A Hierarchical Cyclic Pursuit Scheme for Vehicle Networks [1]

Stephen L. Smith [2], Mireille E. Broucke, and Bruce A. Francis

Department of Electrical and Computer Engineering
University of Toronto, ON, Canada, M5S 3G4

Abstract

The *agreement problem* is studied whereby a group of mobile agents achieves convergence to a common point. A hierarchical cyclic pursuit scheme is introduced, and it is shown that this scheme yields a very significant increase in the rate of convergence to a common point when compared to traditional cyclic pursuit. A second scheme is introduced in which there are more communication links between vehicles. It is shown that this scheme produces a rate of convergence greater than the traditional scheme but significantly less than the hierarchical scheme.

Key words: Autonomous vehicles; hierarchies; cooperative control; multi-agent systems.

1 Introduction

The control of multi-vehicle systems is a topic of current interest in the control community. Potential applications of this research are abundant, and include planetary exploration, automated highway systems, and rescue missions. As an example, consider a fleet of rovers deployed on the surface of Mars. It is impractical to remotely control each rover individually. Rather, it would be desirable for the rovers to explore their surroundings autonomously, working as a cohesive unit. Only at set times would a supervisory controller intervene, telling the rovers to switch to a new behavior. What is described in this paper is a strategy for achieving one behavior for a multi-vehicle system.

Much of the current work involves the use of simple local control strategies in order to achieve a desired global (or group) behavior. One such behavior is the convergence of a group of vehicles (agents) to a common point. This is a type of *agreement problem*, also known as a *rendezvous* or *consensus problem*. Some recent approaches to solving these types of problems include [3–10]. As in [1,4,9], we assume a fixed topology (i.e., a time-invariant sensor graph). Other references, for example [3,5–8,10], consider dynamic topologies. The approaches in [1,3,4] and this paper, are based on a strategy called *cyclic pursuit*, which can be described as follows. A group of n agents, modeled as point masses, are given a number from 1 to n. The position of each of the n agents can be described in the complex plane by the point $z_i = x_i + jy_i$, $i = 1, \dots, n$, where $j = \sqrt{-1}$. The strategy is for agent i to chase agent $i+1$. The i^{th} agent's velocity points in the direction of agent $i+1$ and the magnitude of the velocity is equal to the distance between agent i and $i+1$. The model for cyclic pursuit is given by

$$\begin{aligned} \dot{z}_i &= z_{i+1} - z_i, \quad i = 1, ..., n-1 \\ \dot{z}_n &= z_1 - z_n. \end{aligned} \tag{1}$$

Under this scheme the agents will converge to their stationary centroid.

The scheme above assumes that each agent is equipped with an isotropic sensor with an infinite range. Lin et al. [3], consider sensors with a finite range, and directional sensors which can see agents only within some cone of view. Based on these sensor models, control strategies are developed to ensure convergence to a point. Marshall et al. [4], study a similar pursuit strategy but with wheeled vehicles that are subject to a nonholonomic constraint (kinematic unicycles). For models of this type there are two control inputs, namely the forward and angular velocities. The strategy is to pursue the next agent with linear velocity proportional to the distance to the next agent, and angular velocity proportional to the difference between the desired and actual heading. By appropriate choice of gains on the velocities, the vehicles can either spiral in to a point, converge to a cir-

[1] This paper was not presented at any IFAC meeting. This work was in part supported by the National Sciences and Engineering Research Council of Canada (NSERC).
[2] Corresponding author. Tel.: +1.416.978.4124; fax: +1.416.978.0804
E-mail: {ssmith,broucke,francis}@control.utoronto.ca

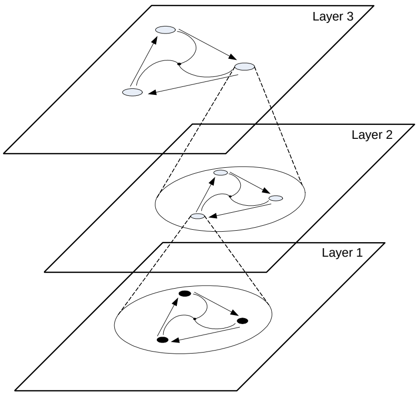

Fig. 1. Three layers of hierarchy in cyclic pursuit.

cle of some radius, or diverge. These are two examples of the application of local control strategies to achieve a global group behavior. There are many other results, and a more complete review of these can be found in [3] and [4].

