 = S_1M_1$.
(ii) Let $M_2 := Q_2MS_2$. Then, $Q_2M = M_2Q_2$.

If an M-invariant subspace $\mathcal{S} \subset \mathcal{X}$ has an M-invariant complement $\mathcal{S}^c$, it is called *M-decoupling*. If M is diagonalizable, then M-invariance and M-decoupling are equivalent.

Lemma 4. (Gohberg et al. [1986, Thm. 3.1.2]). Let M be a diagonalizable matrix, and let $\mathcal{S}$ be an M-invariant subspace. Then, $\mathcal{S}$ is an M-decoupling subspace.

Let $M : \mathcal{X} \to \mathcal{X}$ be a linear map with minimal polynomial ψ. The unstable subspace of M is $\mathcal{X}^+(M) := \text{Ker}\,\psi^+(M)$, where ψ^+ contains all factors of ψ in the closed right complex half-plane $\overline{\mathbb{C}}^+$. Similarly, if $\mathcal{S} \subset \mathcal{X}$, the unstable modal subspace of $\widetilde{M} : \mathcal{S} \to \mathcal{S}$ is denoted $\mathcal{S}^+(\widetilde{M})$.

Lemma 5. Let $\mathcal{S} \subset \mathcal{X}$ be an M-invariant subspace with complement $\mathcal{S}^c$. Define projection maps S_2, Q_2 on $\mathcal{S}^c$, and let $M_2 := Q_2 M S_2$. Then, $(\mathcal{S}^c)^+(M_2) = Q_2 \mathcal{X}^+(M)$.

Lastly, the minimal M-invariant subspace containing $\mathcal{S}$ is denoted $\langle M|\mathcal{S}\rangle := \mathcal{S} + M\mathcal{S} + \cdots + M^{n-1}\mathcal{S}$. A system (A, B) has *controllable subspace* $\mathcal{C} := \langle A|\operatorname{Im} B\rangle$, the minimal A-invariant subspace containing the image of B.

4. PATTERNED MATRICES AND SYSTEMS

In this section, we discuss some initial results that are required in the proof of Theorem 2. In particular, we define and analyze the condensed form. Owing to space constraints, we only present a few proof sketches.

Formally, a *patterned matrix* M satisfies a commuting relationship $VM = MU$, which we denote $M \in \mathfrak{C}(V, U)$. The matrices V and U are called the pattern's *base matrices*. In this paper, always assume that M, V, and U are real-valued. We also use the shorthand $\mathfrak{C}(V) := \mathfrak{C}(V, V)$.

The condensed form comes about through a structure in patterned matrices that is exposed by transforming the base matrices V and U into their Jordan forms $\widehat{V} := \Gamma_V^{-1} V \Gamma_V$ and $\widehat{U} := \Gamma_U^{-1} U \Gamma_U$. Suppose every Jordan block in $\widehat{V}$ and $\widehat{U}$ is given by one of the r distinct choices $J^1, \ldots, J^r$, and suppose each J^i appears ν^i times in $\widehat{V}$ and μ^i times in $\widehat{U}$. Also suppose J^i has size $n^i \times n^i$, and is associated with eigenvalue $\gamma^i \in \mathbb{C}$. The Jordan blocks can be ordered as

$$\widehat{V} = \begin{bmatrix} I_{\nu^1} \otimes J^1 & & \\ & \ddots & \\ & & I_{\nu^r} \otimes J^r \end{bmatrix}, \widehat{U} = \begin{bmatrix} I_{\mu^1} \otimes J^1 & & \\ & \ddots & \\ & & I_{\mu^r} \otimes J^r \end{bmatrix}. \quad (4)$$

Furthermore, there exists a *conjugate permutation* $\{\varepsilon^1, \ldots, \varepsilon^r\}$ of $\{1, \ldots, r\}$ such that each $J^{\varepsilon^i} = \bar{J}^i$, $\nu^{\varepsilon^i} = \nu^i$, and $\mu^{\varepsilon^i} = \mu^i$ (for $i = 1, \ldots, r$). We can choose the Jordan transformations such that

$$\begin{aligned} \Gamma_U &= [\Gamma_U^1 \ \cdots \ \Gamma_U^r] \quad \text{with } \Gamma_U^{\varepsilon^i} = \bar{\Gamma}_U^i \in \mathbb{C}^{m \times (\mu^i n^i)} \\ \Gamma_V &= [\Gamma_V^1 \ \cdots \ \Gamma_V^r] \quad \text{with } \Gamma_V^{\varepsilon^i} = \bar{\Gamma}_V^i \in \mathbb{C}^{n \times (\nu^i n^i)} \end{aligned} \quad (5)$$

and it can be shown that additionally

$$\Gamma_V^{-1} = \operatorname{col}(\daleth_V^1, \ldots, \daleth_V^r) \quad \text{with } \daleth_V^{\varepsilon^i} = \bar{\daleth}_V^i. \quad (6)$$

This grouping and partitioning exposes the structure in a patterned matrix $M \in \mathfrak{C}(V, U)$, by adapting a result from Gantmacher [1959, §VIII.1].

