Control of a Class of Patterned Systems

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ABSTRACT
Engineering systems are often architected to consist of a number of interconnected parts that interact in distinct patterns. Because most control design methods only provide general, unpatterned control laws, a compelling open question is how to synthesise distributed control laws that adhere to a system's interconnection pattern. This paper addresses patterned control synthesis for systems with interconnection patterns. The pattern is encoded algebraically through commuting relationships of the system's state space matrices. We show that a number of classic control problems are amenable to a patterned synthesis. Moreover, we show that these patterned control problems have the same solvability conditions as their unpatterned counterparts. That is, a patterned control law can be found whenever any control law can be found. Our findings suggest that patterned systems *naturally* admit patterned controllers.

KEYWORDS
Distributed control, linear systems, geometric control, stabilisation

1. Introduction

This paper studies the control of a class of multivariate distributed systems called *patterned linear systems*. In a distributed system, the subsystems (or agents) are interconnected according to a specific structure. In a *patterned* linear system, that interconnection structure induces relationships among elements of the system's matrices. These relationships together are called a 'pattern'. This pattern must in turn be carried over to control laws synthesised for the system. The requirement of preserving strict interconnection constraints between subsystems in control laws is well known to be difficult, frequently rendering control problems unsolvable (Wang & Davison, 1973). Patterned systems provide a way around some of these difficulties by relaxing the strict requirement of interconnection *constraints* while still retaining the interconnection structures. That is, patterned control provides a means for distributed control without requiring pure decentralization up front. This approach allows us to apply a traditional control framework — the geometric framework (Wonham, 1979) — to distributed control problems. Traditional control, however, does not guarantee any particular form in synthesised controllers, whereas patterned controllers must follow the same

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relationships as the open-loop system matrices. This is the main issue treated by this paper.

Distributed and decentralised systems have been widely studied. A sampling of research (by no means inclusive) related to our work is given by Belabbas (2013); Commault, Dion, and Hovelaque (1997); Denis and Looze (1999); Elbanna (1988); Hovd, Braatz, and Skogestad (1997); Hovd and Skogestad (1994); Lin (1974); Lunze (1986, 1989); Massioni and Verhaegen (2009); Rotkowitz and Lall (2006); Sundareshan and Elbanna (1991); van der Schaft (1987). Many of the current techniques for distributed control follow the progression of first proposing a controller, and second determining whether it is viable for a particular problem. In contrast, our patterned control methodology first determines whether a control problem is solvable at all, and second synthesises a controller to solve it. Through this progression, the main contribution of our work is to develop a *synthesis-based* control framework for a class of distributed systems, matching the standard framework used in most other control problems. In this paper, we study several standard problems of geometric control theory, showing that those same problems can be solved for patterned distributed systems.

Patterned linear systems were first introduced by Hamilton and Broucke (2012) for the case of scalar subsystems or agents. The pattern of subsystem interconnections was encoded algebraically in terms of polynomials of a base matrix. A geometric approach was adopted to solve many of the standard control synthesis problems (Wonham, 1979) with the additional requirement of obtaining a patterned feedback.

The framework of patterned linear systems was extended to multivariate agents by Sniderman, Broucke, and D'Eleuterio (2013, 2015a), Consolini and Tosques (2014), and Holmes and Broucke (2016). To realise this extension, a new algebraic encoding of the system pattern was adopted in terms of commuting relationships of the system matrices, rather than polynomials of a base matrix. First, Sniderman et al. (2013) studied systems with the particular block circulant pattern, and showed that many standard control problems could be solved by block circulant feedbacks under the same necessary and sufficient conditions. To extend the class of block circulant systems, Sniderman et al. (2015a) looked at patterns encoded by any single commuting relationship between the system matrices and a base matrix, and showed that the standard notion of controllability does not suffice to allow for pole placement with a patterned feedback matrix, in contrast with classical pole placement in linear systems. Rather, patterned pole placement operates under the necessary and sufficient condition of the controllability of a so-called 'reduced system'. Consolini and Tosques (2014) encoded patterns using the automorphism group of a certain system graph for systems whose patterns are completely described by symmetries. These symmetries yield a family of commuting relationships between the system matrices and a (matrix) representation of the automorphism group of the graph. However, a patterned pole placement theorem was not postulated, and therefore only a subset of synthesis problems could be solved. Holmes and Broucke (2016) adopted the framework of Consolini and Tosques (2014) to solve the Patterned Pole Placement Problem. The approach followed that of Sniderman et al. (2015a) to study the controllability of a reduced system (which effectively quotients out the graph symmetries).

In this paper, we follow the framework of Sniderman et al. (2015a) to consider a class of patterned systems where the system matrices satisfy a single commuting relationship with a base matrix that is diagonalisable. A key question arises: do the standard results of linear control theory apply to this class, without requiring additional conditions (as found by Holmes and Broucke (2016); Sniderman et al. (2015a))? We answer this question affirmatively for patterned systems whose base matrices are diagonalisable. With this class of patterned systems, the difficulties encountered by Holmes and Broucke (2016); Sniderman et al. (2015a) to achieve a patterned pole placement theorem do not arise. Instead, we arrive at the appealing outcome that all the standard results of linear control theory carry over to this class of patterned linear systems without any changes in the necessary and sufficient conditions. This finding echoes the results of Hamilton and Broucke (2012), where a similar outcome was obtained for patterned linear systems with scalar agents.

While this paper is directly related to those of Hamilton and Broucke (2012), Sniderman et al. (2013, 2015a), Consolini and Tosques (2014), and Holmes and Broucke (2016), our work also finds roots in earlier research studying systems whose matrices satisfy commuting relationships. First, Hazewinkel and Martin (1983) contributed early work involving commuting relationships. Inspired by a seminal paper on block circulant systems (Brockett & Willems, 1974), they studied a more general class of systems with symmetries, and characterised those systems' controllability and stabilisability in terms of subsystems determined via the symmetry algebra, producing results similar in style to those of Holmes and Broucke (2016). Second, Fagnani and Willems (1994) developed a method to embed symmetries into the behavioural control approach, and found that symmetric controllers exist whenever standard controllers exist. Our work is complementary to the above two, and differs in a two key ways: our patterns involve a diagonalisable base matrix, while the above symmetries use a number of diagonalisable *and* invertible matrices; and we study several geometric control problems other than pole placement and stabilisation, which are not explicitly covered in prior works.

This paper is organised as follows. Section 2 contains standard background on linear algebra and linear systems, as well as notation. Section 3 introduces patterned matrices, and lays out the main mathematical tools for keeping track of patterns when solving control problems. First, we discuss the algebra of patterns in terms of commuting relationships with certain matrices, as well as the geometry of patterns in terms of certain invariant subspaces under those matrices. To tie the algebra and geometry together, we show how these commuting relationships carry through restrictions to these invariant subspaces, thereby providing a way to preserve patterns in the state space decompositions of geometric control. Then, we discuss the spectra of patterned matrices, and examine patterned solutions to linear equations of patterned matrices. With this machinery in hand, we turn to patterned control systems in Section 4. After defining initial concepts, Sections 5 and 6 discuss what the fundamental properties of controllability and observability mean for patterned systems. Section 5 also presents methods for pole placement and stabilisation of patterned systems using patterned feedbacks. Using these techniques, Sections 7–10 study several other multivariable controller synthesis problems on patterned systems, including the Output Stabilisation Problem, Disturbance Decoupling Problem, Stabilisation by Measurement Feedback Problem, and Restricted Regulator Problem. Section 11 provides a numerical example to demonstrate the patterned synthesis technique for a pattern that occurs often in distributed systems. Section 12 gives concluding remarks to further assess the significance of the findings.

2. Mathematical Preliminaries

This section presents some mathematical background and notation. Given a matrix A, denote its transpose by A^{T} and its complex conjugate by $\overline{A}$. If A is square, denote its spectrum by $\sigma(A)$; also denote its 'distinct spectrum' — the set containing one of each distinct eigenvalue — by $\sigma_{\mathrm{d}}(A) \subset \sigma(A)$. Given a vector space $\mathcal{X}$ with independent subspaces $\mathcal{R}, \mathcal{S} \subset \mathcal{X}$ (so $\mathcal{R} \cap \mathcal{S} = \{0\}$), denote their direct sum by $\mathcal{R} \oplus \mathcal{S}$.

It is assumed that the reader is already familiar with the tools of linear geometric control theory (Basile & Marro, 1991; Wonham, 1979). Let $\mathcal{X}$ be a vector space with subspace $\mathcal{S} \subset \mathcal{X}$ and complementary subspace $\mathcal{S}^c \subset \mathcal{X}$, so $\mathcal{X} = \mathcal{S} \oplus \mathcal{S}^c$. Geometric control involves state space decompositions that are based on two standard projection maps: the insertion and the natural projection. The *insertion map* on $\mathcal{S}$, denoted $S : \mathcal{S} \to \mathcal{X}$, maps $x \in \mathcal{S}$ to the corresponding element $x \in \mathcal{X}$; that is, $Sx := x$. The *natural projection* on $\mathcal{S}$ along $\mathcal{S}^c$, denoted $Q : \mathcal{X} \to \mathcal{S}$, maps $x \in \mathcal{X}$ to its component in $\mathcal{S}$; that is, using the unique decomposition $x = s + r$ with $s \in \mathcal{S}$ and $r \in \mathcal{S}^c$, $Qx := s$. These maps satisfy $QS = I_{\mathcal{S}}$, where $I_{\mathcal{S}}$ is the identity map on $\mathcal{S}$. Also, $\mathcal{S} = \operatorname{Im} S$ and $\mathcal{S}^c = \operatorname{Ker} Q$.

Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map, and suppose that $\mathcal{S}$ is A-invariant; that is, $A\mathcal{S} \subset \mathcal{S}$. The *restriction* of A to $\mathcal{S}$, denoted by the linear map $A_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$, is the unique solution of the equation $AS = SA_{\mathcal{S}}$; thus, $A_{\mathcal{S}}$ performs the action of A on $\mathcal{S}$, and is not defined off $\mathcal{S}$. In a basis for $\mathcal{X}$ adapted to $\mathcal{S}$, A has the matrix representation $\left[\begin{smallmatrix} A_1 & * \\ 0 & A_2 \end{smallmatrix}\right]$, where A_1 is a matrix representation of $A_{\mathcal{S}}$ (denoted $A_1 \equiv A_{\mathcal{S}}$). This matrix representation can be produced by the above projection maps: define insertions and natural projections on $\mathcal{S}$ by S_1, Q_1 and on $\mathcal{S}^c$ by S_2, Q_2, and define the coordinate transformation $T = [S_1 \; S_2]$. Then, $T^{-1} = \operatorname{col}(Q_1, Q_2)$, and $T^{-1}AT$ yields a matrix with the above form, where $A_1 = Q_1 A S_1$ and $A_2 = Q_2 A S_2$. The following relationships can be deduced:

$$AS_1 = S_1 A_1 \tag{1}$$
$$Q_2 A = A_2 Q_2 \,. \tag{2}$$

If $\mathcal{S}^c$ is also A-invariant, then the matrix representation in the adapted basis will be block diagonal — i.e., the top-right block '$*$' will be 0 — and (1)–(2) will also hold for S_2 and Q_1.

Let $\mathbb{C} = \mathbb{C}^m \sqcup \mathbb{C}^c$ be a disjoint partition of the complex plane (so $\mathbb{C}^m \cap \mathbb{C}^c = \emptyset$). Split the minimal polynomial of A as $\psi(s) = \psi^m(s)\psi^c(s)$ according to its factors in $\mathbb{C}^m$ and $\mathbb{C}^c$. The corresponding modal subspaces are $\mathcal{X}^m(A) := \operatorname{Ker} \psi^m(A)$ and $\mathcal{X}^c(A) := \operatorname{Ker} \psi^c(A)$. The following result relates this modal decomposition to other state space decompositions of $\mathcal{X}$.

Lemma 2.1 (Wonham, 1979)**.** *Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map. Let $\mathcal{S} \subset \mathcal{X}$ be an A-invariant subspace, and define A_1 and A_2 as above. Let $\mathbb{C} = \mathbb{C}^m \sqcup \mathbb{C}^c$ be a disjoint partition of $\mathbb{C}$. Then, $\mathcal{X}^m(A) \subset \mathcal{S}$ if and only if $\sigma(A_2) \subset \mathbb{C}^c$, and $\mathcal{X}^m(A) \cap \mathcal{S} = 0$ if and only if $\sigma(A_1) \subset \mathbb{C}^c$.*

3. Patterned Matrices and Decoupling Subspaces

Our framework for control of patterned systems is based on an interplay between the algebraic properties of commuting matrices and the geometric properties of invariant subspaces. In this section, we introduce some of the main tools for solving the problem of patterned control synthesis.

Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ be a field, and define matrices $V \in \mathbb{F}^{n\times n}$ and $U \in \mathbb{F}^{m\times m}$.

Definition 3.1. *A matrix $A \in \mathbb{F}^{n\times m}$ is called a patterned matrix if it satisfies the commuting relationship $VA = AU$. This relationship is denoted by $A \in \mathfrak{C}(V, U)$, and V and U are called the base matrices of the pattern. If $V = U$, then we use the shorthand $\mathfrak{C}(V) := \mathfrak{C}(V, V)$. If the field is important, we write $\mathfrak{C}_{\mathbb{F}}(V, U)$.*

A pattern prescribes a specific form in a matrix, and so is useful for encoding a number of common distributed system structures. We present two particular examples here, a ring and a bidirectional chain, that fit into the model used in this paper.

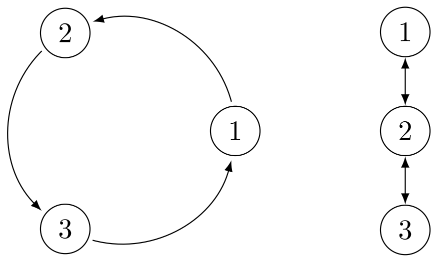

Figure 1. Distributed systems with diagonalisable base matrices: ring (left) and bidirectional chain (right).