Williams et al. [11], study the problem of achieving an overall formation with n groups of homogeneous vehicle formations. In each group there is one leader, which is the only vehicle that can communicate to the other groups. By using a simple linear control law, and by modeling the vehicles as point masses, hierarchical formations, such as groups achieving a common heading, are attained.

In this brief paper, the concept of hierarchy is applied to cyclic pursuit. The simplest hierarchical cyclic pursuit scheme, which we call a two layer hierarchical scheme, can be described as follows. A collection of N_2 agents is divided into n_2 groups, each containing n_1 agents ($n_1 \times n_2 = N_2$). The local control strategy is chosen such that the agents within each group are in cyclic pursuit. In addition, the centroid of each group is pursuing the centroid of the next in order (i.e., the centroids are also in cyclic pursuit). This idea can be extended to more layers of hierarchy as shown in Fig. 1. In Sections 4 and 5 this discussion will be formalized, and it will be shown that this scheme yields a very significant increase in the rate of convergence of a group of vehicles to their centroid when compared to traditional cyclic pursuit (1). The hierarchical scheme requires more communication links between agents than the traditional scheme. Because of this, in Section 6 the rate of convergence of the hierarchical scheme is compared to an alternate scheme with an equal number of communication links. It is shown that the hierarchical scheme still yields a significantly greater rate of convergence than this alternate scheme.

2 Background in circulant matrices

In order to proceed we require a few mathematical tools. This section gives a summary of the theory of circulant matrices and is based on [2] by Davis. Consider an n-tuple $(c_1, c_2, \ldots, c_n)$ of real numbers. This n-tuple along with its $n-1$ cyclic permutations can be used to form the rows of the matrix

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_2 & c_3 & \ldots & c_1 \end{bmatrix} =: \operatorname{circ}(c_1, c_2, \ldots, c_n). \tag{2}$$

Let P denote the special $n \times n$ circulant matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} = \operatorname{circ}(0, 1, 0, \ldots, 0). \tag{3}$$

Matrix C can be written in terms of P and the polynomial

$$q_C(s) = c_n s^{n-1} + c_{n-1} s^{n-2} + \cdots + c_2 s + c_1 s^0$$

as $C = q_C(P)$. The matrix P is in companion form and its characteristic polynomial is $s^n - 1$. The eigenvalues of P are the n^{th} roots of unity $(1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega := \mathrm{e}^{2\pi j/n})$. To diagonalize C we define a matrix containing the eigenvalues of P as $\Omega = \operatorname{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1})$. The corresponding eigenvectors can be used to form the columns of the matrix

$$F = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \ldots & \omega^{(n-1)(n-1)} \end{bmatrix}. \tag{4}$$

From the definition of Ω and F we have that $PF = F\Omega$, and $P = F\Omega F^*$. By pre-multiplying $PF = F\Omega$ by P we see that $P^2 F = F\Omega^2$ from which it follows that $P^3 F = F\Omega^3$, $P^4 F = F\Omega^4$, and so on. Therefore, we have that $CF = Fq_C(\Omega)$ or $C = F\Lambda F^*$, where Λ is the diagonal matrix taking the form

$$\Lambda = \operatorname{diag}\left(q_C(1), q_C(\omega), q_C(\omega^2), \ldots, q_C(\omega^{n-1})\right),$$

and '$*$' represents complex conjugate. In this manner, a circulant matrix C can be diagonalized to reveal its eigenvalues.