Theorem 6. Suppose $V \in \mathbb{R}^{n \times n}$ and $U \in \mathbb{R}^{m \times m}$ have Jordan forms as in (4)–(5). Let $M \in \mathbb{R}^{n \times m}$, and define $\widehat{M} := \Gamma_V^{-1} M \Gamma_U$. Then, $M \in \mathfrak{C}(V, U)$ if and only if

$$\widehat{M} = \begin{bmatrix} \widehat{M}^{11} & \cdots & \widehat{M}^{1r} \\ \vdots & & \vdots \\ \widehat{M}^{r1} & \cdots & \widehat{M}^{rr} \end{bmatrix}, \ \widehat{M}^{ij} = \begin{bmatrix} \widehat{M}_{11}^{ij} & \cdots & \widehat{M}_{1\mu^i}^{ij} \\ \vdots & & \vdots \\ \widehat{M}_{\nu^i 1}^{ij} & \cdots & \widehat{M}_{\nu^i \mu^i}^{ij} \end{bmatrix} \quad (7)$$

where $\widehat{M}^{ij} \in \mathbb{C}^{(\nu^i n^i) \times (\mu^j n^j)}$ satisfies the following: if $\gamma^i \neq \gamma^j$, then $\widehat{M}^{ij} = 0$; if $\gamma^i = \gamma^j$, then every $\widehat{M}_{kl}^{ij} \in \mathbb{C}^{n^i \times n^j}$ is an U△T matrix; and for all i and j, $\widehat{M}^{\varepsilon^i \varepsilon^j} = \overline{\widehat{M}}^{ij}$.

If a patterned system $\dot{x} = Ax + Bu$ with $A \in \mathfrak{C}(V)$ and $B \in \mathfrak{C}(V, U)$ has diagonalizable base matrices V and U, then Theorem 6 can be applied to synthesize patterned controllers [Sniderman et al., 2017] as follows.

Algorithm 7.

1. Using the ordering (4)–(5), the transformed matrices $\widehat{A}$ and $\widehat{B}$ in Theorem 6 are block diagonal, since $\widehat{A}^{ij} = 0$ and $\widehat{B}^{ij} = 0$ for $i \neq j$. Thus, the original system splits into r decoupled modal subsystems.
2. Since V and U are diagonalizable, the Jordan blocks J^i are all scalars, so also the U△T matrices in (7) are scalars. This means the diagonal blocks $(\widehat{A}^{ii}, \widehat{B}^{ii})$ have no nontrivial commuting relationships — they are *unpatterned* [Sniderman et al., 2017, Lem. 3.13].
3. (A, B) has thus been split into r unpatterned subsystems $(\widehat{A}^{ii}, \widehat{B}^{ii})$. Synthesize a controller for each one.
4. Stack the unpatterned controllers into a block diagonal matrix, and transform it to the original coordinates using Theorem 6. The resulting feedback is patterned.

Algorithm 7 allows us to reproduce many standard control results for patterned systems with diagonalizable base matrices. However, the situation when base matrices are not diagonalizable remains an open problem. The difficulty arises in step 3: the U△T blocks in $\widehat{A}^{ij}$ and $\widehat{B}^{ij}$ are not scalar in general, so the constituent subsystems are not unpatterned. For controller synthesis, the U△T structure must be preserved in order to recover a patterned feedback.

Given $M \in \mathfrak{C}(V, U)$, the pattern manifests in repeated elements in the U△T subblocks $\widehat{M}_{kl}^{ij}$. Our **key idea** is to remove this structural repetition, essentially eliminating the redundancy inherent in U△T matrices. We form a new matrix — the *condensed form* — by extracting the entry on the main diagonal of each block $\widehat{M}_{kl}^{ij}$. Define $\nu := \nu^1 + \cdots + \nu^r$ and $\mu := \mu^1 + \cdots + \mu^r$.

Definition 8. Let $M \in \mathfrak{C}(V, U)$, and follow the Jordan block grouping of (4)–(5). Choose any nonsingular matrices $\breve{\Gamma}_V \in \mathbb{C}^{\nu \times \nu}$ and $\breve{\Gamma}_U \in \mathbb{C}^{\mu \times \mu}$ that satisfy

$$\begin{aligned} \breve{\Gamma}_U &= [\breve{\Gamma}_U^1 \ \cdots \ \breve{\Gamma}_U^r] \quad \text{with } \breve{\Gamma}_U^{\varepsilon^i} = \bar{\breve{\Gamma}}_U^i \in \mathbb{C}^{\mu \times \mu^i} \\ \breve{\Gamma}_V &= [\breve{\Gamma}_V^1 \ \cdots \ \breve{\Gamma}_V^r] \quad \text{with } \breve{\Gamma}_V^{\varepsilon^i} = \bar{\breve{\Gamma}}_V^i \in \mathbb{C}^{\nu \times \nu^i}. \end{aligned} \quad (8)$$

Then, the *condensed form* $\breve{M} \in \mathbb{R}^{\nu \times \mu}$ of M is defined as

$$\breve{M} := \breve{\Gamma}_V \begin{bmatrix} \breve{M}^1 & & \\ & \ddots & \\ & & \breve{M}^r \end{bmatrix} \breve{\Gamma}_U^{-1}, \ \breve{M}^i = \begin{bmatrix} \breve{m}_{11}^i & \cdots & \breve{m}_{1\mu^i}^i \\ \vdots & & \vdots \\ \breve{m}_{\nu^i 1}^i & \cdots & \breve{m}_{\nu^i \mu^i}^i \end{bmatrix} \quad (9)$$

where each $\breve{m}_{kl}^i$ is the diagonal entry of the U△T block $\widehat{M}_{kl}^{ii}$. Also, $\breve{M}^{\varepsilon^i} = \overline{\breve{M}}^i$ for every $i = 1, \ldots, r$.