Example 3.2 (Ring)**.** Consider a system made up of three parts, whose interconnections form a *ring* (as in Figure 1 (left)). The parts are identical, each having the same n states and m inputs, and the interconnections among the parts are also identical; for example, the coupling between parts '1' and '2' is the same as the coupling between parts '2' and '3', and between '3' and '1'. This manifests as a block structure in all the system matrices, made up of diagonal bands of identical blocks (wrapped around cyclically):

$$M = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_3 & M_1 & M_2 \\ M_2 & M_3 & M_1 \end{bmatrix}, \quad M_i \in \mathbb{F}^{n\times m}.$$

This block structure is called *block circulant*, and can be uniquely described by a commuting relationship involving the fundamental permutation matrix

$$\Pi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Specifically, M is block circulant if and only if $(\Pi_3 \otimes I_n)M = M(\Pi_3 \otimes I_m)$ (where '$\otimes$' represents the Kronecker product), and it follows that every ring system is $(\Pi_3 \otimes I_n)$-patterned. ◀

Example 3.3 (Bidirectional Chain)**.** Consider a system made up of three parts, whose interconnections form a *bidirectional chain* (as in Figure 1 (right)). Again, the parts are identical and have identical interconnections. This again manifests as a block structure in all the system matrices; one possible such structure is given by

$$M = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_1 + M_3 & M_2 \\ M_3 & M_2 & M_1 \end{bmatrix}, \quad M_i \in \mathbb{F}^{n \times m},$$

where the coupling in the middle term is in the style of a mass-spring system. This block structure can be uniquely described by a commuting relationship involving the matrix

$$\widetilde{V} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Specifically, M has the above block structure if and only if $(\widetilde{V} \otimes I_n)M = M(\widetilde{V} \otimes I_m)$ (where '$\otimes$' denotes the Kronecker product), and it follows that every bidirectional chain system is $(\widetilde{V} \otimes I_n)$-patterned. ◀

Knowing that many common distributed systems can be encoded by patterns, the central question of this paper is how to keep sight of a pattern while performing the standard decompositions and syntheses for control design. The key idea in linking these notions lies in finding suitable invariant subspaces that also have invariant complements. These subspaces are called *decoupling*, in that they fully decouple a state space into two invariant parts. Invoking these decoupling subspaces is the crucial step that ties together the algebraic properties of commuting matrices and the geometric notions of linear geometric control (Basile & Marro, 1991; Wonham, 1979).

Definition 3.4. *A subspace $\mathcal{S} \subset \mathcal{X}$ is called V-decoupling if it is V-invariant, and it has a V-invariant complement $\mathcal{S}^c$. That is, $V\mathcal{S} \subset \mathcal{S}$ and $V\mathcal{S}^c \subset \mathcal{S}^c$.*

In general, a V-invariant subspace is not V-decoupling. However, if V is diagonalisable, then this will be the case (Gohberg, Lancaster, & Rodman, 1986, Theorem 3.2.1). For this reason, we focus only on patterns with diagonalisable base matrices, and always assume that this property holds. Such an assumption is not unfounded; it appears in a number of other recent papers (Consolini & Tosques, 2014, 2015; Deutscher, 2013; Massioni & Verhaegen, 2009) (and is less restrictive than some of them). In fact, some such restriction appears in all work to date regarding control with commuting relationships, except for some preliminary results in (Sniderman et al., 2015a).

Assumption 3.5. *Throughout the paper (even when not explicitly stated), we assume that all base matrices are diagonalisable. For instance, if $M \in \mathfrak{C}(V, U)$, then V and U are diagonalisable.*

The next two results state that decoupling and diagonalisability are maintained in common subspace operations and restrictions.

Lemma 3.6 (Gohberg et al., 1986)**.** *Let V be diagonalisable and let $\mathcal{S} \subset \mathcal{X}$ be V-invariant (or -decoupling). Then $V_{\mathcal{S}}$ is diagonalisable.*

Lemma 3.7. *Let V be a diagonalisable matrix, and let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{X}$ be V-decoupling subspaces. Then, $\mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ are also V-decoupling subspaces.*

Proof. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are V-invariant, therefore $\mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ are also V-invariant (Gohberg et al., 1986, Proposition 1.8.1). Since V is diagonalisable, therefore $\mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ also have V-invariant complements (Gohberg et al., 1986, Theorem 3.2.1), so they are V-decoupling by definition. $\square$

Next, consider $A \in \mathfrak{C}(V)$, and let $\mathcal{S}$ be an A-invariant subspace. Is $\mathcal{S}$ also V-invariant (or V-decoupling)? Not generally. Fortunately, several specific A-invariant subspaces are V-decoupling, including those needed to solve the control problems in this paper.

Lemma 3.8. *Let $A \in \mathfrak{C}(V)$, $B \in \mathfrak{C}(V, U)$, and $K \in \mathfrak{C}(U, V)$. Let ρ be a polynomial, and let $\mathcal{S} \subset \mathcal{X}$ be a V-decoupling subspace. Then,*

(i) $\operatorname{Im} B$ *and* $\operatorname{Ker} K$ *are V-decoupling subspaces.*
(ii) $\operatorname{Im} \rho(A)$ *and* $\operatorname{Ker} \rho(A)$ *are V-decoupling subspaces.*
(iii) $A\mathcal{S}$ (the image of $\mathcal{S}$ under A) and $A^{-1}\mathcal{S} := \{x \in \mathcal{X} \mid Ax \in \mathcal{S}\}$ (the preimage of $\mathcal{S}$ under A) are V-decoupling subspaces.

Proof.

(i) Let $x \in \operatorname{Im} B$, so $x = Bv$ for some v. Then, $Vx = V(Bv) = B(Uv) \in \operatorname{Im} B$. Let $x \in \operatorname{Ker} K$, so $Kx = 0$. Then, $K(Vx) = U(Kx) = 0$, so $Vx \in \operatorname{Ker} K$.
(ii) $\rho(A) \in \mathfrak{C}(V)$ by Lemma A.1, so $\operatorname{Im} \rho(A)$ and $\operatorname{Ker} \rho(A)$ are V-invariant by (i).
(iii) First, $V(M\mathcal{S}) = M(V\mathcal{S}) \subset M\mathcal{S}$, so $M\mathcal{S}$ is V-invariant. Second, let $x \in M^{-1}\mathcal{S}$. Since $Mx \in \mathcal{S}$ and $V\mathcal{S} \subset \mathcal{S}$, it follows that $M(Vx) = V(Mx) \in \mathcal{S}$, so $Vx \in M^{-1}\mathcal{S}$. Thus, $M^{-1}\mathcal{S}$ is V-invariant. $\square$

Having stated the main properties of decoupling subspaces, we next explore their utility in maintaining patterns in restrictions of patterned matrices. In particular, if A is a V-patterned matrix, and $\mathcal{S}$ is an A-invariant and V-decoupling subspace, then the restriction $A_{\mathcal{S}}$ will commute with the restriction $V_{\mathcal{S}}$.

Lemma 3.9. *Let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{X}$ be V-decoupling subspaces. Let S_1, Q_1 and S_2, Q_2 be the insertions and natural projections on $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Also define the restrictions $V_1 := V_{\mathcal{S}_1}$ and $V_2 := V_{\mathcal{S}_2}$. For $i, j = 1, 2$,*

(i) If $A \in \mathfrak{C}(V)$, then $Q_i A S_j \in \mathfrak{C}(V_i, V_j)$.
(ii) If $B \in \mathfrak{C}(V, U)$, then $Q_i B \in \mathfrak{C}(V_i, U)$.
(iii) If $K \in \mathfrak{C}(U, V)$, then $K S_i \in \mathfrak{C}(U, V_i)$.

Proof. Using (1) and (2), $(Q_i A S_j) V_j = Q_i A V S_j = Q_i V A S_j = V_i (Q_i A S_j)$, giving (i). The proofs of (ii) and (iii) are similar. $\square$

The next result shows how matrix patterns are preserved through state space decompositions. This result exploits the fact that the restriction of a patterned matrix to a V-decoupling subspace retains the essential algebraic property of commuting with certain matrices, an immediate result of Lemma 3.9(i). Thus, commuting becomes the key enabling property in patterned decompositions.

Theorem 3.10 (Patterned Representation Theorem)**.** *Let $A \in \mathfrak{C}(V)$, and let $\mathcal{S} \subset \mathcal{X}$ be an A-invariant and V-decoupling subspace, with V-invariant complement $\mathcal{S}^c \subset \mathcal{X}$.*

Then, A has a matrix representation

$$\begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \tag{3}$$

where $A_1 \equiv A_{\mathcal{S}}$. Moreover, $A_1 \in \mathfrak{C}(V_{\mathcal{S}})$ and $A_2 \in \mathfrak{C}(V_{\mathcal{S}^c})$.

Proof. Let $\{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$ be a preferred basis for $\mathcal{X}$, so $\mathcal{S} = \operatorname{span}\{x_1, \dots, x_k\}$ and $\mathcal{S}^c = \operatorname{span}\{x_{k+1}, \dots, x_n\}$. From this basis, define the insertions and natural projections $S_1 : \mathcal{S} \to \mathcal{X}$, $S_2 : \mathcal{S}^c \to \mathcal{X}$, $Q_1 : \mathcal{X} \to \mathcal{S}$, and $Q_2 : \mathcal{X} \to \mathcal{S}^c$. The coordinate transformation $T = [S_1 \ S_2]$ gives the standard representation

$$T^{-1}AT = \begin{bmatrix} Q_1AS_1 & * \\ 0 & Q_2AS_2 \end{bmatrix}.$$

Then, $A_1 := Q_1AS_1 = Q_1S_1A_{\mathcal{S}} = A_{\mathcal{S}}$ (using (1)) and $A_2 := Q_2AS_2$, giving (3). Further, $A_1 \in \mathfrak{C}(V_{\mathcal{S}})$ and $A_2 \in \mathfrak{C}(V_{\mathcal{S}^c})$ by Lemma 3.9(i). □

Lemma 3.9 and Theorem 3.10 start with patterned matrices and show that their restrictions have patterns involving the restrictions of the base matrices. Conversely, we next start with matrices that commute with these restrictions, and show that they can be lifted back up to full patterned matrices.

Lemma 3.11 (Patterned Lifting Lemma)**.** *Let $\mathcal{S} \subset \mathcal{X}$ be a V-decoupling subspace with insertion $S_1 : \mathcal{V} \to \mathcal{X}$ and natural projection $Q_1 : \mathcal{X} \to \mathcal{V}$ (along a V-invariant complement).*

(i) If $A_1 \in \mathfrak{C}(V_{\mathcal{S}})$, then $S_1A_1Q_1 \in \mathfrak{C}(V)$.
(ii) If $B_1 \in \mathfrak{C}(V_{\mathcal{S}}, U)$, then $S_1B_1 \in \mathfrak{C}(V, U)$.
(iii) If $K_1 \in \mathfrak{C}(U, V_{\mathcal{S}})$, then $K_1Q_1 \in \mathfrak{C}(U, V)$.

Proof. Using (1) and (2),

$$(S_1A_1Q_1)V = S_1A_1V_{\mathcal{S}}Q_1 = S_1V_{\mathcal{S}}A_1Q_1 = V(S_1A_1Q_1)$$

giving (i). The proofs of (ii) and (iii) are similar. □

Transitioning between a full space and a subspace is one of the main ideas in linear geometric control. The ability to perform these transitions without sacrificing commuting relationships, as shown by Theorem 3.10 and Lemma 3.11, provides the method by which feedback laws developed through linear geometric control will be able to preserve a control system's pattern. The matrices that occur in these decompositions will always involve restrictions of V-patterned matrices to V-decoupling subspaces.

On the other hand, we have already mentioned that block diagonalisation is also an important factor in control syntheses, and is still required for two steps in our framework. Block diagonalisation of patterned matrices is based on the Jordan forms of its base matrices via a classic result in Gantmacher (1959), stated here as Theorem A.2. Block diagonalisation has been the central idea of most previous research (e.g., (Brockett & Willems, 1974; Hovd et al., 1997; Hovd & Skogestad, 1994; Lunze, 1986; Massioni & Verhaegen, 2009) for block circulant and related systems): these papers start by block diagonalising the system, then designing a feedback law on each of the blocks, and finally transforming back to original coordinates. A novelty of this work is that our design approach begins with the standard control theoretic decompositions on the full system, whereas block diagonalisation is suppressed to the lowest level step in control design. This approach is intuitively appealing as it decouples the control problem from the pattern — it allows standard techniques to be used without consideration of the system's pattern. Block diagonalisation still remains the mechanism by which we explicitly synthesise patterned controllers, specifically to solve the Patterned Pole Placement Problem and patterned linear equations. Next, we present a particular ordering of the eigenvalues of V and U that facilitates block diagonalisation.

Remark 3.12 (Ordering Conventions)**.** Throughout the paper, superscript indices refer to items ordered according to this remark, and subscript indices refer to items with no particular ordering. Let V and U be diagonalisable, and denote the set of their distinct eigenvalues by $\sigma_{\mathrm{d}}(V) \cup \sigma_{\mathrm{d}}(U) = \{\delta^1, \dots, \delta^r\}$, where each δ^i has multiplicity n^i in $\sigma(V)$ and m^i in $\sigma(U)$. (Note that some n^i or m^i may be zero, reflecting that δ^i does not appear in the corresponding spectrum.) Order all the eigenvalues of V and U as follows:

$$\begin{aligned} \sigma(V) &= \{\overbrace{\delta^1, \dots, \delta^1}^{n^1 \text{ times}}, \; \dots, \overbrace{\delta^r, \dots, \delta^r}^{n^r \text{ times}}\}, \\ \sigma(U) &= \{\underbrace{\delta^1, \dots, \delta^1}_{m^1 \text{ times}}, \; \dots, \underbrace{\delta^r, \dots, \delta^r}_{m^r \text{ times}}\}. \end{aligned}$$

Next, partition Γ_V and Γ_U in the same way, so

$$\Gamma_V = \begin{bmatrix} \Gamma_V^1 & \cdots & \Gamma_V^r \end{bmatrix}, \qquad \Gamma_U = \begin{bmatrix} \Gamma_U^1 & \cdots & \Gamma_U^r \end{bmatrix} \tag{4}$$

where each Γ_V^i (n^i columns) and Γ_U^i (m^i columns) is constructed from eigenvectors corresponding to eigenvalue δ^i; these full sets of eigenvectors exist because V and U are diagonalisable (Gohberg et al., 1986). Finally, diagonalise V and U as $\hat{V} := \Gamma_V^{-1} V \Gamma_V = \mathrm{diag}(\delta^1 I_{n^1}, \dots, \delta^r I_{n^r})$ and $\hat{U} := \Gamma_U^{-1} U \Gamma_U = \mathrm{diag}(\delta^1 I_{m^1}, \dots, \delta^r I_{m^r})$, so the eigenvalues appear in the same order.