2.1 Block circulant matrices

Consider the matrix (2) but with each real number entry replaced by an $m \times m$ matrix D_i:

$$D = \mathrm{circ}(D_1, D_2, \ldots, D_n).$$

The matrix D is of dimension $nm \times nm$ and has a block circulant form. This matrix can be written in terms of the $n \times n$ circulant matrix P as

$$D = D_n \otimes P^{n-1} + D_{n-1} \otimes P^{n-2} + \cdots + D_2 \otimes P + D_1 \otimes I,$$

where $\otimes$ is the Kronecker product. The matrix D can be block diagonalized using F, from (4), yielding

$$\Lambda := (F \otimes I_m)^* D (F \otimes I_m),$$

where I_m is the $m \times m$ identity matrix. It can be shown that the block diagonal matrix Λ has the following entries along its diagonal:

$$D_1 + \omega^{i-1} D_2 + \omega^{2(i-1)} D_3 + \cdots + \omega^{(n-1)(i-1)} D_n,$$

for $i = 1, \ldots, n$.

3 Traditional cyclic pursuit

Consider n agents, modeled as point masses, numbered from 1 to n performing cyclic pursuit as described by (1). This system can be written in vector form as

$$\dot{z} = A_1 z. \tag{5}$$

The matrix A_1 can be written as $A_1 = P - I$, where P is given in (3). The eigenvalues of P are the n^{th} roots of unity; thus, the eigenvalues of A_1 are the n^{th} roots of unity shifted left by one. The eigenvector for the zero eigenvalue satisfies $A_1 v = 0$, or $Pv = v$, so all of v's components are equal. For simplicity take them all to be 1. The following theorem results from the above observations.

Theorem 1 (Bruckstein et al. [1]) *Consider the cyclic pursuit scheme in* (1). *For every initial condition, the centroid of the agents $z_1(t), \ldots, z_n(t)$ is stationary and every $z_i(t)$ converges to this centroid.*

The zero eigenvalue of A_1 dictates that the agents converge to their centroid and thus the rate of convergence of the agents is determined by the nonzero eigenvalue with the smallest absolute real part.

Now consider the situation where each agent follows a displacement of the next agent:

$$\begin{aligned} \dot{z}_i &= (z_{i+1} + c_i) - z_i, \quad i = 1, ..., n-1 \\ \dot{z}_n &= (z_1 + c_n) - z_n. \end{aligned}$$

In vector form, this can be written as $\dot{z} = A_1 z + c$. Premultiplying by v^T we obtain $v^T \dot{z} = v^T c$. Since v is a vector of 1's, if we denote the centroid of the agents as $\bar{z}$, we get that

$$\dot{\bar{z}} = \frac{1}{n} \sum_{i=1}^{n} c_i. \tag{6}$$

By properly selecting the c_i's we can create hierarchy within cyclic pursuit.

4 Two layer hierarchy

We will start by looking at the two layer hierarchical scheme, as described in Section 1, where N_2 agents are divided into n_2 groups of n_1 agents ($n_1 \times n_2 = N_2$). Each agent will be described by the subscripts $z_{p,q}$ where $p = 1, \ldots, n_2$ is the group index, and $q = 1, \ldots, n_1$ is the agent index. Therefore, the two layer hierarchy system can be written as,

$$\begin{array}{ll} \text{group } 1 & \left\{ \begin{array}{l} \dot{z}_{1,1} = z_{1,2} - z_{1,1} + d_{1,1} \\ \dot{z}_{1,2} = z_{1,3} - z_{1,2} + d_{1,2} \\ \quad \vdots \\ \dot{z}_{1,n_1} = z_{1,1} - z_{1,n_1} + d_{1,n_1} \end{array} \right\} \\ & \qquad \vdots \\ \text{group } n & \left\{ \begin{array}{l} \dot{z}_{n_2,1} = z_{n_2,2} - z_{n_2,1} + d_{n_2,1} \\ \dot{z}_{n_2,2} = z_{n_2,3} - z_{n_2,2} + d_{n_2,2} \\ \quad \vdots \\ \dot{z}_{n_2,n_1} = z_{n_2,1} - z_{n_2,n_1} + d_{n_2,n_1}. \end{array} \right\} \end{array} \tag{7}$$

where the $d_{p,q}$'s are the displacements. We require the centroids of the groups to be in cyclic pursuit. Therefore, the desired equations of motion for the centroids of each group are