Remark 9. A subtle distinction between the construction of $\breve{M}$ and $\widehat{M}$ regards the manner in which certain blocks are defined to be zero. In $\widehat{M}$, a block is fixed at zero when the corresponding eigenvalues are distinct, while in $\breve{M}$, an entry is fixed at zero when the corresponding *Jordan blocks* are distinct (different eigenvalues or block sizes). By defining $\breve{M}$ this way, we obtain a mapping between the eigenvalues of M and $\breve{M}$ (given in Lemma 13). As we will show, this mapping allows us to place any patterned system's poles by a similar method to Algorithm 7.

The next result shows that the condensed form $\breve{M}$ is patterned in its own right: for suitable base matrices, a matrix that satisfies the resulting commuting relationship is the condensed form of some patterned matrix in $\mathfrak{C}(V, U)$.

Lemma 10. Given $V \in \mathbb{R}^{n\times n}$ and $U \in \mathbb{R}^{m\times m}$, there exist diagonalizable matrices $\breve{V} \in \mathbb{R}^{\nu\times\nu}$ and $\breve{U} \in \mathbb{R}^{\mu\times\mu}$ such that the following hold:

(i) If $M \in \mathfrak{C}(V,U)$, then $\breve{M} \in \mathfrak{C}(\breve{V},\breve{U})$;
(ii) If $\breve{M} \in \mathfrak{C}(\breve{V},\breve{U})$, then $\breve{M}$ is the condensed form of some $M \in \mathfrak{C}(V,U)$.

Proof. Choose any nonsingular matrices $\breve{\Gamma}_V$ and $\breve{\Gamma}_U$ in the form of (8). Choose any distinct $\breve{\gamma}^1,\ldots,\breve{\gamma}^r \in \mathbb{C}$ such that $\breve{\gamma}^{\varepsilon^i} = \overline{\breve{\gamma}}^i$ for each $i = 1,\ldots,r$, and define

$$\begin{aligned}\breve{V} &= \breve{\Gamma}_V \operatorname{diag}(\breve{\gamma}^1 I_{\nu^1},\ldots,\breve{\gamma}^r I_{\nu^r})\breve{\Gamma}_V^{-1}\\ \breve{U} &= \breve{\Gamma}_U \operatorname{diag}(\breve{\gamma}^1 I_{\mu^1},\ldots,\breve{\gamma}^r I_{\mu^r})\breve{\Gamma}_U^{-1}\end{aligned} \tag{10}$$

which are real-valued and diagonalizable by construction.

Suppose $M \in \mathfrak{C}(V,U)$. Take the condensed form $\breve{M} = \breve{\Gamma}_V W \breve{\Gamma}_U^{-1}$ as in (9), and partition $W = [W^{ij}]$. Then, W is in the form of (7) because $W^{ij} = 0$ for $i \neq j$ (since $\breve{\gamma}^i \neq \breve{\gamma}^j$), and because each W^{ij}_{kl} is a scalar. By Theorem 6, $\breve{M} \in \mathfrak{C}(\breve{V},\breve{U})$, giving (i).

Next, suppose $\breve{M} \in \mathfrak{C}(\breve{V},\breve{U})$. Noting that $\breve{\Gamma}_V = \Gamma_{\breve{V}}$ and $\breve{\Gamma}_U = \Gamma_{\breve{U}}$ by construction, let $\operatorname{diag}(\breve{M}^1,\ldots,\breve{M}^r) := \breve{\Gamma}_V^{-1}\breve{M}\breve{\Gamma}_U$ by Theorem 6. Define

$$M := \Gamma_V \begin{bmatrix} \breve{M}^1 \otimes I_{n^1} & & \\ & \ddots & \\ & & \breve{M}^r \otimes I_{n^r}\end{bmatrix} \Gamma_U^{-1}. \tag{11}$$

Again using Theorem 6, $M \in \mathfrak{C}(V,U)$, giving (ii). □

Since Lemma 10 gives diagonalizable base matrices $\breve{V}$ and $\breve{U}$, we can always synthesize controllers by applying Algorithm 7 to the condensed form $(\breve{A},\breve{B})$. After applying the algorithm, a remaining step is to expand these condensed controllers back to their "full-system" counterparts. The idea is simply to expand each entry of the condensed form into an U△T block.

Lemma 11. Let $\breve{M}$ be in condensed form (with respect to $\mathfrak{C}(V,U)$), and consider $\breve{\Gamma}_V^{-1}\breve{M}\breve{\Gamma}_U = \operatorname{diag}(\breve{M}^1,\ldots,\breve{M}^r)$ as in (9). Define M as in (11). Then, $M \in \mathfrak{C}(V,U)$.