If V and U are real, then their spectra are symmetric, so there exists a 'conjugate permutation' $\{\varepsilon^1, \dots, \varepsilon^r\}$ of $\{1, \dots, r\}$ such that for all $i = 1, \dots, r$,

$$\delta^{\varepsilon^i} = \bar{\delta}^i, \quad n^{\varepsilon^i} = n^i, \quad m^{\varepsilon^i} = m^i. \tag{5}$$

Choose the eigenvectors of Γ_V and Γ_U to follow this conjugate permutation, so $\Gamma_V^{\varepsilon^i} = \bar{\Gamma}_V^i$ and $\Gamma_U^{\varepsilon^i} = \bar{\Gamma}_U^i$ for each $i = 1, \dots, r$.

The most important result on block diagonalisation, explaining how a real patterned matrix can be turned into a block diagonal matrix and vice versa, is given in the statement below. The proof can be found in the appendix, along with more supporting results of this type.

Lemma 3.13. *Let V and U be real and follow the ordering of Remark 3.12. Let matrices A and $\hat{A}$ satisfy $\hat{A} = \Gamma_V^{-1} A \Gamma_U$. Then, $A \in \mathfrak{C}_{\mathbb{R}}(V, U)$ if and only if $\hat{A} = \mathrm{diag}(\hat{A}^1, \dots, \hat{A}^r)$, where $\hat{A}^i \in \mathbb{C}^{n^i \times m^i}$ and $\hat{A}^{\varepsilon^i} = \bar{\hat{A}}^i$ for each $i = 1, \dots, r$.*

An application of the above result is to find patterned solutions to linear equations of patterned matrices. The proof is in the appendix.

Lemma 3.14. *Suppose $A \in \mathfrak{C}(V,U)$ and $B \in \mathfrak{C}(W,U)$ (with W, V, and U all diagonalisable, following Assumption 3.5). Also suppose that the equation $XA = B$ has a solution X. Then, there exists a patterned solution $X \in \mathfrak{C}(W,V)$.*

At this point, the foundations have been laid in the areas of patterned matrices, decoupling subspaces, and restrictions to subspaces and lifts to the full state space of patterned matrices. Patterned control systems can now be studied from a geometric viewpoint.

4. Patterned Systems

Consider the linear time-invariant system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ y(t) &= Cx(t) \\ z(t) &= Dx(t) \end{aligned} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $w(t) \in \mathbb{R}^s$ represents a disturbance, $y(t) \in \mathbb{R}^p$ is the measurement vector, and $z(t) \in \mathbb{R}^q$ is the output vector. The state space, input space, measurement space, and output space are denoted by $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{Z}$, respectively. Assume a real system with matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{q\times n}$, and $E \in \mathbb{R}^{n\times s}$. If all matrices in the system are patterned — that is, $A \in \mathfrak{C}(V)$, $B \in \mathfrak{C}(V,U)$, $C \in \mathfrak{C}(Y,V)$, $D \in \mathfrak{C}(Z,V)$, and $E \in \mathfrak{C}(V,W)$ — then (6) is called a *V-patterned system.*

The *open-loop poles* of the system are the eigenvalues of A; for patterned systems, they form a so-called V-patterned spectrum, explained below. The stable and unstable subspaces of A are denoted $\mathcal{X}^-(A)$ and $\mathcal{X}^+(A)$. Finally, we refer to a system as *unpatterned* if no commuting relationships are explicitly being considered — that is, if the system matrices' elements are not explicitly constrained by certain algebraic relationships, meaning that *any* controller would be a valid choice for the system.

5. Controllability

Consider the patterned system (A,B) from (6), and let $\mathcal{B} := \operatorname{Im} B$. The *controllable subspace* $\langle A|\mathcal{B}\rangle$ of (A,B) is given by $\langle A|\mathcal{B}\rangle = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}$. This section establishes that the controllable subspace captures exactly the portion of a patterned system that can be controlled by patterned feedback, in the same way as it does for a general system with 'unpatterned' feedback.

Lemma 5.1. *The controllable subspace of the V-patterned system (A,B) is a V-decoupling subspace.*

Proof. $A \in \mathfrak{C}(V)$ and $B \in \mathfrak{C}(V,U)$. By Lemma A.1, $A^{i-1}B \in \mathfrak{C}(V,U)$, and by Lemma 3.8(i), $\operatorname{Im}(A^{i-1}B)$ is a V-decoupling subspace for all $i \in \mathbb{N}$. Using Lemma 3.7, it follows that $\langle A|\mathcal{B}\rangle$ is a V-decoupling subspace. □

It is well known that the pair (A, B) is controllable if and only if the spectrum of $A + BK$ can be arbitrarily assigned to any symmetric set of poles by choice of state feedback $K : \mathcal{X} \to \mathcal{U}$. For a patterned system, the question arises as to what possible poles can be achieved by choice of *patterned* state feedback. This question is addressed by the next result.

Definition 5.2 (Patterned Spectrum)**.** *Suppose V is a real and diagonalisable matrix with distinct eigenvalues $\sigma_{\mathrm{d}}(V) = \{\delta^1, \dots, \delta^r\}$ and multiplicities n^i, $i = 1, \dots, r$ (following Remark 3.12). A spectrum $\mathscr{L}$ is called V-patterned if it can be ordered and partitioned as $\mathscr{L} = \mathscr{L}^1 \uplus \cdots \uplus \mathscr{L}^r$, where $\operatorname{card}(\mathscr{L}^i) = n^i$ for each i, and $\mathscr{L}^i = \overline{\mathscr{L}}^j$ whenever $\delta^i = \bar{\delta}^j$.*

Lemma 5.3. *Let $A \in \mathfrak{C}(V)$. Then, $\sigma(A)$ is V-patterned.*

Proof. Following Remark 3.12, suppose $\sigma_{\mathrm{d}}(V) = \{\delta^1, \dots, \delta^r\}$ with multiplicities n^i. Define $\widehat{A} := \Gamma_V^{-1} A \Gamma_V$. By Lemma 3.13, $\widehat{A} = \operatorname{diag}(\widehat{A}^1, \dots, \widehat{A}^r)$, where $\operatorname{card} \sigma(\widehat{A}^i) = n^i$, and $\widehat{A}^i = \overline{\widehat{A}}^j$ whenever $\delta^i = \bar{\delta}^j$. Taking $\mathscr{L}^i = \sigma(\widehat{A}^i)$, it follows immediately that $\sigma(A)$ is V-patterned. □

This result says that any patterned matrix has a similarly patterned spectrum, and so it follows that pole placement in patterned systems can only achieve patterned spectra. In other words, if a patterned system (A, B) is controllable, then the eigenvalues of $A + BK$ can be assigned to any V-patterned spectrum by patterned feedback $K \in \mathfrak{C}(U, V)$, and cannot be assigned to any non-V-patterned spectrum. This result was first stated for block circulant systems by Brockett and Willems (1974), and is generalised here to apply to any patterned system with diagonalisable base matrices. A similar result is found in (Consolini & Tosques, 2014) for patterns with a group structure. A more general result is found in (Sniderman et al., 2015a), but our use of Assumption 3.5 allows for a much simpler synthesis method for explicit pole placement, as well as a cleaner proof.

Theorem 5.4 (Patterned Pole Placement)**.** *Let (A, B) be a patterned system satisfying Assumption 3.5. Then, (A, B) is controllable if and only if for every V-patterned spectrum $\mathscr{L}$, there exists a patterned feedback $K \in \mathfrak{C}(U, V)$ such that $\sigma(A + BK) = \mathscr{L}$.*

The proof of Theorem 5.4, found in the appendix, provides an explicit method by which the poles of a controllable patterned system can be placed into any patterned spectrum by patterned feedback. The underlying algorithm is outlined here:

1. Block diagonalise the system matrices using Lemma 3.13. This decomposes the full system into r unpatterned subsystems. Also split the desired patterned spectrum into r subspectra, as per Definition 5.2.
2. Place the poles of each subsystem into the corresponding subspectrum using unpatterned feedbacks, by any standard method. The only restriction on these unpatterned feedbacks is to maintain the complex conjugate relationships between subsystems.
3. Transform the system back into standard coordinates. The resulting feedback law and closed-loop system will be patterned by Lemma 3.13, and the system will have the desired closed-loop spectrum.

5.1. *Controllable Decomposition*

Any system can be transformed using a basis that separates its controllable and uncontrollable parts. If the system is patterned, then each of these parts will also be patterned — the resulting commuting relationships are found using the Patterned Representation Theorem 3.10, as given in the First Decomposition Theorem below.

Theorem 5.5 (First Decomposition Theorem)**.** *Let (A, B) be a patterned system satisfying Assumption 3.5, and let its controllable subspace be $\mathcal{C} := \langle A|\mathcal{B}\rangle$ with V-invariant complement $\mathcal{C}^c$. Then, there exists a coordinate transformation $T : \mathcal{X} \to \mathcal{X}$ such that the transformed system matrices have the form*

$$T^{-1}AT = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{7}$$

where $A_1 \equiv A_{\mathcal{C}} \in \mathfrak{C}(V_{\mathcal{C}})$, $A_2 \in \mathfrak{C}(V_{\mathcal{C}^c})$, and $B_1 \in \mathfrak{C}(V_{\mathcal{C}}, U)$. Moreover, the pair (A_1, B_1) is controllable.

Proof. First, by Lemma 5.1, $\mathcal{C}$ is V-invariant and has a V-invariant complement $\mathcal{C}^c$. Choose projection maps S_1, S_2, Q_1, and Q_2 according to Theorem 3.10, and define the coordinate transformation $T = [S_1 \ S_2]$ with $T^{-1} = \operatorname{col}(Q_1, Q_2)$. The form of $T^{-1}AT$ follows from Theorem 3.10, and

$$T^{-1}B = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} B =: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} .$$

Then, all the desired commuting relationships follow from Lemma 3.9(i)–(ii). Also, since $\mathcal{B} \subset \langle A|\mathcal{B}\rangle$, therefore $B_2 = Q_2 B = 0$ and $S_1 Q_1 B = B$, and so $B = S_1 B_1$. Finally, the pair (A_1, B_1) is controllable if $S_1\langle A_1|\mathcal{B}_1\rangle = \langle A|\mathcal{B}\rangle$ (where $\mathcal{B}_1 := \operatorname{Im} B_1$). Using (1), it follows that $S_1\langle A_1|\mathcal{B}_1\rangle = S_1\mathcal{B}_1 + S_1 A_1 \mathcal{B}_1 + \cdots + S_1 A_1^{n-1}\mathcal{B}_1 = S_1\mathcal{B}_1 + A S_1 \mathcal{B}_1 + \cdots + A^{n-1} S_1 \mathcal{B}_1 = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B} = \langle A|\mathcal{B}\rangle$. □

5.2. *Stabilisability*

A patterned system (A, B) is stabilisable by patterned feedback if there exists a patterned matrix K that places its poles in a stable patterned spectrum, $\sigma(A + BK) \subset \mathbb{C}^-$. Stabilisability by patterned feedback can be determined by the same condition as standard stabilisability.

Theorem 5.6 (Patterned Stabilisability)**.** *Let (A, B) be a patterned system satisfying Assumption 3.5. Then, (A, B) can be stabilised by a patterned state feedback $K \in \mathfrak{C}(U, V)$ if and only if $\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle$.*

Proof. *(If)* Suppose $\mathcal{X}^+(A) \subset \mathcal{C}$. By the First Decomposition Theorem 5.5 there exists a coordinate transformation $(\xi_1, \xi_2) := T^{-1}x$ such that the transformed system is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{8}$$

where $A_1 \in \mathfrak{C}(V_{\mathcal{C}})$, $B_1 \in \mathfrak{C}(V_{\mathcal{C}}, U)$, and (A_1, B_1) is controllable. By Lemma 3.6, $V_{\mathcal{C}}$ is diagonalisable, so we can apply Theorem 5.4 to obtain $K_1 \in \mathfrak{C}(U, V_{\mathcal{C}})$ such that $\sigma(A_1 + B_1K_1) \subset \mathbb{C}^-$.

By Lemma 5.1, $\mathcal{C}$ has a V-invariant complement $\mathcal{C}^c$. Let Q_1 be the natural projection on $\mathcal{C}$ along $\mathcal{C}^c$, and define $K := K_1Q_1$. By Lemma 3.11(iii), $K \in \mathfrak{C}(U, V)$. Substituting $u = Kx = K_1\xi_1$ into (8), the closed-loop system satisfies $\sigma(A + BK) = \sigma(A_1 + B_1K_1) \uplus \sigma(A_2)$. Since $\mathcal{X}^+(A) \subset \mathcal{C}$, Lemma 2.1 implies $\sigma(A_2) \subset \mathbb{C}^-$. We conclude $\sigma(A + BK) \subset \mathbb{C}^-$ using the patterned feedback $K \in \mathfrak{C}(U, V)$.

(Only If) The solvability condition is identical to that for general stabilisability. Since this condition is necessary for the existence of any feedback, it is also necessary for the existence of a patterned feedback in the class $\mathfrak{C}(U, V)$. □

The proof of Theorem 5.6 illustrates the algorithm to synthesise a patterned feedback, for the Patterned Stabilisation Problem and for most further control problems considered in this paper (other than Disturbance Decoupling):

1. Decompose the system, to isolate the part whose behaviour we want to modify. The full system's pattern induces a pattern on the isolated part of the system.
2. Place the poles of the isolated part of the system by patterned feedback, following the method of Theorem 5.4.
3. Reassemble the original patterned system. The patterned controller for the isolated part of the system becomes a patterned controller for the full system.

For stabilisation, the part of the system that we want to modify is its entire controllable part (or at least the unstable part therein), so a patterned controller is synthesised as follows: split the system into its controllable part and its uncontrollable part (using Theorem 3.10); place the poles of the controllable part into a stable patterned spectrum using a patterned feedback (using Theorem 5.4); and lift this patterned feedback controller back to the full system (using Lemma 3.11). In some further control problems, the first step might be more involved. For example, Stabilisation by Measurement Feedback (Theorem 9.3) will split the system first into an observable part, and second into a controllable part therein; and Output Stabilisation (Theorem 7.5) needs a *friend* matrix in order to split the system into a part that appears in the output and a part that doesn't. Even in these more complicated cases, the overarching methodology is the same.