$$\begin{aligned} \dot{\bar{z}}_p &= \bar{z}_{p+1} - \bar{z}_p, \quad p = 1, \ldots, n_2 - 1 \\ \dot{\bar{z}}_{n_2} &= \bar{z}_1 - \bar{z}_{n_2}, \end{aligned} \tag{8}$$

where the centroid of the p^{th} group is defined as

$$\bar{z}_p := \frac{1}{n_1} \sum_{q=1}^{n_1} z_{p,q}.$$

The question is how to choose the displacements, $d_{p,q}$'s, to achieve the motion in the centroids described by (8). From (6), the dynamics of group p's centroid is

$$\dot{\bar{z}}_p = \frac{1}{n_1} \sum_{q=1}^{n_1} d_{p,q}.$$

Combining this with (8) we get that

$$\sum_{q=1}^{n_1} d_{p,q} = n_1(\bar{z}_{p+1} - \bar{z}_p). \quad (9)$$

Looking at (9) and summing over all of the $d_{p,q}$'s we can see that

$$\sum_{p=1}^{n_2}\sum_{q=1}^{n_1} d_{p,q} = n_1 \sum_{p=1}^{n_2} \dot{\bar{z}}_p = 0,$$

and therefore the centroid of the N_2 agents is stationary.

Several different $d_{p,q}$'s can be chosen that will satisfy (9). One such choice is

$$d_{p,q} = z_{p+1,q} - z_{p,q}. \quad (10)$$

This means that the q^{th} agent in the p^{th} group chases the q^{th} agent in the $(p+1)^{th}$ group. By substituting the expression for the $d_{p,q}$'s into (7) it can be seen that each agent has a communication link to two other agents ($z_{p+1,q}$ and $z_{p,q+1}$). Therefore, with this scheme, there is a total of $2N_2$ communication links. This system can be further examined by looking at the vector form

$$\dot{z} = Bz + Dz,$$

where B is the block diagonal matrix describing the cyclic pursuit within the groups, and is given by

$$B = \text{diag}(A_1, \ldots, A_1),\ (n_2 \text{ blocks}),$$

where $A_1 = (P - I)_{n_1 \times n_1}$ as in (5). The eigenvalues of B are n_2 sets of the n_1^{th} roots of unity, shifted left by 1. The matrix D represents the $d_{p,q}$'s and has the form

$$\begin{aligned} D &= \text{circ}(-1, 1, 0, \ldots, 0)_{n_2 \times n_2} \otimes I_{n_1} \\ &= (P - I)_{n_2 \times n_2} \otimes I_{n_1}. \end{aligned} \quad (11)$$

The matrix $I_{n_1} =: S$ in (11) represents the sensor connections of each agent in one group to the agents in the next group. A '1' in the fg^{th} position of the S matrix, $f, g = 1, \ldots, n_1$, indicates that the f^{th} agent in each group senses the g^{th} agent in the next group (modulo n_2). Therefore, $S = I$ indicates that the f^{th} agent of each group sees the f^{th} agent in the next group.

If we compute $A_2 := B + D$, it has the block circulant structure (with each block being of size $n_1 \times n_1$)

$$A_2 = \text{circ}(A_1 - I, I, 0, \ldots, 0)_{N_2 \times N_2}. \quad (12)$$

This matrix can be block diagonalized to obtain the following matrices along the diagonal:

$$(A_1 - I) + \omega^{r-1} I, \quad r = 1, \ldots, n_2,$$

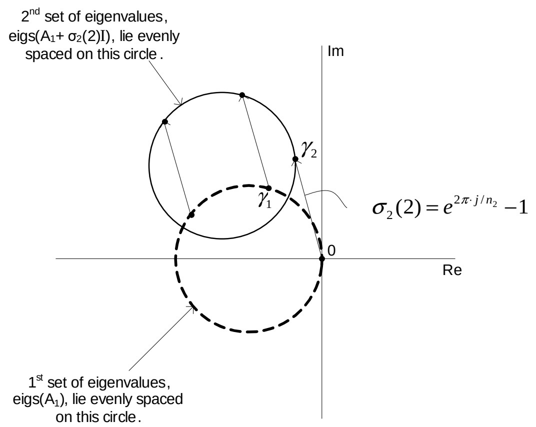

Fig. 2. Finding the γ-value of A_2.