4.1 Patterned Controllability and Pole Placement

A square patterned matrix cannot necessarily attain an arbitrary spectrum of eigenvalues. Rather, the attainable spectra are restricted as follows.

Definition 12. (Patterned Spectrum). Let $V \in \mathbb{R}^{n\times n}$ have Jordan blocks as in (4). A spectrum $\mathscr{L}$ is called *V-patterned* if it can be ordered and partitioned as $\mathscr{L} = \mathscr{L}^1 \uplus \cdots \uplus \mathscr{L}^r$, where

$$\mathscr{L}^i = \{\, \underbrace{\lambda^i_1,\ldots,\lambda^i_1}_{n^i \text{ times}},\ \cdots\ ,\ \underbrace{\lambda^i_{\nu^i},\ldots,\lambda^i_{\nu^i}}_{n^i \text{ times}} \,\}$$

and $\mathscr{L}^{\varepsilon^i} = \overline{\mathscr{L}}^i$ for each $i = 1,\ldots,r$.

Lemma 13. (Sniderman et al. [2015a, Lem. 5.3 & 5.8]). Let $M \in \mathfrak{C}(V)$. The eigenvalues of M form a V-patterned spectrum, and the eigenvalues of $\breve{M}$ are the same without the "n^i times" repetitions. That is, $\sigma(M)$ is in the form of Definition 12, and

$$\sigma(\breve{M}) = \{\, \lambda^1_1,\ldots,\lambda^1_{\nu^1},\ \cdots\ ,\ \lambda^r_1,\ldots,\lambda^r_{\nu^r} \,\}.$$

A fundamental problem is to identify conditions when the poles of a patterned system can be placed into a patterned spectrum by a patterned feedback. This question was answered by Sniderman et al. [2015a] by introducing a new notion of patterned controllability.

Definition 14. A (V,U)-patterned system (A,B) is called *patterned controllable* if its condensed form $(\breve{A},\breve{B})$ is controllable in the usual sense.

Theorem 15. (Patterned Pole Placement [Sniderman et al., 2015a, Thm. 6.2]). A (V,U)-patterned system (A,B) is patterned controllable if and only if for every V-patterned spectrum $\mathscr{L}$, there exists a patterned feedback $K \in \mathfrak{C}(U,V)$ such that $\sigma(A+BK) = \mathscr{L}$.

In certain cases, the standard notion of controllability *is* sufficient for patterned pole placement (so the new notion of patterned controllability is not needed). This occurs for patterned systems with diagonalizable base matrices.

Corollary 16. ([Sniderman et al., 2017, Thm. 5.4]). Let (A,B) be a (V,U)-patterned system with V and U diagonalizable. (A,B) is controllable if and only if for every V-patterned spectrum $\mathscr{L}$, there exists a patterned feedback $K \in \mathfrak{C}(U,V)$ such that $\sigma(A+BK) = \mathscr{L}$.

5. DECOMPOSITIONS FOR PATTERNED CONTROL

In this section, we present tools for decomposing patterned systems. We follow the standard approach for linear systems [Wonham, 1979], the main distinction being that we require decoupling subspaces (see §3) instead of invariant subspaces. The following results show that patterns can be preserved through decompositions without losing the underlying commuting relationships.

Lemma 17. Let $\mathcal{S}_1,\mathcal{S}_2 \subset \mathcal{X}$ be V-decoupling subspaces, with insertion maps S_i and natural projection maps Q_i. Also define the restrictions $V_i := V_{\mathcal{S}_i}$. Then, for $i,j = 1,2$,

(i) If $A \in \mathfrak{C}(V)$, then $Q_i A S_j \in \mathfrak{C}(V_i,V_j)$.
(ii) If $B \in \mathfrak{C}(V,U)$, then $Q_i B \in \mathfrak{C}(V_i,U)$.
(iii) If $K \in \mathfrak{C}(U,V)$, then $K S_j \in \mathfrak{C}(U,V_j)$.

Proof. Using Lemma 3, $(Q_i A S_j)V_j = Q_i(AV)S_j = Q_i(VA)S_j = V_i(Q_i A S_j)$. (ii)–(iii) are similar. □

Theorem 18. (Patterned Representation Theorem). Let $M \in \mathfrak{C}(V)$, and let $\mathcal{S} \subset \mathcal{X}$ be an M-invariant and V-decoupling subspace with V-invariant complement $\mathcal{S}^c$. Then, M has a matrix representation $\left[\begin{smallmatrix} M_1 & * \\ 0 & M_2\end{smallmatrix}\right]$ (as in (3)), where $M_1 \in \mathfrak{C}(V_{\mathcal{S}})$ and $M_2 \in \mathfrak{C}(V_{\mathcal{S}^c})$.

Proof. Define insertions and natural projections S_1,Q_1 and S_2,Q_2 on $\mathcal{S}$ and $\mathcal{S}^c$, respectively, and define $M_1 := Q_1 M S_1$ and $M_2 := Q_2 M S_2$. The coordinate transformation $T := [S_1\ S_2]$ gives the matrix representation (3). By Lemma 17(i), $M_1 \in \mathfrak{C}(V_{\mathcal{S}})$ and $M_2 \in \mathfrak{C}(V_{\mathcal{S}^c})$. □

The next result shows the opposite direction: a patterned matrix can be recovered from a patterned restriction.