6. Observability

Consider the patterned system (C, A) from (6). The *unobservable subspace* $\mathcal{N}$ of (C, A) is given by $\mathcal{N} = \mathrm{Ker}(C) \cap \mathrm{Ker}(CA) \cap \cdots \cap \mathrm{Ker}(CA^{n-1})$. This section establishes that the unobservable subspace captures exactly the portion of a patterned system that cannot be seen by a patterned observer, in the same way as it does for a general system with an unpatterned observer.

Lemma 6.1. *The unobservable subspace of the V-patterned pair (C, A) is a V-decoupling subspace.*

Proof. $A \in \mathfrak{C}(V)$ and $C \in \mathfrak{C}(Y, V)$. By Lemma A.1, $CA^{i-1} \in \mathfrak{C}(Y, V)$, and by Lemma 3.8(i), $\mathrm{Ker}(CA^{i-1})$ is a V-decoupling subspace for all $i \in \mathbb{N}$. Using Lemma 3.7, it follows that $\mathcal{N}$ is a V-decoupling subspace. □

The duality between controllability and observability (Wonham, 1979, Lemma 3.4) gives the following result about the observability of patterned systems, following immediately from Theorem 5.4.

Theorem 6.2. *Let (C, A) be a patterned system satisfying Assumption 3.5. Then, (C, A) is observable if and only if for every V-patterned spectrum $\mathscr{L}$, there exists a patterned feedback $K \in \mathfrak{C}(V, Y)$ such that $\sigma(A + KC) = \mathscr{L}$.*

6.1. *Observable Decomposition*

Any system can be transformed using a basis that separates its unobservable and observable parts. If the system is patterned, then each of these parts will also be patterned — the resulting commuting relationships are found using the Patterned Representation Theorem 3.10, as given in the Second Decomposition Theorem below. The proof is by duality with the First Decomposition Theorem 5.5.

Theorem 6.3 (Second Decomposition Theorem)**.** *Let (C, A) be a patterned system satisfying Assumption 3.5, and let its unobservable subspace be $\mathcal{N}$ with V-invariant complement $\mathcal{N}^c$. Then, there exists a coordinate transformation $T : \mathcal{X} \to \mathcal{X}$ such that the transformed system matrices have the form*

$$T^{-1}AT = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \qquad CT = [0 \ C_2] \tag{9}$$

where $A_1 \equiv A_{\mathcal{N}} \in \mathfrak{C}(V_{\mathcal{N}})$, $A_2 \in \mathfrak{C}(V_{\mathcal{N}^c})$, and $C_2 \in \mathfrak{C}(Y, V_{\mathcal{N}^c})$. Moreover, the pair (C_2, A_2) is observable.

6.2. *Detectability*

A patterned system (C, A) is detectable by patterned feedback if there exists a patterned matrix K for which $\sigma(A + LC) \subset \mathbb{C}^-$. Detectability by patterned feedback can be determined by the same condition as standard detectability. The proof follows by considering the dual of Theorem 5.6.

Theorem 6.4 (Patterned Detectability)**.** *Let (C, A) be a patterned system satisfying Assumption 3.5. Then, (C, A) is detectable by a patterned state feedback $K \in \mathfrak{C}(V, Y)$ if and only if $\mathcal{N} \subset \mathcal{X}^-(A)$.*

Proof. *(If)* Suppose $\mathcal{N} \subset \mathcal{X}^-(A)$, which is equivalent to $(\mathcal{X}^-(A))^\perp \subset \mathcal{N}^\perp$, which evaluates to $\mathcal{X}^+(A^{\mathrm{T}}) \subset \langle A^{\mathrm{T}} | \operatorname{Im} C^{\mathrm{T}} \rangle$. Therefore, $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is patterned stabilisable by Theorem 5.6, so there exists $K \in \mathfrak{C}(Y^{\mathrm{T}}, V^{\mathrm{T}})$ for which $\sigma(A^{\mathrm{T}} + C^{\mathrm{T}}K) \subset \mathbb{C}^-$. Take $L := -K^{\mathrm{T}} \in \mathfrak{C}(V, Y)$, giving

$$\sigma(A - LC) = \sigma\big((A - LC)^{\mathrm{T}}\big) = \sigma(A^{\mathrm{T}} + C^{\mathrm{T}}K) \subset \mathbb{C}^- .$$

By definition, (A, C) is patterned detectable.

(Only If) The solvability condition is identical to that for general stabilisability. Since this condition is necessary for the existence of any feedback, it is also necessary for the existence of a patterned feedback in the class $\mathfrak{C}(U, V)$. □

7. Output Stabilisation

Consider the system (D, A, B) from (6), where D corresponds to the system output $z(t)$. The *Output Stabilisation Problem* (OSP) is to find a state feedback $u(t) = Kx(t)$ such that $z(t) \to 0$ as $t \to \infty$; in geometric terms, $\mathcal{X}^+(A + BK) \subset \operatorname{Ker} D$. The OSP can be stated equivalently for patterned systems with patterned feedback, as given in Problem 7.1 below, and this section shows that the standard OSP and the Patterned OSP are equally solvable for patterned systems; that is, if any output-stabilising feedback exists, then a patterned output-stabilising feedback exists.

Problem 7.1 (Patterned OSP)**.** Let (D, A, B) be a patterned system satisfying Assumption 3.5. Find a patterned state feedback $K : \mathcal{X} \to \mathcal{U}$, $K \in \mathfrak{C}(U, V)$, such that $\mathcal{X}^+(A + BK) \subset \operatorname{Ker} D$.

Solving the OSP requires the notion of *controlled invariant subspaces*: a subspace $\mathcal{V} \subset \mathcal{X}$ is controlled invariant if there exists $F : \mathcal{X} \to \mathcal{U}$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$; then, F is called a *friend* of $\mathcal{V}$. The set of all controlled invariant subspaces in $\mathcal{X}$ is denoted by $\mathscr{I}(\mathcal{X})$. The notions of controlled invariance and friends carry over to patterned systems: the next result shows that it is always possible to find a patterned friend for a V-decoupling controlled invariant subspace.

Lemma 7.2. *Let (A, B) be a patterned system. If $\mathcal{V} \subset \mathcal{X}$ is a V-decoupling subspace, and $\mathcal{V} \in \mathscr{I}(\mathcal{X})$, then there exists $F \in \mathfrak{C}(U, V)$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.*

Proof. Suppose $\dim \mathcal{V} = k \leq n$. Let $\mathcal{W}$ be a V-invariant complement of $\mathcal{V}$. Let S_1, Q_1 and S_2, Q_2 be the insertion and natural projection maps on $\mathcal{V}$ and $\mathcal{W}$, respectively. By (Wonham, 1979, Lemma 4.2), $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$. This equation can be written in matrix form as

$$AS_1 = P - BR\,, \tag{10}$$

where $P \in \mathbb{R}^{n \times k}$, $R \in \mathbb{R}^{m \times k}$, and $\operatorname{Im} P \subset \mathcal{V}$. Define $T = [S_1\ S_2]$; then, $T^{-1} = \operatorname{col}(Q_1, Q_2)$. Also let $V_1 := V_{\mathcal{V}}$ and $V_2 := V_{\mathcal{W}}$. Then $T^{-1}AS_1 = T^{-1}P - T^{-1}BR$ decomposes as

$$A_1 = P_1 - B_1 R \tag{11}$$

$$A_2 = \quad - B_2 R \tag{12}$$

where $A_i := Q_i A S_1 \in \mathfrak{C}(V_i, V_1)$, $B_i := Q_i B \in \mathfrak{C}(V_i, U)$, and $P_i := Q_i P$ for $i = 1, 2$; the commuting relationships follow from Lemma 3.9(i)–(ii), and $P_2 = Q_2 P = 0$ since $\operatorname{Im} P \subset \mathcal{V}$. Thus, finding P and R that satisfy (10) is equivalent to finding P_1 and R that satisfy (11)–(12). By Lemma 3.6, V_1 and V_2 are diagonalisable, so by Lemma 3.14, (12) can be solved for $R \in \mathfrak{C}(U, V_1)$. Then, $P_1 := A_1 + B_1 R$, and $P_1 \in \mathfrak{C}(V_1)$ by Lemma A.1. Since $\operatorname{Im} P \subset \mathcal{V}$, therefore $S_1 Q_1 P = P$, so $P := S_1 P_1$. Now define the friend $F = RQ_1$. Observe that $(A + BF)S_1 = AS_1 + BRQ_1S_1 = AS_1 + BR = P$, so $(A + BF)\mathcal{V} = \operatorname{Im} P \subset \mathcal{V}$. Also, $F \in \mathfrak{C}(U, V)$ by Lemma 3.11(iii). We conclude F is a patterned friend of the V-decoupling controlled invariant subspace $\mathcal{V}$. $\square$

Given a controlled invariant subspace $\mathcal{V}$ with a friend F, any system can be transformed into a basis which splits it into the parts in $\mathcal{V}$ and the parts not in $\mathcal{V}$. If the system, subspace, and friend are all patterned, then each of the transformed parts will also be patterned, based on the Patterned Representation Theorem 3.10. These commuting relationships are given in the Third Decomposition Theorem below. The proof follows immediately from Theorem 3.10 and Lemma 7.2.

Theorem 7.3 (Third Decomposition Theorem)**.** *Let (A, B) be a patterned system, and let $\mathcal{V} \subset \mathcal{X}$ be a V-decoupling and controlled invariant subspace with V-invariant complement $\mathcal{W}$. Then, there exists a state and feedback transformation (T, F), where $T : \mathcal{X} \to \mathcal{X}$ and $F \in \mathfrak{C}(U, V)$ is a patterned friend of $\mathcal{V}$, such that the transformed system matrices have the form*

$$T^{-1}(A + BF)T = \begin{bmatrix} \widetilde{A}_1 & * \\ 0 & \widetilde{A}_2 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{13}$$

where $\widetilde{A}_1 \equiv (A + BF)_{\mathcal{V}} \in \mathfrak{C}(V_{\mathcal{V}})$, $\widetilde{A}_2 \in \mathfrak{C}(V_{\mathcal{W}})$, $B_1 \in \mathfrak{C}(V_{\mathcal{V}}, U)$, and $B_2 \in \mathfrak{C}(V_{\mathcal{W}}, U)$.

The OSP implements the above decomposition using a part of the system that does not show up in the output, given by $\mathcal{V}^\star := \sup \mathscr{I}(\operatorname{Ker} D)$ — the largest controlled invariant subspace contained in $\operatorname{Ker} D$. For $\mathcal{V}^\star$ to be usable in the Patterned OSP, it must also be a V-decoupling subspace; this is verified in the next result. Then, Lemma 7.2 and Theorem 7.3 guarantee that $\mathcal{V}^\star$ can be used to split a patterned system into patterned subsystems, allowing a patterned output-stabilising feedback to be found.

Lemma 7.4. *Let (D, A, B) be a patterned system. Then, $\mathcal{V}^\star := \sup \mathscr{I}(\operatorname{Ker} D)$ is a V-decoupling subspace.*

Proof. Consider the recursive algorithm in (Wonham, 1979, Theorem 4.3):

$$\begin{aligned} \mathcal{V}^0 &= \operatorname{Ker}(D) \\ \mathcal{V}^i &= \operatorname{Ker}(D) \cap A^{-1}(\mathcal{B} + \mathcal{V}^{i-1}) \,. \end{aligned}$$

The sequence is nonincreasing and has a lower bound of $\{0\}$, so it must have a fixed point. By Wonham (1979, Theorem 4.3), that fixed point is $\mathcal{V}^\star$. Also, using Lemma 3.7 and Lemma 3.8(iii), each $\mathcal{V}^i$ is a V-decoupling subspace. Hence, the fixed point $\mathcal{V}^\star$ is also a V-decoupling subspace. $\square$

The Patterned OSP is solved next, using the same solvability condition as the standard, unpatterned OSP.

Theorem 7.5. *Let (D, A, B) be a patterned system. The Patterned OSP is solvable if and only if*

$$\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{V}^\star \tag{14}$$

where $\mathcal{V}^\star := \sup \mathscr{I}(\operatorname{Ker} D)$.

Proof. *(If)* Suppose (14) holds. By Lemma 7.4, $\mathcal{V}^\star$ is a controlled invariant and V-decoupling subspace. Let $\mathcal{W}$ be a V-invariant complement. Then we can apply the Third Decomposition Theorem 7.3 with $(\xi_1, \xi_2) = T^{-1}x$, $v = u - Fx$, and $F \in \mathfrak{C}(U, V)$

a patterned friend of $\mathcal{V}^\star$, to get

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}\widetilde{A}_1 & *\\ 0 & \widetilde{A}_2\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix} v \tag{15}$$

where $\widetilde{A}_2 \in \mathfrak{C}(V_{\mathcal{W}})$ and $B_2 \in \mathfrak{C}(V_{\mathcal{W}}, U)$. We claim that the subsystem $(\widetilde{A}_2, B_2)$ is stabilisable. Let $Q_2 : \mathcal{X} \to \mathcal{W}$ be the natural projection on $\mathcal{W}$ along $\mathcal{V}^\star$. Then $Q_2\mathcal{V}^\star = 0$. By Lemma 2.1 of (Wonham, 1979), $\langle A + BF|\mathcal{B}\rangle = \langle A|\mathcal{B}\rangle$, and by Lemma 6.2 of (Wonham, 1979), $\mathcal{X}^+(A + BF) + \langle A|\mathcal{B}\rangle = \mathcal{X}^+(A) + \langle A|\mathcal{B}\rangle$. Combining these facts with (14), we have

$$\begin{aligned}\mathcal{X}^+(A + BF) &\subset \mathcal{X}^+(A + BF) + \langle A|\mathcal{B}\rangle = \mathcal{X}^+(A) + \langle A|\mathcal{B}\rangle\\ &\subset \langle A|\mathcal{B}\rangle + \mathcal{V}^\star = \langle A + BF|\mathcal{B}\rangle + \mathcal{V}^\star\,.\end{aligned}$$

It can also be shown that $Q_2\mathcal{X}^+(A + BF) = \mathcal{W}^+(\widetilde{A}_2)$ and $Q_2\langle A + BF|\mathcal{B}\rangle = \langle \widetilde{A}_2|\mathcal{B}_2\rangle$, where $\mathcal{B}_2 = \operatorname{Im} B_2$. Putting it all together, we have

$$\mathcal{W}^+(\widetilde{A}_2) = Q_2\mathcal{X}^+(A + BF) \subset Q_2(\langle A + BF|\mathcal{B}\rangle + \mathcal{V}^\star) = \langle \widetilde{A}_2|\mathcal{B}_2\rangle\,.$$

This is precisely the condition for stabilisability of (A_2, B_2). Thus, by Theorem 5.6, there exists $K_2 \in \mathfrak{C}(U, V_{\mathcal{W}})$ such that $\sigma(\widetilde{A}_2 + B_2K_2) \subset \mathbb{C}^-$. Lifting K_2 and applying Lemmas 3.11 and A.1, we obtain the overall patterned feedback $K = F + K_2Q_2 \in \mathfrak{C}(U, V)$. It follows from Lemma 2.1 that $\mathcal{X}^+(A + BK) \subset \mathcal{V}^\star$. Thus, K solves the Patterned OSP.