where $\omega = \mathrm{e}^{2\pi j/n_2}$. The eigenvalues of A_2 are the union of the eigenvalues of these matrices. That is, we have n_2 sets of n_1 eigenvalues, the r^{th} set being comprised of the eigenvalues of A_1 shifted by $\omega^{r-1} - 1$. This can be written more compactly as

$$\text{eigs}(A_2) = \bigcup_{r=1}^{n_2} \text{eigs}\left(A_1 + (\mathrm{e}^{2\pi j(r-1)/n_2} - 1)I\right). \quad (13)$$

4.1 Rate of convergence to the centroid

By examining (13) it can be seen that A_2 has one eigenvalue at zero and all others lie in the open left half-plane. The matrices on the right hand side of (13) are circulant, thus when A_2 is block diagonalized, each block is circulant. But since the blocks are circulant, they can be further diagonalized, thereby fully diagonalizing A_2. Therefore, the zero eigenvalue of A_2 dictates that the agents converge to their centroid rather than to the origin. The rate of convergence of the agents to their centroid is determined by the nonzero eigenvalue of A_2 with the smallest absolute real part. In order to simplify the subsequent discussion the following definition is introduced.

Definition 2 *The* γ-value *of a set of eigenvalues which lie in the left half-plane is the nonzero eigenvalue with the smallest absolute real part.*

The eigenvalues of A_1 can be written as

$$\lambda_p = \mathrm{e}^{2\pi j(p-1)/n_1} \quad p = 1, \ldots, n_1.$$

If we define $\sigma_2(r) := \mathrm{e}^{2\pi j\cdot(r-1)/n_2} - 1$, the real part takes values in the range

$$-2 \leq \Re(\sigma_2(r)) \leq 0 \quad \forall r = 1, \ldots, n_2, \quad (14)$$

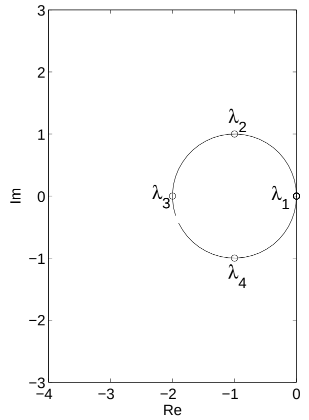

Fig. 3. The eigenvalues of A_1 for $n_1 = 4$.

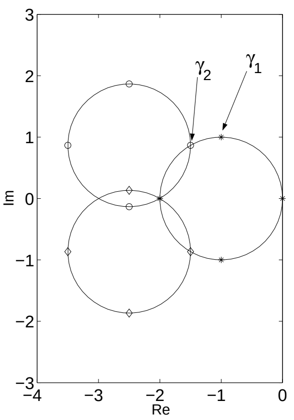

Fig. 4. Eigenvalue structure for $n_1 = 4$, and $n_2 = 3$.

and (13) can be written as

$$\text{eigs}(A_2) = \bigcup_{r=1}^{n_2} \text{eigs}\left(A_1 + \sigma_2(r)I\right).$$

Now, looking for the γ-value of eigs(A_2), we know from (14) that $\sigma_2(r)$ shifts the r^{th} set of eigenvalues of A_1 to the left by some amount between 0 and 2. In order to find the γ-value of eigs(A_2) we need to find the set of eigenvalues which is shifted by the smallest amount to the left, and then find the γ-value of that set.

The set of eigenvalues of A_1 that is not shifted at all is the first ($r = 1$) set which has the shift $\sigma_2(1) = 0$. The eigenvalues of this set are simply the eigenvalues of A_1. The rightmost eigenvalue of A_1 lies at zero ($\lambda_1 = 0$), and thus the next eigenvalue to the left of 0 provides the γ-value, which is $\lambda_2 = \mathrm{e}^{2\pi j/n_1} - 1 =: \gamma_1$ (or equivalently λ_{n_1}). The eigenvalues of A_1 are shown in Fig. 3.