Lemma 19. (Patterned Lifting Lemma). Let $\mathcal{S} \subset \mathcal{X}$ be a V-decoupling subspace with insertion S_1 and natural projection Q_1 (along a V-invariant complement). Then,

(i) If $A_1 \in \mathfrak{C}(V_{\mathcal{S}})$, then $S_1 A_1 Q_1 \in \mathfrak{C}(V)$.

(ii) If $B_1 \in \mathfrak{C}(V_\mathcal{S}, U)$, then $S_1 B_1 \in \mathfrak{C}(V, U)$.
(iii) If $K_1 \in \mathfrak{C}(U, V_\mathcal{S})$, then $K_1 Q_1 \in \mathfrak{C}(U, V)$.

Proof. Using Lemma 3, $(S_1 A_1 Q_1)V = S_1(A_1 V_\mathcal{S})Q_1 = S_1(V_\mathcal{S} A_1)Q_1 = V(S_1 A_1 Q_1)$. (ii)–(iii) are similar. □

Lastly, the images and kernels of patterned matrices are invariant under their associated base matrices.

Lemma 20. Let $M \in \mathfrak{C}(V, U)$. Then, $\operatorname{Im} M$ is V-invariant, and $\operatorname{Ker} M$ is U-invariant.

Proof. First, if $x \in \operatorname{Im} M$, then $x = Mv$ for some v, and $Vx = V(Mv) = M(Uv) \in \operatorname{Im} M$. Second, if $x \in \operatorname{Ker} M$, then $M(Ux) = V(Mx) = 0$, so $Ux \in \operatorname{Ker} M$. □

The Patterned Representation Theorem 18 and Lifting Lemma 19 require V*-decoupling* subspaces, which do not always arise from the standard system decompositions. Fortunately, this restriction is not a limitation when using the condensed form: by Lemma 10, $\breve{M}$ has diagonalizable base matrices; by Lemma 4, all invariant subspaces under diagonalizable matrices are decoupling; and by Lemma 20, the subspaces in the standard system decompositions are invariant. In sum, for the condensed form, the standard system decompositions can always be performed on decoupling subspaces, thereby preserving a system's pattern.

6. PATTERNED STABILIZATION (THM. 2 PROOF)

(If.) Suppose condition (1) holds, and define $\breve{V}$ and $\breve{U}$ as in (10). By Lemma 10, $(\breve{A}, \breve{B})$ is a $(\breve{V}, \breve{U})$-patterned system. We will find a patterned feedback $\breve{K} \in \mathfrak{C}(\breve{U}, \breve{V})$ such that $\sigma(\breve{A} + \breve{B}\breve{K}) \subset \mathbb{C}^-$, and then we will expand $\breve{K}$ into a stabilizing feedback $K \in \mathfrak{C}(U, V)$ for the full system.

Since $\breve{A}^{k-1} \in \mathfrak{C}(\breve{V})$ and $\breve{B} \in \mathfrak{C}(\breve{V}, \breve{U})$, it is easily verified that $\breve{A}^{k-1}\breve{B} \in \mathfrak{C}(\breve{V}, \breve{U})$. By Lemma 20, $\operatorname{Im}(\breve{A}^{k-1}\breve{B})$ is $\breve{V}$-invariant, as is the controllable subspace $\breve{\mathcal{C}} := \langle \breve{A} | \operatorname{Im} \breve{B} \rangle$.

Since $\breve{V}$ is diagonalizable, therefore $\breve{\mathcal{C}}$ is $\breve{V}$-decoupling by Lemma 4, with $\breve{V}$-invariant complement $\breve{\mathcal{C}}^c$. Let $\breve{S}_1, \breve{Q}_1$ and $\breve{S}_2, \breve{Q}_2$ be the insertion and natural projection maps on $\breve{\mathcal{C}}$ and $\breve{\mathcal{C}}^c$, respectively, and define $\breve{T} := \begin{bmatrix} \breve{S}_1 & \breve{S}_2 \end{bmatrix}$. Then,

$$\breve{T}^{-1}\breve{A}\breve{T} = \begin{bmatrix} \breve{A}_1 & * \\ 0 & \breve{A}_2 \end{bmatrix}, \breve{T}^{-1}\breve{B} = \begin{bmatrix} \breve{B}_1 \\ 0 \end{bmatrix}; \; \breve{T}^{-1}\breve{V}\breve{T} = \begin{bmatrix} \breve{V}_1 & 0 \\ 0 & \breve{V}_2 \end{bmatrix} \quad (12)$$

where $(\breve{A}_1, \breve{B}_1)$ is $(\breve{V}_1, \breve{U})$-patterned by Theorem 18, and is controllable since (12) is the standard controllable decomposition on $(\breve{A}, \breve{B})$. Therefore, we can apply Corollary 16 (or use Algorithm 7) to synthesize $\breve{K}_1 \in \mathfrak{C}(\breve{U}, \breve{V}_1)$ such that $\sigma(\breve{A}_1 + \breve{B}_1\breve{K}_1) \subset \mathbb{C}^-$. Also, $\sigma(\breve{A}_2) \subset \mathbb{C}^-$ from (1) [Wonham, 1979, Lem. 4.5].