(Only If) Condition (14) is exactly the necessary condition for the general OSP. Since it is necessary for the existence of a general state feedback, it is also necessary for the existence of a patterned state feedback. □

8. Disturbance Decoupling

Consider the system (D, A, B, E) from (6), where E corresponds to a disturbance $w(t)$ that is not directly measurable by the controller. The *Disturbance Decoupling Problem* (DDP) is to find a state feedback $u(t) = Kx(t)$ so that the output $z(t)$ is unchanged for any disturbance $w(t)$. This control problem can be stated in the same manner for patterned systems with patterned feedback, and is given in geometric terms as follows.

Problem 8.1 (Patterned DDP)**.** Let (D, A, B, E) be a patterned system satisfying Assumption 3.5. Find a patterned state feedback $K : \mathcal{X} \to \mathcal{U}$, $K \in \mathfrak{C}(U, V)$, such that $\langle A + BK|\operatorname{Im} E\rangle \subset \operatorname{Ker} D$.

The Patterned DDP and the standard, unpatterned DDP are solvable by the same condition, as shown next. The proof is a minor variation of Wonham (1979, Theorem 4.2), so it is omitted here.

Theorem 8.2. *The Patterned DDP is solvable if and only if $\mathcal{E} \subset \mathcal{V}^\star$, where $\mathcal{E} = \operatorname{Im} E$ and $\mathcal{V}^\star = \sup \mathscr{I}(\operatorname{Ker} D)$.*

The above solution to the Patterned DDP does not depend on E being a patterned matrix. It is interesting that a patterned feedback can block out disturbances that do not follow the system's pattern and underlying structure.

9. Stabilisation by Measurement Feedback

Consider the system (C, A, B) from (6). The *Stabilisation by Measurement Feedback Problem* (SMFP) is to find a measurement feedback $u(t) = K'y(t)$ such that $x(t) \to 0$ as $t \to \infty$. Generally, this measurement feedback is not found directly; instead, a state feedback $u(t) = Kx(t)$ is found first, where K is constrained to only use states that can be seen in the measurement y. This constraint is the same for patterned systems as for unpatterned systems, and is given in geometric terms in Problem 9.1 below. This section shows that the standard SMFP and the Patterned SMFP are equally solvable for patterned systems; that is, if any measurement feedback exists, then a patterned measurement feedback exists.

Problem 9.1 (Patterned SMFP)**.** Let (C, A, B) be a patterned system satisfying Assumption 3.5. Find a patterned state feedback $K : \mathcal{X} \to \mathcal{U}$, $K \in \mathfrak{C}(U, V)$, such that

$$\operatorname{Ker} C \subset \operatorname{Ker} K \tag{16}$$

$$\sigma(A + BK) \subset \mathbb{C}^- \,. \tag{17}$$

Condition (17) guarantees the stability of the closed-loop system, and condition (16) 'masks out' any state information that is not available in the measurement y, which ensures that the state feedback K can later be turned into a measurement feedback K'. This mask is characterised geometrically by $\mathcal{L} := \langle A | \operatorname{Ker} C \rangle$, the smallest A-invariant subspace containing $\operatorname{Ker} C$. Then, any feedback K found on the subsystem corresponding to $\mathcal{L}$ will satisfy $\mathcal{L} \subset \operatorname{Ker} K$, fulfilling condition (16).

Lemma 9.2. *Let (C, A) be a patterned system satisfying Assumption 3.5. Then, the subspace $\mathcal{L} := \langle A | \operatorname{Ker} C \rangle = \operatorname{Ker} C + A \operatorname{Ker} C + \cdots + A^{n-1} \operatorname{Ker} C$ is a V-decoupling subspace.*

Proof. $A \in \mathfrak{C}(V)$ and $C \in \mathfrak{C}(Y, V)$. By Lemma A.1, $A^i \in \mathfrak{C}(V)$ for all $i \in \mathbb{N}$, and by Lemma 3.8(i), $\operatorname{Ker} C$ is a V-decoupling subspace; therefore, by Lemma 3.8(iii), $A^i \operatorname{Ker} C$ is a V-decoupling subspace. Using Lemma 3.7, it follows that $\mathcal{L}$ is a V-decoupling subspace. □

With the guarantee that the masking subspace $\mathcal{L} := \langle A | \operatorname{Ker} C \rangle$ is a V-decoupling subspace, the Patterned SMFP has the following sufficient conditions for the existence of a solution — the same conditions as for the standard, unpatterned SMFP.

Theorem 9.3. *The Patterned SMFP is solvable if*

$$\mathcal{X}^+(A) \subset \langle A | \mathcal{B} \rangle \tag{18}$$

$$\mathcal{X}^+(A) \cap \langle A | \operatorname{Ker} C \rangle = 0 \,. \tag{19}$$

Proof. Let $\mathcal{L} := \langle A | \operatorname{Ker} C \rangle$. By Lemma 9.2, $\mathcal{L}$ is a V-decoupling subspace with a V-invariant complement $\mathcal{L}^c$. Since $\mathcal{L}$ is A-invariant, the system can be decomposed as in the Second Decomposition Theorem 6.3, where we replace $\mathcal{N}$ with $\mathcal{L}$ and $\mathcal{N}^c$ with

$\mathcal{L}^c$. Applying the coordinate transformation $(\xi_1, \xi_2) = T^{-1}x$, the transformed system is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{20}$$

where $A_2 \in \mathfrak{C}(V_{\mathcal{L}^c})$ and $B_2 \in \mathfrak{C}(V_{\mathcal{L}^c}, U)$ by Lemma 3.9(i)–(ii). We claim the subsystem (A_2, B_2) is stabilisable. Let $Q_2 : \mathcal{X} \to \mathcal{L}^c$ be the natural projection on $\mathcal{L}^c$ along $\mathcal{L}$. Using (18) we have

$$(\mathcal{L}^c)^+(A_2) = Q_2\mathcal{X}^+(A) \subset Q_2\langle A|\mathcal{B}\rangle = \langle A_2|\mathcal{B}_2\rangle\,.$$

By Theorem 5.6, there exists a patterned feedback $K_2 \in \mathfrak{C}(U, V_{\mathcal{L}^c})$ that stabilises (A_2, B_2). Lifting K_2 results in the overall feedback matrix $K := K_2Q_2$ with $K \in \mathfrak{C}(U, V)$ by Lemma 3.11(iii).

Finally we show K solves the Patterned SMFP. Substituting $u = Kx = K_2\xi_2$ into (20), the closed-loop system satisfies $\sigma(A+BK) = \sigma(A_1) \uplus \sigma(A_2+B_2K_2)$. By Lemma 2.1 and (19), $\sigma(A_1) \subset \mathbb{C}^-$, and by construction, $\sigma(A_2 + B_2K_2) \subset \mathbb{C}^-$. Thus, $\sigma(A+BK) \subset \mathbb{C}^-$, satisfying requirement (17) of the SMFP. Also, $\operatorname{Ker} C \subset \mathcal{L} = \operatorname{Ker} Q_2 \subset \operatorname{Ker} K$, satisfying requirement (16). We conclude that K solves the Patterned SMFP. □

The SMFP asks for a measurement feedback $u = K'y$; however, the above solution to the Patterned SMFP gave a state feedback $u = Kx$. Since $\operatorname{Ker} C \subset \operatorname{Ker} K$ (from requirement (16)), the state feedback K can be turned into a measurement feedback K' that solves the equation $K'C = K$; also, since $K \in \mathfrak{C}(U, V)$ and $C \in \mathfrak{C}(Y, V)$, a patterned solution $K' \in \mathfrak{C}(U, Y)$ can be found using Lemma 3.14. Thus, the patterned measurement feedback $u = K'y$ solves the Patterned SMFP. These comments can also be carried forward to patterned observers for detectable systems.

10. Output Stabilisation by Measurement Feedback

Consider the system (D, C, A, B) from (6). The *Output Stabilisation by Measurement Feedback Problem* (OSMFP) is to find a measurement feedback $u(t) = K'y(t)$ such that $z(t) \to 0$ as $t \to \infty$. As in the SMFP, the geometric solution to the OSMFP instead finds a state feedback $u(t) = Kx(t)$, where $\operatorname{Ker} K \supset \mathcal{L}$ for a 'masking subspace' $\mathcal{L} \supset \operatorname{Ker} C$ that blocks out any state information not seen in the measurement. This turns the OSMFP into the *Restricted Regulator Problem* (RRP), which can be stated geometrically for patterned systems in the same way as for unpatterned systems. We show that the Patterned RRP and the standard, unpatterned RRP are equally solvable; that is, if any feedback exists, then a patterned feedback exists.

Problem 10.1 (Patterned RRP)**.** Let (D, A, B) be a patterned system, and let $\mathcal{L}$ be an A-invariant and V-decoupling subspace. Find a patterned state feedback $K : \mathcal{X} \to \mathcal{U}$, $K \in \mathfrak{C}(U, V)$, such that

$$\mathcal{L} \subset \operatorname{Ker} K \tag{21}$$

$$\mathcal{X}^+(A + BK) \subset \operatorname{Ker} D\,. \tag{22}$$

The Patterned RRP can be solved by the same conditions as the unpatterned RRP, as long as all subspaces chosen a priori are V-decoupling. That is, if any output-stabilising state feedback solves the RRP, then some patterned feedback solves it as well.

Theorem 10.2. *The Patterned RRP is solvable if and only if there exists a V-decoupling and controlled invariant subspace $\mathcal{V} \in \mathscr{I}(\operatorname{Ker} D)$ such that*

$$A(\mathcal{L} \cap \mathcal{V}) \subset \mathcal{L} \cap \mathcal{V} \tag{23}$$

$$\mathcal{X}^+(A) \cap \mathcal{L} \subset \mathcal{L} \cap \mathcal{V} \tag{24}$$

$$\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{V}\,. \tag{25}$$

Proof. *(If)* Suppose (23)–(25) hold for some V-decoupling subspace $\mathcal{V} \in \mathscr{I}(\operatorname{Ker} D)$. By Lemma 3.7, $\mathcal{L} + \mathcal{V}$ and $\mathcal{L} \cap \mathcal{V}$ are V-decoupling subspaces, and have V-invariant complements $\mathcal{R}$ and $\mathcal{W}$, respectively; that is,

$$\mathcal{X} = (\mathcal{L} + \mathcal{V}) \oplus \mathcal{R}\,, \qquad \mathcal{X} = (\mathcal{L} \cap \mathcal{V}) \oplus \mathcal{W}\,.$$

Also, $\tilde{\mathcal{L}} := \mathcal{L} \cap \mathcal{W}$ and $\tilde{\mathcal{V}} := \mathcal{V} \cap \mathcal{W}$ are V-decoupling subspaces by Lemma 3.7. By the modular distributive rule (Wonham, 1979, (0.3.1)), $\mathcal{L} = (\mathcal{L} \cap \mathcal{V}) \oplus \tilde{\mathcal{L}}$ and $\mathcal{V} = (\mathcal{L} \cap \mathcal{V}) \oplus \tilde{\mathcal{V}}$. Thus, the state space $\mathcal{X}$ splits into four V-decoupling subspaces:

$$\mathcal{X} = (\mathcal{L} \cap \mathcal{V}) \oplus \tilde{\mathcal{L}} \oplus \tilde{\mathcal{V}} \oplus \mathcal{R}\,. \tag{26}$$

Let S_i and Q_i, $i = 1, \ldots, 4$, be the insertion and natural projection maps of these spaces, in the same order as (26).

Next, a patterned friend F' of $\mathcal{V}$ will be found such that $\mathcal{L}$ and $\mathcal{V}$ are both $(A+BF')$-invariant. We start by choosing any patterned friend $F \in \mathfrak{C}(U, V)$ using Lemma 7.2, and then take $F' := FS_3Q_3$. By Lemmas 3.9(iii) and 3.11(iii), $F' \in \mathfrak{C}(U, V)$. Also, $F'\mathcal{L} = FS_3(Q_3\mathcal{L}) = 0$ since $\mathcal{L} \subset \operatorname{Ker} Q_3$. Thus, $(A + BF')\mathcal{L} = A\mathcal{L} \subset \mathcal{L}$, so $\mathcal{L}$ is $(A+BF')$-invariant. Lastly, $\mathcal{V}$ is also $(A+BF')$-invariant: using $F'\mathcal{L} = 0$, $\tilde{\mathcal{V}} = S_3Q_3\tilde{\mathcal{V}}$, and (23),

$$\begin{aligned}
(A + BF')\mathcal{V} &= (A + BF')(\mathcal{L} \cap \mathcal{V}) + (A + BF')\tilde{\mathcal{V}} \\
&= A(\mathcal{L} \cap \mathcal{V}) + (A + BFS_3Q_3)(S_3Q_3\tilde{\mathcal{V}}) \\
&= A(\mathcal{L} \cap \mathcal{V}) + (AS_3 + BFS_3)(Q_3\tilde{\mathcal{V}}) \\
&= A(\mathcal{L} \cap \mathcal{V}) + (A + BF)(S_3Q_3\tilde{\mathcal{V}}) \\
&\subset (\mathcal{L} \cap \mathcal{V}) + \mathcal{V} = \mathcal{V}\,.
\end{aligned}$$

Therefore, $F' \in \mathfrak{C}(U, V)$ is a patterned friend of $\mathcal{V}$ for which $\mathcal{L}$, $\mathcal{V}$, $\mathcal{L} \cap \mathcal{V}$, and $\mathcal{L} + \mathcal{V}$ are all $(A + BF')$-invariant. Now we apply the feedback transformation (T, F'), where $T = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \end{bmatrix}$. Let $\xi = T^{-1}x$ and $v = u - F'x$. Then

$$\dot{\xi} = \begin{bmatrix} \tilde{A}_1 & * & * & * \\ 0 & \tilde{A}_2 & 0 & * \\ 0 & 0 & \tilde{A}_3 & * \\ 0 & 0 & 0 & \tilde{A}_4 \end{bmatrix} \xi + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} v \tag{27}$$

where the zeros arise from the $(A + BF')$-invariance of $\mathcal{L}$, $\mathcal{V}$, $\mathcal{L} \cap \mathcal{V}$, and $\mathcal{L} + \mathcal{V}$. Also, all the blocks are patterned as in Lemma 3.9(i)–(ii); in particular, $\tilde{A}_4 \in \mathfrak{C}(V_{\mathcal{R}})$ and

$B_4 \in \mathfrak{C}(V_{\mathcal{R}}, U)$.