Next, define $\breve{K} = \breve{K}_1\breve{Q}_1$, so $\breve{K} \in \mathfrak{C}(\breve{U}, \breve{V})$ by Lemma 19(iii). Follow Lemma 11 to expand $\breve{K}$ into $K \in \mathfrak{C}(U, V)$. By direct calculation, the condensed form of $A + BK$ is $\breve{A} + \breve{B}\breve{K}$. Then, by Lemma 13, the eigenvalues of $A + BK$ are the same as those of $\sigma(\breve{A} + \breve{B}\breve{K}) = \sigma(\breve{A}_1 + \breve{B}_1\breve{K}_1) \uplus \sigma(\breve{A}_2)$. It follows from above that $\sigma(A + BK) \subset \mathbb{C}^-$, as desired.

(Only If.) Suppose there exists $K \in \mathfrak{C}(U, V)$ such that $\sigma(A + BK) \subset \mathbb{C}^-$. By Lemma 13, $\sigma(\breve{A} + \breve{B}\breve{K}) \subset \mathbb{C}^-$, so $(\breve{A}, \breve{B})$ is stabilizable and condition (1) is met. □

Theorem 2 shows that the standard notion of stabilization is not sufficient to achieve *patterned* stabilization, so a natural question is when the standard notion *can* be used. As with controllability in Corollary 16, a sufficient condition is that the base matrices are diagonalizable [Sniderman et al., 2017, Thm. 5.6].

Corollary 21. Let (A, B) be a (V, U)-patterned system with V and U diagonalizable. Then, (A, B) is stabilizable by patterned feedback if and only if $\mathcal{X}^+(A) \subset \langle A | \operatorname{Im} B \rangle$.

7. PATTERNED STABILIZATION BY MEASUREMENT FEEDBACK

Consider the system $\dot{x} = Ax + Bu$ as above (with $A \in \mathfrak{C}(V)$ and $B \in \mathfrak{C}(V, U)$), and incorporate a patterned measurement $y = Cx$ with $C \in \mathfrak{C}(Y, V)$ for some matrix Y. Only y is available for control, rather than the full state x as above. The Stabilization by Measurement Feedback Problem (SMFP) asks whether the system can be stabilized by a *measurement feedback* $u = Ky$.

Problem 22. (Patterned SMFP). Given a (V, U, Y)-patterned system (A, B, C), find a patterned measurement feedback $K \in \mathfrak{C}(U, Y)$ such that $\sigma(A + BKC) \subset \mathbb{C}^-$.

Our solution to the Patterned SMFP follows the technique of the standard SMFP. We first find a state feedback $u = K'x$ instead of the desired measurement feedback, with the caveat that K' must only use states that appear in the measurement — in other words, $\operatorname{Ker} C \subset \operatorname{Ker} K'$. This constraint is characterized geometrically by $\mathcal{L} := \langle A | \operatorname{Ker} C \rangle$ (the smallest A-invariant subspace containing $\operatorname{Ker} C$). Then, the standard SMFP is solvable if $\mathcal{X}^+(A) \subset \mathcal{C}$ and $\mathcal{X}^+(A) \cap \mathcal{L} = \{0\}$. The Patterned SMFP uses similar sufficient conditions on the condensed form.

Theorem 23. A patterned system (A, B, C) is stabilizable by patterned measurement feedback if

$$\breve{\mathcal{X}}^+(\breve{A}) \subset \breve{\mathcal{C}} \quad (13)$$

$$\breve{\mathcal{X}}^+(\breve{A}) \cap \breve{\mathcal{L}} = \{0\}\,. \quad (14)$$

where $\breve{\mathcal{C}} := \langle \breve{A} | \operatorname{Im} \breve{B} \rangle$ and $\breve{\mathcal{L}} := \langle \breve{A} | \operatorname{Ker} \breve{C} \rangle$.

Proof. Suppose conditions (13)–(14) hold, and define $\breve{V}$, $\breve{U}$, and $\breve{Y}$ as in (10). By Lemma 10, $(\breve{A}, \breve{B}, \breve{C})$ is a $(\breve{V}, \breve{U}, \breve{Y})$-patterned system. We will find a patterned feedback $\breve{K} \in \mathfrak{C}(\breve{U}, \breve{Y})$ such that $\sigma(\breve{A} + \breve{B}\breve{K}\breve{C}) \subset \mathbb{C}^-$, and then we will expand $\breve{K}$ into a stabilizing measurement feedback $K \in \mathfrak{C}(U, Y)$ for the full system.