We claim $(\widetilde{A}_4, B_4)$ is stabilisable. Using (25) and the property $\mathcal{X}^+(A+BF') \subset \langle A+BF'|\mathcal{B}\rangle + \mathcal{V}$ (Wonham, 1979, Lemma 6.2),

$$\mathcal{R}^+(\widetilde{A}_4) = Q_4\mathcal{X}^+(A+BF') \subset Q_4\langle A+BF'|\mathcal{B}\rangle + Q_4\mathcal{V} = \langle \widetilde{A}_4|\mathcal{B}_4\rangle\,.$$

Therefore, by Theorem 5.6, there exists $K_4 \in \mathfrak{C}(U, V_{\mathcal{R}})$ such that $\sigma(\widetilde{A}_4 + B_4K_4) \subset \mathbb{C}^-$. Lifting K_4 and using Lemmas 3.11 and A.1(iii), we obtain the overall patterned feedback $K = F' + K_4Q_4 \in \mathfrak{C}(U, V)$. We show K solves the Patterned RRP. First, recall $F'\mathcal{L} = 0$. Also $Q_4\mathcal{L} = 0$, so $K\mathcal{L} = 0$, which gives (21). Second, from (26) we have $\mathcal{X} = \mathcal{V} \oplus (\widetilde{\mathcal{L}} \oplus \mathcal{R})$. Using Lemma 2.1, if $\sigma(\widetilde{A}_2) \uplus \sigma(\widetilde{A}_4 + B_4K_4) \subset \mathbb{C}^-$, then we get (22). From above we already know $\sigma(\widetilde{A}_4 + B_4K_4) \subset \mathbb{C}^-$, so it only remains to show $\sigma(\widetilde{A}_2) \subset \mathbb{C}^-$. To that end, recall that $A\mathcal{L} \subset \mathcal{L}$ and $F'\mathcal{L} = 0$, so $(A+BK)_{\mathcal{L}} = A_{\mathcal{L}}$. Then using (24),

$$\mathcal{L}^+((A+BK)_{\mathcal{L}}) = \mathcal{L}^+(A_{\mathcal{L}}) \subset \mathcal{X}^+(A) \cap \mathcal{L} \subset \mathcal{L} \cap \mathcal{V}\,.$$

Decomposing $\mathcal{L} = (\mathcal{L} \cap \mathcal{V}) \oplus \widetilde{\mathcal{L}}$, it follows that $\sigma(\widetilde{A}_2) = \sigma((A+BK)_{\widetilde{\mathcal{L}}}) \subset \mathbb{C}^-$ by Lemma 2.1. Thus, $\sigma((A+BK)_{\widetilde{\mathcal{L}} \oplus \mathcal{R}}) \subset \mathbb{C}^-$, so $\mathcal{X}^+(A+BK) \subset \mathcal{V}$ by Lemma 2.1, giving (22). Therefore, K solves the Patterned RRP.

(Only If) Suppose there exists $K \in \mathfrak{C}(U, V)$ such that (21)–(22) hold. Define $\mathcal{V} := \mathcal{X}^+(A+BK)$, a V-decoupling subspace by Lemma 3.8. Since $\mathcal{V}$ is a modal subspace, $(A+BK)\mathcal{V} \subset \mathcal{V}$. From (22), also $\mathcal{V} \subset \operatorname{Ker} D$. Thus, $\mathcal{V} \in \mathscr{I}(\operatorname{Ker} D)$ is a V-decoupling and controlled invariant subspace, as required.

Since $\mathcal{L} \subset \operatorname{Ker} K$ by (21), $(A+BK)_{\mathcal{L}} = A_{\mathcal{L}}$. Then using the property $\mathcal{X}^+(A) \cap \mathcal{L} = \mathcal{X}^+(A+BK) \cap \mathcal{L} = \mathcal{V} \cap \mathcal{L}$ (Wonham, 1979, Lemma 6.1), condition (24) holds. Second, since $\mathcal{L}$ and $\mathcal{X}^+(A)$ are both A-invariant, condition (23) also holds. Third, using the property $\langle A|\mathcal{B}\rangle + \mathcal{X}^+(A+BK) = \langle A|\mathcal{B}\rangle + \mathcal{X}^+(A)$ (Wonham, 1979, Lemma 6.2), we have $\mathcal{X}^+(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{X}^+(A) = \langle A|\mathcal{B}\rangle + \mathcal{V}$ by definition of $\mathcal{V}$, so condition (25) holds. $\square$

The Patterned OSMFP asks for a measurement feedback $u = K'y$; however, in converting the Patterned OSMFP to the Patterned RRP, the above solution provided a state feedback $u = Kx$. In that solution, K was 'restricted' such that $\operatorname{Ker} K \supset \mathcal{L}$ for some V-decoupling subspace $\mathcal{L}$. If also $\mathcal{L} \supset \operatorname{Ker} C$, then the patterned state feedback K can be transformed into a patterned measurement feedback $K' \in \mathfrak{C}(U, Y)$ by solving the equation $K'C = K$, using Lemma 3.14. This operation depends on the choice of the V-decoupling subspace $\mathcal{L}$; one possible choice is $\mathcal{L} = \langle A|\operatorname{Ker} C\rangle$, as in the SMFP. Then, the patterned measurement feedback $u = K'y$ solves the Patterned OSMFP.

11. Numerical Example

Patterns appear in many physical systems and many control applications, ranging from formation control of satellites to power generation in wind farms to chemical reactions in cellular structures. Each of these applications deserves a full treatment of its own; the example presented here is purely pedagogical, provided as a demonstration of our framework. We consider one of the most ubiquitous distributed system structures — a *ring* system — whose matrices all have a block circulant pattern.

Consider the patterned system (D, C, A, B) as in (6), given by the block circulant matrices

$$A = \left[\begin{array}{rr|rr|rr|rr} -1 & 5 & 3 & 1 & -9 & -3 & 3 & 1 \\ 0 & -3 & 0 & -5 & 0 & 1 & 0 & -5 \\ \hline 3 & 1 & -1 & 5 & 3 & 1 & -9 & -3 \\ 0 & -5 & 0 & -3 & 0 & -5 & 0 & 1 \\ \hline -9 & -3 & 3 & 1 & -1 & 5 & 3 & 1 \\ 0 & 1 & 0 & -5 & 0 & -3 & 0 & -5 \\ \hline 3 & 1 & -9 & -3 & 3 & 1 & -1 & 5 \\ 0 & -5 & 0 & 1 & 0 & -5 & 0 & -3 \end{array}\right], \quad B = \left[\begin{array}{r|r|r|r} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right],$$

$$C = \left[\begin{array}{rr|rr|rr|rr} 7 & 5 & -1 & 1 & -5 & -3 & -1 & 1 \\ \hline -1 & 1 & 7 & 5 & -1 & 1 & -5 & -3 \\ \hline -5 & -3 & -1 & 1 & 7 & 5 & -1 & 1 \\ \hline -1 & 1 & -5 & -3 & -1 & 1 & 7 & 5 \end{array}\right], \quad D = \left[\begin{array}{rr|rr|rr|rr} 5 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ \hline -1 & 0 & 5 & 0 & -1 & 0 & 1 & 0 \\ \hline 1 & 0 & -1 & 0 & 5 & 0 & -1 & 0 \\ \hline -1 & 0 & 1 & 0 & -1 & 0 & 5 & 0 \end{array}\right].$$

The system's base matrices are determined by the fundamental permutation matrix Π_4, as in Example 3.2 — specifically, $A \in \mathfrak{C}(\Pi_4 \otimes I_2)$, $B \in \mathfrak{C}(\Pi_4 \otimes I_2, \Pi_4 \otimes I)$, and $C, D \in \mathfrak{C}(\Pi_4 \otimes I, \Pi_4 \otimes I_2)$ (where '$\otimes$' denotes the Kronecker product). The control objective is to stabilise the system output $z = Dx$ using a patterned measurement feedback $u = \widetilde{K}y$ with $\widetilde{K} \in \mathfrak{C}(\Pi_4 \otimes I)$ — this is the Patterned OSMFP, as in Section 10.

The eigenvalues of A are given by the $(\Pi_4 \otimes I_2)$-patterned spectrum $\sigma(A) = \{-16, -12, -4, -4, -4, 8, 8, 8\}$. The controllable subspace $\mathcal{C} := \langle A|\mathcal{B}\rangle$, unstable subspace $\mathcal{X}^+(A) := \operatorname{Ker}(A - 8I)$, unobservable subspace $\mathcal{N}$, and supremal controlled invariant subspace $\mathcal{V}^\star := \sup \mathscr{I}(\operatorname{Ker} D)$ of the system are all $(\Pi_4 \otimes I_2)$-decoupling, and given by

$$\mathcal{C} = \operatorname{Im} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathcal{X}^+(A) = \operatorname{Im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathcal{N} = \operatorname{Im} \begin{bmatrix} 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -3 & 0 & -1 \\ -2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 3 & 0 & -1 \end{bmatrix}, \quad \mathcal{V}^\star = \operatorname{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The columns of each matrix form a basis for the corresponding subspace. Denote the i^{th} basis vector of $\mathcal{C}$ by c_i, of $\mathcal{N}$ by n_i, of $\mathcal{V}^\star$ by v_i, and of $\mathcal{X}^+(A)$ by a_i.

If the system were fully stabilisable and detectable, a patterned feedback could be found using the techniques of the Patterned SMFP; this would stabilise the full system, including the output z. However, the unstable mode a_3 is not controllable, so the full system cannot be stabilised by any feedback, patterned or otherwise. Instead, since $a_3 \in \mathcal{V}^\star$, this uncontrollable unstable mode does not show up in the output z, so the output can be stabilised by turning the OSMFP into a Patterned RRP, as in Section 10. Formulating the Patterned RRP requires a choice of two $(\Pi_4 \otimes I_2)$-decoupling subspaces: we take $\mathcal{L} := \langle A|\operatorname{Ker} C\rangle = \mathcal{N}$ and $\mathcal{V} := \mathcal{V}^\star$. Using these subspaces, we verify the solvability conditions (23)–(25) of the Patterned RRP, given in Theorem 10.2: first,

$\mathcal{L} \cap \mathcal{V} = \text{span}\{n_4\}$ is A-invariant, satisfying (23); second, $\mathcal{X}^+(A) \cap \mathcal{L} = \text{span}\{n_4\} = \mathcal{L} \cap \mathcal{V}$, satisfying (24); and third, $\langle A|\mathcal{B}\rangle + \mathcal{V} = \mathcal{X} \supset \mathcal{X}^+(A)$, satisfying (25), and confirming that the Patterned RRP is solvable.

Following the method of Theorem 10.2, a patterned output-stabilising feedback can be found by placing poles in the subsystem corresponding to the $(\Pi_4 \otimes I_2)$-decoupling subspace $\mathcal{R} := (\mathcal{L} + \mathcal{V})^\perp = \text{span}\{r_1, r_2, r_3\}$, where $r_1 = (1, 0, -1, 0, 1, 0, -1, 0)$, $r_2 = (3, 2, 0, 0, -3, -2, 0, 0)$, and $r_3 = (0, 0, 3, 2, 0, 0, -3, -2)$. $\mathcal{R}$ is the fourth subspace in the state space decomposition (26), which has insertion maps

$$\begin{aligned} S_1 &= \begin{bmatrix} n_4 \end{bmatrix}, \qquad & S_2 &= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}, \\ S_3 &= \begin{bmatrix} v \end{bmatrix}, \qquad & S_4 &= \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}, \end{aligned}$$

where $v := v_1 + v_2$. Define the coordinate transformation $T = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \end{bmatrix}$; then, the natural projection maps Q_1, Q_2, Q_3, and Q_4 are uniquely defined by $T^{-1} = \text{col}(Q_1, Q_2, Q_3, Q_4)$, where Q_1 is the first row, Q_2 is the following three rows, Q_3 is the fifth row, and Q_4 is the final three rows. With these projection maps, the system decomposition can be carried out, beginning by finding a patterned friend F of $\mathcal{V}$. Following Lemma 7.2, a friend is given by the block circulant matrix

$$F = \frac{1}{2}\left[\begin{array}{cc|cc|cc|cc} 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \end{array}\right] \in \mathfrak{C}(\Pi_4 \otimes I, \Pi_4 \otimes I_2)\,.$$

As in Theorem 10.2, F is modified to another friend $F' = FS_3Q_3$, so $F'\mathcal{L} = 0$ and $(A + BF')\mathcal{V} \subset \mathcal{V}$. In this example, $F' = F$. Then, the feedback transformation $v = u - F'x$ gives $\dot{x} = Ax + Bu = (A + BF')x + Bv$, and $\mathcal{L} + \mathcal{V}$ is $(A + BF')$-invariant. The system is decomposed as

$$T^{-1}(A+BF')T = \left[\begin{array}{c|ccc|c|ccc} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & -4 & & & 0 & 0 & 8 & 0 \\ & & -4 & & 0 & 0 & 0 & 8 \\ & & & -4 & 4 & 0 & 0 & 0 \\ \hline & & & & -12 & 0 & 0 & 0 \\ \hline & & & & & -16 & & \\ & & & & & & 8 & \\ & & & & & & & 8 \end{array}\right], \quad T^{-1}B = \frac{1}{26}\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline -6 & 0 & 6 & 0 \\ 0 & -6 & 0 & 6 \\ 13 & 13 & 13 & 13 \\ \hline 0 & 0 & 0 & 0 \\ \hline 13 & -13 & 13 & -13 \\ 4 & 0 & -4 & 0 \\ 0 & 4 & 0 & -4 \end{array}\right],$$

$$DT = \left[\begin{array}{c|ccc|c|ccc} 0 & 8 & 0 & 4 & 0 & 8 & 12 & 0 \\ 0 & 0 & 8 & 4 & 0 & -8 & 0 & 12 \\ 0 & -8 & 0 & 4 & 0 & 8 & -12 & 0 \\ 0 & 0 & -8 & 4 & 0 & -8 & 0 & 12 \end{array}\right].$$

Number the four subsystems from top-left to bottom-right (using the blocks along the diagonal of $T^{-1}(A + BF')T$), so the i^{th} subsystem is $(D_i, \widetilde{A}_i, B_i)$. All the submatrices are patterned, commuting with various restrictions of $\Pi_4 \otimes I_2$ as in Lemma 3.9; in particular, $\widetilde{A}_4 \in \mathfrak{C}((\Pi_4 \otimes I_2)_\mathcal{R})$ and $B_4 \in \mathfrak{C}((\Pi_4 \otimes I_2)_\mathcal{R}, \Pi_4 \otimes I)$. Also, the system output clearly shows up only in D_2 and D_4, and the second subsystem is stable since $\sigma(\widetilde{A}_2) = \{-4, -4, -4\}$; therefore, stabilising the entire output is equivalent to stabilising the fourth subsystem $(\widetilde{A}_4, B_4)$. Checking the unstable and controllable modes of this subsystem, it can be verified that $\mathcal{R}^+(\widetilde{A}_4) \subset \langle \widetilde{A}_4|\mathcal{B}_4\rangle$, and so this pair is stabilisable.

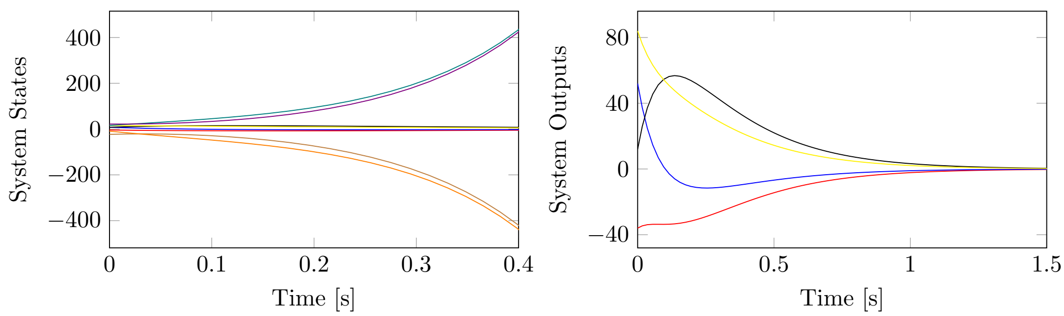

Figure 2. Closed-loop system behaviour for numerical example.

Using Theorem 5.6, a patterned stabilising feedback is given by

$$K_4 = 52 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathfrak{C}(\Pi_4 \otimes I, (\Pi_4 \otimes I_2)_{\mathcal{R}})$$

which places the closed-loop eigenvalues of the fourth subsystem as $\sigma(\tilde{A}_4 + B_4 K_4) = \{-16, -8, -8\}$, a stable $(\Pi_4 \otimes I_2)_{\mathcal{R}}$-patterned spectrum. Lifting K_4 to the full system using Lemma 3.11(iii) gives a patterned matrix $K_4 Q_4 \in \mathfrak{C}(\Pi_4 \otimes I, \Pi_4 \otimes I_2)$ that stabilises the output of the feedback-transformed system $(D, A + BF', B)$. The overall state feedback solving the Patterned RRP for the original system (D, A, B) is then given by the block circulant matrix

$$K = F' + K_4 Q_4 = \frac{1}{2} \left[\begin{array}{rr|rr|rr|rr} -12 & -9 & 0 & -1 & 12 & 7 & 0 & -1 \\ \hline 0 & -1 & -12 & -9 & 0 & -1 & 12 & 7 \\ \hline 12 & 7 & 0 & -1 & -12 & -9 & 0 & -1 \\ \hline 0 & -1 & 12 & 7 & 0 & -1 & -12 & -9 \end{array}\right]$$

which is also patterned, $K \in \mathfrak{C}(\Pi_4 \otimes I, \Pi_4 \otimes I_2)$. To confirm that K solves the Patterned RRP, check the requirements (21)–(22) (from Problem 10.1): first, it can be deduced from the state space decomposition that $\mathcal{L} \subset \mathcal{L} + \mathcal{V} \subset \operatorname{Ker} K$, satisfying (21); second, it can be calculated that $\mathcal{X}^+(A + BK) \subset \mathcal{V} \subset \operatorname{Ker} D$, satisfying (22). Thus, the state feedback $u = Kx$ solves the Patterned RRP.

To turn this state feedback into a patterned measurement feedback, a matrix $K' \in \mathfrak{C}(\Pi_4 \otimes I)$ can be found such that $K'C = K$. This equation is solvable since $\operatorname{Ker} C \subset \operatorname{Ker} K$, signifying that K only uses states that can be seen in the measurement. Using Lemma 3.14, a patterned solution is given by the block circulant matrix

$$K' = \frac{1}{8} \begin{bmatrix} -5 & -1 & 3 & -1 \\ -1 & -5 & -1 & 3 \\ 3 & -1 & -5 & -1 \\ -1 & 3 & -1 & -5 \end{bmatrix} \in \mathfrak{C}(\Pi_4 \otimes I)$$

and $u = K'y = Kx$. This patterned feedback stabilises the output of the system, solving the Patterned OSMFP and meeting the system's control objective. As the plots in Figure 2 show, while certain closed-loop system states remain unstable with this feedback, the system output has been stabilised.

12. Concluding Remarks

In this paper we showed that distributed systems can be controlled in a way that respects and preserves their interconnection structures. These interconnection structures are manifest as patterns in the system matrices. These patterns, in turn, are encoded algebraically in commuting relationships. The algebra of patterns is linked to geometry via decoupling subspaces, thereby forming a bridge between matrix structure and subspace structure. Our results fit within the large body of research on systems with symmetries, strongly suggesting that commuting properties are an integral consideration in carrying out a patterned control design. In addition to the numerical example of Section 11, our prior work includes another numerical example of Patterned Output Stabilisation (Sniderman et al., 2013), as well as a full application of our framework to the formation control of balloons (Sniderman, Broucke, & D'Eleuterio, 2015b).

The control design tools we used are standard and familiar to any researcher versed in geometric control theory. While control and analysis of distributed systems (particularly block circulant systems) have traditionally been approached through block diagonalisation, we have found that commuting properties also provide a direct and intuitive means for control design that connect well with the standard methods, allowing decompositions in particular to be done precisely the same way for patterned systems as for general linear systems. As a result, the solvability conditions of each patterned control design problem are almost completely independent of the pattern itself; we did not have to modify any of the standard geometric control conditions in order to guarantee recovery of a patterned control law. In other words, not only can a patterned system be controlled by patterned feedback, but it can be done in the 'usual' way. In short, our work suggests that patterned systems *naturally* admit patterned controllers.

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References

Basile, G., & Marro, G. (1991). *Controlled and Conditioned Invariants in Linear System Theory.* Englewood Cliffs: Prentice-Hall.

Belabbas, M.-A. (2013). Sparse Stable Systems. *Systems & Control Letters*, *62*(10), 981–987.

Brockett, R. W., & Willems, J. L. (1974, sep). Discretized Partial Differential Equations: Examples of Control Systems Defined on Modules. *Automatica*, *10*(5), 507–515.

Commault, C., Dion, J. M., & Hovelaque, V. (1997). A Geometric Approach for Structured Systems: Application to Disturbance Decoupling. *Automatica*, *33*(3), 403–409.

Consolini, L., & Tosques, M. (2014). Static Output Feedback of Equivariant Linear Systems with Applications to the Control of Interconnected Agents. In *53rd IEEE Conference on Decision and Control* (pp. 3988–3993). Los Angeles: IEEE.

Consolini, L., & Tosques, M. (2015). On the Distributed Stabilization and Consensus of Symmetrically Interconnected Agents. In *2015 European Control Conference* (pp. 1926–1931).

Denis, N., & Looze, D. P. (1999). H_∞ Controller Design for Systems with Circulant Symmetry. In *38th IEEE Conference on Decision and Control* (pp. 3144–3149). IEEE.

Deutscher, J. (2013). Krylov Subspace Methods for Block Patterned Linear Systems. In *2013 European Control Conference* (Vol. 1, pp. 827–832). Zurich.

Elbanna, R. M. (1988). *Some New Results on the Stabilization and State Estimation in Large-Scale Systems by Decentralized and Multilevel Schemes* (Ph.D. Thesis). University of Arizona.

Fagnani, F., & Willems, J. C. (1994). Interconnections and Symmetries of Linear Differential Systems. *Mathematics of Control, Signals, and Systems*, *7*, 167–186.

Gantmacher, F. R. (1959). *The Theory of Matrices.* Chelsea Publishing.

Gohberg, I., Lancaster, P., & Rodman, L. (1986). *Invariant Subspaces of Matrices with Applications.* Wiley.

Hamilton, S. C., & Broucke, M. E. (2012). *Geometric Control of Patterned Linear Systems.* Lecture Notes in Control and Information Sciences: Springer.

Hazewinkel, M., & Martin, C. F. (1983). Symmetric Linear Systems: An Application of Algebraic Systems Theory. *International Journal of Control*, *37*(6), 1371–1384.

Holmes, C., & Broucke, M. E. (2016). Pattern Preserving Pole Placement and Stabilization for Linear Systems. In *American Control Conference.*

Hovd, M., Braatz, R. D., & Skogestad, S. (1997). SVD Controllers for H2-, H∞- and μ-Optimal Control. *Automatica*, *33*(3), 433–439.

Hovd, M., & Skogestad, S. (1994, jun). Control of Symmetrically Interconnected Plants. *Automatica*, *30*(6), 957–973.

Lin, C.-T. (1974). Structural Controllability. *IEEE Transactions on Automatic Control*, *19*(3), 201–208.

Lunze, J. (1986). Dynamics of Strongly Coupled Symmetric Composite Systems. *International Journal of Control*, *44*(6), 1617–1640.

Lunze, J. (1989, jul). Stability Analysis of Large-Scale Systems Composed of Strongly Coupled Similar Subsystems. *Automatica*, *25*(4), 561–570.

Massioni, P., & Verhaegen, M. (2009, jan). Distributed Control for Identical Dynamically Coupled Systems: A Decomposition Approach. *IEEE Transactions on Automatic Control*, *54*(1), 124–135.

Rotkowitz, M., & Lall, S. (2006). A Characterization of Convex Problems in Decentralized Control. *IEEE Transactions on Automatic Control*, *51*(2), 1984–1996.

Sniderman, A. C., Broucke, M. E., & D'Eleuterio, G. M. T. (2013, dec). Block Circulant Control: A Geometric Approach. In *52nd IEEE Conference on Decision and Control* (pp. 2097–2102). Florence: IEEE.

Sniderman, A. C., Broucke, M. E., & D'Eleuterio, G. M. T. (2015a). Controllability Is Not Sufficient for Pole Placement in Patterned Systems. In *54th IEEE Conference on Decision and Control.* Osaka: IEEE.

Sniderman, A. C., Broucke, M. E., & D'Eleuterio, G. M. T. (2015b). Formation Control of Balloons: A Block Circulant Approach. In *American Control Conference* (pp. 1463–1468). Chicago: IEEE.

Sundareshan, M. K., & Elbanna, R. M. (1991). Qualitative Analysis and Decentralized Controller Synthesis for a Class of Large-Scale Systems with Symmetrically Interconnected Subsystems. *Automatica*, *27*(2), 383–388.

van der Schaft, A. (1987). Symmetries in Optimal Control. *SIAM Journal on Control and Optimization*, *25*(2), 245–259.

Wang, S.-H., & Davison, E. J. (1973). On the Stabilization of Decentralized Control Systems. *IEEE Transactions on Automatic Control*, *34*(4), 473–478.

Wonham, W. M. (1979). *Linear Multivariable Control: A Geometric Approach* (Third ed.). New York, NY: Springer-Verlag.

Appendix A. Block Diagonalising Patterned Matrices

There are two places in our framework where patterned matrices are explicitly synthesised: in solutions to linear equations (Lemma 3.14), and in pole placement (Theorem 5.4). This appendix provides further background on commuting matrices and their block diagonalisation in order to complete those proofs. Recall the notational conventions from Section 3. First, we require the following basic properties of commuting matrices.

Lemma A.1. *The following algebraic properties hold:*

(i) *If $A, B \in \mathfrak{C}(V, U)$, then $A + B \in \mathfrak{C}(V, U)$.*
(ii) *If $A \in \mathfrak{C}(W, V)$ and $B \in \mathfrak{C}(V, U)$, then $AB \in \mathfrak{C}(W, U)$.*
(iii) *If $A \in \mathfrak{C}(V, U)$, then $A^{\mathrm{T}} \in \mathfrak{C}(U^{\mathrm{T}}, V^{\mathrm{T}})$.*
(iv) *If $A \in \mathfrak{C}(V, U)$ is nonsingular, then $A^{-1} \in \mathfrak{C}(U, V)$.*

Let the Jordan forms of V and U be given by

$$\widehat{V} := \Gamma_V^{-1} V \Gamma_V = \operatorname{diag}(\widehat{V}_1, \ldots, \widehat{V}_\nu), \qquad \widehat{U} := \Gamma_U^{-1} U \Gamma_U = \operatorname{diag}(\widehat{U}_1, \ldots, \widehat{U}_\mu)$$

where each $\widehat{V}_i$ and $\widehat{U}_j$ is a Jordan block. Block diagonalisation of commuting matrices is based on these Jordan blocks.