In an analogous fashion to the proof of Theorem 2, the subspace $\breve{\mathcal{L}} := \langle \breve{A} | \operatorname{Ker} \breve{C} \rangle$ is $\breve{V}$-decoupling, with a $\breve{V}$-invariant complement $\breve{\mathcal{L}}^c$. Let $\breve{S}_1, \breve{Q}_1$ and $\breve{S}_2, \breve{Q}_2$ be insertion and natural projection maps on $\breve{\mathcal{L}}$ and $\breve{\mathcal{L}}^c$, and define $\breve{T} := \begin{bmatrix} \breve{S}_1 & \breve{S}_2 \end{bmatrix}$. We obtain the decomposition (12), where $(\breve{A}_2, \breve{B}_2)$ is $(\breve{V}_2, \breve{U})$-patterned by Theorem 18. This subsystem is also stabilizable: using (14) and Lemma 5,

$$(\breve{\mathcal{L}}^c)^+(\breve{A}_2) = \breve{Q}_2\breve{\mathcal{X}}^+(\breve{A}) \subset \breve{Q}_2\langle \breve{A} | \operatorname{Im} \breve{B} \rangle = \langle \breve{A}_2 | \operatorname{Im} \breve{B}_2 \rangle\,.$$

Apply Corollary 21 to stabilize $(\breve{A}_2, \breve{B}_2)$ using a patterned state feedback $\breve{K}_2' \in \mathfrak{C}(\breve{U}, \breve{V}_2)$, so $\sigma(\breve{A}_2 + \breve{B}_2\breve{K}_2') \subset \mathbb{C}^-$. From condition (14), it also follows that $\sigma(\breve{A}_1) \subset \mathbb{C}^-$

[Wonham, 1979, Lem. 4.5]. Next, define $\breve{K}' = \breve{K}_2'\breve{Q}_2 \in \mathfrak{C}(\breve{U}, \breve{V})$ by Lemma 19(iii), and notice that

$$\operatorname{Ker} \breve{C} \subset \breve{\mathcal{L}} = \operatorname{Ker} \breve{Q}_2 \subset \operatorname{Ker}(\breve{K}_2'\breve{Q}_2) = \operatorname{Ker} \breve{K}' .$$

Therefore, the equation $\breve{K}' = \breve{K}\breve{C}$ has a patterned solution $\breve{K} \in \mathfrak{C}(\breve{U}, \breve{Y})$ [Sniderman et al., 2017, Lem. 3.14]. This measurement feedback $\breve{K}$ gives closed-loop poles

$$\sigma(\breve{A} + \breve{B}\breve{K}\breve{C}) = \sigma(\breve{A}_1) \uplus \sigma(\breve{A}_2 + \breve{B}_2\breve{K}_2') \subset \mathbb{C}^-$$

for the condensed system. Lastly, following Lemma 11, let $K \in \mathfrak{C}(U, Y)$ be a matrix with condensed form $\breve{K}$. By Lemma 13, $\sigma(A + BKC) \subset \mathbb{C}^-$, so the system has been stabilized by patterned measurement feedback. □

8. NUMERICAL EXAMPLE

Let (A, B, C) be the following (V, U, Y)-patterned system:

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 0 & 3 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 & 1 & -1 & 0 \\ -6 & 5 & -1 & -3 & 9 \\ 0 & 0 & -1 & 0 & 0 \\ -2 & 2 & 5 & 0 & -1 \\ 0 & 0 & 2 & 0 & -3 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 4 & 0 & -1 & -1 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$Y = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -7 & 7 & -4 & -5 & 12 \\ -5 & 5 & -2 & -3 & 8 \\ -2 & 2 & -2 & -2 & 4 \end{bmatrix}.$$

Notice that $\sigma(A) = \{-3, -1, -1, 2, 2\}$, so the system has two unstable open-loop poles. Our goal is to stabilize the system using a patterned measurement feedback $K \in \mathfrak{C}(U, Y)$, if possible. Following the method of Theorem 23, the first steps are to find the condensed form of the system, and to check conditions (13)–(14). Using transformations

$$\Gamma_V = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \ \Gamma_U = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \ \Gamma_Y = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

with $\breve{\Gamma}_V = I_3$ and $\breve{\Gamma}_U = \breve{\Gamma}_Y = I_2$, the system matrices have condensed forms

$$\breve{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \ \breve{B} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}, \ \breve{C} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It can also be confirmed that $\breve{V} = \operatorname{diag}(1, 1, 2)$ and $\breve{U} = \breve{Y} = \operatorname{diag}(1, 2)$ satisfy Lemma 10, so $\breve{A} \in \mathfrak{C}(\breve{V})$, $\breve{B} \in \mathfrak{C}(\breve{V}, \breve{U})$, and $\breve{C} \in \mathfrak{C}(\breve{Y}, \breve{V})$. The relevant subspaces are

$$\breve{\mathcal{X}}^+(\breve{A}) = \operatorname{Im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \breve{\mathcal{C}} = \operatorname{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \breve{\mathcal{L}} = \operatorname{Im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be seen that the condensed form's single unstable mode is controllable ($\breve{\mathcal{X}}^+(\breve{A}) \subset \breve{\mathcal{C}}$) and is not masked out ($\breve{\mathcal{X}}^+(\breve{A}) \cap \breve{\mathcal{L}} = \{0\}$). Thus, conditions (13)–(14) of Theorem 23 hold, and the Patterned SMFP is solvable. To find a solution, the condensed system will be decomposed based on $\breve{\mathcal{X}} = \breve{\mathcal{L}} \oplus \breve{\mathcal{L}}^c$, where $\breve{\mathcal{L}}^c = \operatorname{Im}[0\ 1\ 0]^{\mathrm{T}}$. Letting $\breve{T}$ be the pertinent coordinate transformation,

$$\breve{T}^{-1}\breve{A}\breve{T} = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -3 & 0 \\ \hline 0 & 0 & 2 \end{array}\right], \ \breve{T}^{-1}\breve{B} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hline 2 & 0 \end{array}\right]; \ \breve{T}^{-1}\breve{V}\breve{T} = \left[\begin{array}{cc|c} 1 & & \\ & 2 & \\ \hline & & 1 \end{array}\right].$$