Theorem A.2 (Gantmacher, 1959)**.** *Let A and $\widehat{A}$ satisfy $\widehat{A} := \Gamma_V^{-1} A \Gamma_U$, and partition $\widehat{A}$ according to the Jordan blocks of V and U:*

$$\widehat{A} := \Gamma_V^{-1} A \Gamma_U = \begin{bmatrix} \widehat{A}_{11} & \cdots & \widehat{A}_{1\mu} \\ \vdots & & \vdots \\ \widehat{A}_{\nu 1} & \cdots & \widehat{A}_{\nu\mu} \end{bmatrix}$$

where $\widehat{A}_{ij}$ has the same number of rows as $\widehat{V}_i$ and the same number of columns as $\widehat{U}_j$. Also suppose that $\widehat{V}_i$ corresponds to eigenvalue δ_i, and $\widehat{U}_j$ corresponds to eigenvalue γ_j. Then, $A \in \mathfrak{C}(V, U)$ if and only if the blocks $\widehat{A}_{ij}$ satisfy the following for all i, j: if $\delta_i = \gamma_j$, then $\widehat{A}_{ij}$ is an upper triangular Toeplitz matrix; and if $\delta_i \neq \gamma_j$, then $\widehat{A}_{ij} = 0$.

Any matrix $A \in \mathfrak{C}(V, U)$ (for any V and U) can be put into the partitioned form of $\widehat{A}$ in Theorem A.2. Combining this result with Assumption 3.5 that V and U are diagonalisable, each upper triangular Toeplitz matrix $\widehat{A}_{ij}$ is a scalar because the Jordan blocks of V and U are all 1×1. Thus, in our case, a pattern $\mathfrak{C}(V, U)$ determines a 'hatted' form in which some entries are fixed at zero, while others can be freely assigned. Furthermore, the partitioning of Theorem A.2 results in a block diagonal matrix if Γ_V and Γ_U are chosen according to the ordering conventions of Remark 3.12. The conventions regarding Γ_V and Γ_U also carry over to their inverses.

Lemma A.3. *Suppose V and U are real matrices, and follow Remark 3.12. Partition $\Gamma_V = \operatorname{row}(\Gamma^1, \ldots, \Gamma^r)$ as in (4), and partition Γ_V^{-1} as $\operatorname{col}(\reflectbox{Γ}^1, \ldots, \reflectbox{Γ}^r)$, where the number of rows of $\reflectbox{$\Gamma^i equals the number of columns of Γ^i. Let $\{\varepsilon^1, \ldots, \varepsilon^r\}$ be the conjugate permutation (5). Then, $\reflectbox{$\Gamma$}^i = \overline{\reflectbox{Γ}}^{\varepsilon^i}$ for each i.*

Proof. Since $\Gamma_V^{-1}\Gamma_V = I$, therefore $\reflectbox{$\Gamma$}^i \Gamma^i = I$, and $\reflectbox{$\Gamma$}^i \Gamma^j = 0$ for all $i \neq j$. Also,

$\overline{\reflectbox{$\Gamma$}^{\varepsilon^i}\Gamma^{\varepsilon^i}} = \bar{I} = I$, and so $\reflectbox{$\Gamma$}^i\Gamma^i = \bar{\reflectbox{Γ}}^{\varepsilon^i}\bar{\Gamma}^{\varepsilon^i}$. From the conventions in Remark 3.12, $\Gamma^i = \bar{\Gamma}^{\varepsilon^i}$, giving $\reflectbox{$\Gamma$}^i\Gamma^i = \bar{\reflectbox{Γ}}^{\varepsilon^i}\Gamma^i$. Thus,

$$\reflectbox{Γ}^i\Gamma_V = \begin{bmatrix}0 & \cdots & \reflectbox{Γ}^i\,\Gamma^i & \cdots & 0\end{bmatrix} = \begin{bmatrix}0 & \cdots & \bar{\reflectbox{Γ}}^{\varepsilon^i}\Gamma^i & \cdots & 0\end{bmatrix} = \bar{\reflectbox{Γ}}^{\varepsilon^i}\Gamma_V\,.$$

Since Γ_V is full rank, it follows that $\reflectbox{$\Gamma$}^i = \bar{\reflectbox{$\Gamma$}}^{\varepsilon^i}$. $\square$

The matrix $\hat{A} := \Gamma_V^{-1}A\Gamma_U$ also inherits the complex conjugate structure described above, allowing us to prove Lemma 3.13.

Proof of Lemma 3.13. *(If)* Let $A \in \mathfrak{C}_{\mathbb{R}}(V,U)$. By the ordering of Remark 3.12 and by Theorem A.2, $\hat{A} = \operatorname{diag}(\hat{A}^1,\dots,\hat{A}^r)$. (Note that the $\hat{A}^i$ blocks may contain multiple $\hat{A}_{(\cdot,\cdot)}$ blocks of Theorem A.2). Partition $\Gamma_V^{-1} = \operatorname{col}(\reflectbox{Γ}^1,\dots,\reflectbox{Γ}^r)$ as in Lemma A.3 and $\Gamma_V = \operatorname{row}(\Gamma^1,\dots,\Gamma^r)$ as in (4), where $\reflectbox{$\Gamma^i has n^i columns and Γ^i has m^i rows. Then,

$$\hat{A} := \begin{bmatrix}\hat{A}^1 & & \\ & \ddots & \\ & & \hat{A}^r\end{bmatrix} = \begin{bmatrix}\reflectbox{Γ}^1 A\Gamma^1 & & \\ & \ddots & \\ & & \reflectbox{Γ}^r A\Gamma^r\end{bmatrix}.$$

By Lemma A.3, $\reflectbox{$\Gamma$}^i = \bar{\reflectbox{$\Gamma$}}^{\varepsilon^i}$, and by Remark 3.12, $\Gamma^i = \bar{\Gamma}^{\varepsilon^i}$. Since A is real, it follows that $\hat{A}^i = \bar{\hat{A}}^{\varepsilon^i}$, for $i = 1,\dots,r$.

(Only If) Let $\hat{A} = \operatorname{diag}(\hat{A}^1,\dots,\hat{A}^r)$, where $\hat{A}^i \in \mathbb{C}^{n^i\times m^i}$ and $\hat{A}^{\varepsilon^i} = \bar{\hat{A}}^i$ for each $i = 1,\dots,r$. Define $A := \Gamma_V\hat{A}\Gamma_U^{-1}$. Then we have

$$\begin{aligned} VA = \Gamma_V\hat{V}\hat{A}\Gamma_U^{-1} &= \Gamma_V\begin{bmatrix}\lambda_1 I_{n^1} & & \\ & \ddots & \\ & & \lambda_r I_{n^r}\end{bmatrix}\begin{bmatrix}\hat{A}^1 & & \\ & \ddots & \\ & & \hat{A}^r\end{bmatrix}\Gamma_U^{-1} \\ &= \Gamma_V\begin{bmatrix}\hat{A}^1 & & \\ & \ddots & \\ & & \hat{A}^r\end{bmatrix}\begin{bmatrix}\lambda_1 I_{m^1} & & \\ & \ddots & \\ & & \lambda_r I_{m^r}\end{bmatrix}\Gamma_U^{-1} = AU \end{aligned}$$

so $A \in \mathfrak{C}(V,U)$. Next, consider

$$A := \Gamma_V\hat{A}\Gamma_U^{-1} = [\Gamma^1 \cdots \Gamma^r]\begin{bmatrix}\hat{A}^1 & & \\ & \ddots & \\ & & \hat{A}^r\end{bmatrix}\begin{bmatrix}\reflectbox{Γ}^1 \\ \vdots \\ \reflectbox{Γ}^r\end{bmatrix} = \Gamma^1\hat{A}^1\reflectbox{Γ}^1 + \cdots + \Gamma^r\hat{A}^r\reflectbox{Γ}^r\,. \quad \text{(A1)}$$

We know $\Gamma^i = \bar{\Gamma}^{\varepsilon^i}$ by Remark 3.12, $\reflectbox{$\Gamma$}^i = \bar{\reflectbox{$\Gamma$}}^{\varepsilon^i}$ by Lemma A.3, and $\hat{A}^i = \bar{\hat{A}}^{\varepsilon^i}$ by assumption. There are two cases: if $\varepsilon^i = i$, then $\Gamma^i\hat{A}^i\reflectbox{$\Gamma^i is real; and if $\varepsilon^i = j \neq i$, then $\overline{\Gamma^i\hat{A}^i\reflectbox{Γ}^i} = \Gamma^j\hat{A}^j\reflectbox{$\Gamma^j and so $\Gamma^i\hat{A}^i\reflectbox{Γ}^i + \Gamma^j\hat{A}^j\reflectbox{$\Gamma^j is real. We conclude A is real, so $A \in \mathfrak{C}_{\mathbb{R}}(V,U)$. $\square$

We now prove two results utilizing block diagonalisation: solving patterned linear equations and patterned pole placement.

Proof of Lemma 3.14. Following the ordering of Remark 3.12, we diagonalise W, V, and U as $\widehat{W} := \Gamma_W^{-1} W \Gamma_W$, etc., and define $\widehat{A} := \Gamma_V^{-1} A \Gamma_U$ and $\widehat{B} := \Gamma_W^{-1} B \Gamma_U$. By Lemma 3.13, $\widehat{A} = \operatorname{diag}(\widehat{A}^1, \dots, \widehat{A}^r)$ with $\widehat{A}^i \in \mathbb{C}^{n^i \times m^i}$, and $\widehat{B} = \operatorname{diag}(\widehat{B}^1, \dots, \widehat{B}^r)$ with $\widehat{B}^i \in \mathbb{C}^{p^i \times m^i}$. Also, if $\mathbb{F} = \mathbb{R}$ then $\widehat{A}^i = \overline{\widehat{A}}^{\varepsilon^i}$ and $\widehat{B}^i = \overline{\widehat{B}}^{\varepsilon^i}$. Therefore, solving the equation $XA = B$ for X is equivalent to solving the equation

$$\begin{bmatrix} \widehat{X}^{11} & \cdots & \widehat{X}^{1r} \\ \vdots & & \vdots \\ \widehat{X}^{r1} & \cdots & \widehat{X}^{rr} \end{bmatrix} \begin{bmatrix} \widehat{A}^1 & & \\ & \ddots & \\ & & \widehat{A}^r \end{bmatrix} = \begin{bmatrix} \widehat{B}^1 & & \\ & \ddots & \\ & & \widehat{B}^r \end{bmatrix} \tag{A2}$$

for $\widehat{X}^{ij} \in \mathbb{C}^{p^i \times n^j}$. This decomposes into

$$\begin{aligned} \widehat{X}^{ij} \widehat{A}^j &= 0\,, && i \neq j \\ \widehat{X}^{ii} \widehat{A}^i &= \widehat{B}^i\,, && i = 1, \dots, r\,. \end{aligned}$$

For the first set of equations we choose the solution $\widehat{X}^{ij} = \underline{0}$, $i \neq j$. For the second set of equations we choose any solution such that $\widehat{X}^{\varepsilon^i \varepsilon^i} = \overline{\widehat{X}}^{ii}$ (note that $\overline{\widehat{X}}^{ii} \widehat{A}^{\varepsilon^i} = \overline{\widehat{X}}^{ii} \overline{\widehat{A}}^i = \overline{\widehat{X}^{ii} \widehat{A}^i} = \overline{\widehat{B}}^i = \widehat{B}^{\varepsilon^i}$).

Define $X = \Gamma_W \operatorname{diag}(\widehat{X}^{11}, \dots, \widehat{X}^{rr}) \Gamma_V^{-1}$. Then,

$$\begin{aligned} XA &= \left(\Gamma_W \begin{bmatrix} \widehat{X}^{11} & & \\ & \ddots & \\ & & \widehat{X}^{rr} \end{bmatrix} \Gamma_V^{-1} \right) \left(\Gamma_V \begin{bmatrix} \widehat{A}^1 & & \\ & \ddots & \\ & & \widehat{A}^r \end{bmatrix} \Gamma_U^{-1} \right) \\ &= \Gamma_W \begin{bmatrix} \widehat{B}^1 & & \\ & \ddots & \\ & & \widehat{B}^r \end{bmatrix} \Gamma_U^{-1} = B\,, \end{aligned}$$

and by Lemma 3.13, $X \in \mathfrak{C}(W, V)$. $\square$

Proof of Theorem 5.4. *(Only If)* Following the notation and ordering of Remark 3.12, $\sigma_{\mathrm{d}}(V) \cup \sigma_{\mathrm{d}}(U) = \{\lambda_1, \dots, \lambda_r\}$, with multiplicities n^i in $\sigma(V)$ and m^i in $\sigma(U)$, and conjugate permutation $\{\varepsilon^1, \dots, \varepsilon^r\}$. Let $\mathscr{L}$ be a V-patterned spectrum and, following Definition 5.2, partition it as $\mathscr{L} = \mathscr{L}^1 \uplus \cdots \uplus \mathscr{L}^r$, where $\operatorname{card}(\mathscr{L}^i) = n^i$ for each $i = 1, \dots, r$.

Define $\widehat{A} := \Gamma_V^{-1} A \Gamma_V$ and $\widehat{B} := \Gamma_V^{-1} B \Gamma_U$. By Lemma 3.13, $\widehat{A} = \operatorname{diag}(\widehat{A}^1, \dots, \widehat{A}^r)$ and $\widehat{B} = \operatorname{diag}(\widehat{B}^1, \dots, \widehat{B}^r)$, where $\widehat{A}^i \in \mathbb{C}^{n^i \times n^i}$ and $\widehat{B}^i \in \mathbb{C}^{n^i \times m^i}$. Since (A, B) is controllable, $(\widehat{A}, \widehat{B})$ is controllable and, in turn, each pair $(\widehat{A}^i, \widehat{B}^i)$ is controllable. Therefore, for each $i = 1, \dots, r$, there exists some matrix $\widehat{K}^i$ such that $\sigma(\widehat{A}^i + \widehat{B}^i \widehat{K}^i) = \mathscr{L}^i$. Also, if $\mathbb{F} = \mathbb{R}$ then $\widehat{A}^i = \overline{\widehat{A}}^{\varepsilon^i}$ and $\widehat{B}^i = \overline{\widehat{B}}^{\varepsilon^i}$ by Lemma 3.13, and $\mathscr{L}^i = \overline{\mathscr{L}}^{\varepsilon^i}$ from Definition 5.2; therefore, the $\widehat{K}^i$ can be chosen such that $\widehat{K}^i = \overline{\widehat{K}}^{\varepsilon^i}$. Now, define the overall feedback $K := \Gamma_V \operatorname{diag}(\widehat{K}^1, \dots, \widehat{K}^r) \Gamma_U^{-1}$. Then, $K \in \mathfrak{C}(V, U)$ by Lemma 3.13, and $\sigma(A + BK) = \mathscr{L}$ by construction, so K is a patterned pole-placing feedback.

(If) The argument is a minor variation of the proof in Wonham (1979, Theorem 2.1), so it is omitted here. $\square$