The top-left subsystem is masked out by $\breve{\mathcal{L}}$, while the bottom-right subsystem is available to the (condensed form's) measurement. As guaranteed in the proof of Theorem 23, the system's instability is relegated to the latter subsystem. In particular, the patterned feedback $[-2\ \ 0]^{\mathrm{T}} \in \mathfrak{C}(1, \breve{U})$ moves the single pole from 2 to -2. Letting $\breve{Q}_c : \mathbb{R}^3 \to \breve{\mathcal{L}}^c$ be the third row of $\breve{T}^{-1}$, we define $\breve{K}' := \left[\begin{smallmatrix} -2 \\ 0 \end{smallmatrix}\right] \breve{Q}_c = \left[\begin{smallmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathfrak{C}(\breve{U}, \breve{V})$ by Lemma 19(iii).

Turn $\breve{K}'$ into a patterned *measurement* feedback by solving the equation $\breve{K}\breve{C} = \breve{K}'$; a patterned solution is $\breve{K} := \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathfrak{C}(\breve{U}, \breve{Y})$. Expand $\breve{K}$ to the full system by turning each entry into an U△T block as in Lemma 11, giving

$$K := \Gamma_U \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \Gamma_Y^{-1} = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 0 & 1 \\ -3 & 3 & 2 \end{bmatrix} \in \mathfrak{C}(U, Y).$$

Finally, $\sigma(A + BKC) = \{-3, -2, -2, -1, -1\}$, confirming that the unstable open-loop poles have been moved from 2 to -2, and we have solved the Patterned SMFP.

In conclusion, we have solved patterned versions of two fundamental control problems, stabilization by state and measurement feedback. These problems had previously been solved for certain patterned systems, but the general case remained unsolved. We have closed these open problems through the construct of the condensed form, which removes the structure of the pattern. Consequently, the new solvability conditions generalize those of standard geometric control to patterned distributed systems.

REFERENCES

Brockett, R.W. and Willems, J.L. (1974). Discretized Partial Differential Equations: Examples of Control Systems Defined on Modules. *Automatica*, 10(5), 507–515.

Consolini, L. and Tosques, M. (2014). Static Output Feedback of Equivariant Linear Systems with Applications to the Control of Interconnected Agents. In *53rd IEEE CDC*, 3988–3993.

Consolini, L. and Tosques, M. (2015). On the Distributed Stabilization and Consensus of Symmetrically Interconnected Agents. In *2015 ECC*, 1926–1931.

Gantmacher, F. (1959). *The Theory of Matrices*. Chelsea.

Gohberg, I., Lancaster, P., and Rodman, L. (1986). *Invariant Subspaces of Matrices with Applications*. Wiley.

Hamilton, S.C. and Broucke, M.E. (2012). *Geometric Control of Patterned Linear Systems*. Springer-Verlag Berlin Heidelberg.

Holmes, C. and Broucke, M.E. (2016). Pattern Preserving Pole Placement and Stabilization for Linear Systems. In *2016 ACC*, 5181–5186.

Massioni, P. and Verhaegen, M. (2009). Distributed Control for Identical Dynamically Coupled Systems: A Decomposition Approach. *IEEE TAC*, 54(1), 124–135.

Ornik, M., Sniderman, A.C., Broucke, M.E., and D'Eleuterio, G.M.T. (2016). Pattern Identification in Distributed Systems. In *55th IEEE CDC*, 6895–6900.

Sniderman, A.C., Broucke, M.E., and D'Eleuterio, G.M.T. (2013). Block Circulant Control: A Geometric Approach. In *52nd IEEE CDC*, 2097–2102.

Sniderman, A.C., Broucke, M.E., and D'Eleuterio, G.M.T. (2015a). Controllability Is Not Sufficient for Pole Placement in Patterned Systems. In *54th IEEE CDC*, 4296–4301.

Sniderman, A.C., Broucke, M.E., and D'Eleuterio, G.M.T. (2015b). Formation Control of Balloons: A Block Circulant Approach. In *2015 ACC*, 1463–1468.

Sniderman, A.C., Broucke, M.E., and D'Eleuterio, G.M.T. (2017). Control of a Class of Patterned Systems. *Int. J. Control* (in press).

Wang, S.H. and Davison, E.J. (1973). On the Stabilization of Decentralized Control Systems. *IEEE TAC*, 34(4), 473–478.

Wonham, W.M. (1979). *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag New York.