The set of eigenvalues which is shifted to the left by the next smallest amount is given by $\sigma_2(2)$ (or equivalently $\sigma_2(n_2)$). The rightmost eigenvalue of this set is given by $\lambda_1 + \sigma_2(2) = \mathrm{e}^{2\pi j/n_2} - 1$, as shown in Fig. 2. Since this eigenvalue is nonzero it is the γ-value of the set: $\gamma_2 := \mathrm{e}^{2\pi j/n_2} - 1$. Both γ_1 and γ_2 are shown in Fig. 4 for $n_1 = 4$ and $n_2 = 3$.

The question arises, which eigenvalue has a smaller absolute real part: γ_1 or γ_2? We have

$$\Re(\gamma_1) = \cos(2\pi/n_1) - 1 \text{ and } \Re(\gamma_2) = \cos(2\pi/n_2) - 1.$$

Therefore, the γ-value is given by γ_1 if $n_1 \geq n_2$ and by γ_2 if $n_1 \leq n_2$. The real part of the γ-value of eigs(A_2) can be written as $\chi := \cos(2\pi/w) - 1$, where $w = \max\{n_1, n_2\}$. In comparing the real part of the γ-value for the two cyclic pursuit schemes, hierarchical and traditional, we get that the increase in the rate of convergence to the centroid when using hierarchical cyclic pursuit is

$$\frac{\text{hierarchical}}{\text{traditional}} = \frac{\cos(2\pi/w) - 1}{\cos(2\pi/N_2) - 1}.$$

Expanding the cos terms to the first order and using the fact that $N_2 = n_1 \times n_2$, and $w = \max\{n_1, n_2\}$, we arrive at the following theorem.

Theorem 3 (Two layers of hierarchy) *Suppose we have two layers of hierarchy in cyclic pursuit, where N_2 agents are divided into n_2 groups, with each group containing n_1 agents. Then the increase in the rate of convergence of the two layer hierarchy scheme, when compared to N_2 agents in traditional cyclic pursuit, is approximated by:*

$$R := \frac{\textit{hierarchical}}{\textit{traditional}} \approx \min\{n_1, n_2\}^2 =: R_2.$$

As the total number of agents becomes large ($N_2 \to \infty$), $R/R_2 \to 1$.

Thus in the special case where $n_1 = n_2 = \sqrt{N_2}$, the N_2 agents will convergence to their centroid approximately N_2 times faster using the hierarchy scheme than using traditional cyclic pursuit.

5 The generalized scheme

In the most general setting, we have L layers of hierarchy (in the previous section we had 2 layers). We call the layer consisting of n_1 agents the first layer. The second layer then consists of n_2 subgroups of n_1 agents, the third layer, n_3 groups of n_2 subgroups of n_1 agents, and so on. In total we have N_L agents, where

$$\prod_{m=1}^{L} n_m = N_L. \tag{15}$$

We would like to write this system in the form

$$\dot{z} = A_L z, \tag{16}$$

where z is a column vector of length N_L and A_L is an N_L by N_L matrix. For $L = 1$, we have $N_L = N_1 = n_1$ and the A_1 matrix is $A_1 = (P - I)_{n_1 \times n_1}$. Each time we add a new layer we would like the behavior in the layer below to remain the same. For example, with one layer, we have traditional cyclic pursuit. When we add another layer and have several groups, we would still like the agents within each individual group to be in cyclic pursuit. We then add sensor connections between each of the groups to achieve cyclic pursuit at the new level (between the centroids of the groups). In looking at the A_2 matrix in (12) this becomes evident. The A_1 matrices along the diagonal represent the cyclic pursuit within each group. The I's along the off-diagonal and $-I$'s along the diagonal represent the sensor connections between groups to create the new layer of hierarchy; each agent in a group takes the position of an agent in the next group, minus its own position (as described in (10)), to create the new layer.

Lemma 4 *An L layer hierarchy scheme can be put into the form of* (16). *The matrix A_L is given by the recursive expression*

$$\begin{aligned} A_1 &= \mathrm{circ}(-1, 1, 0, \ldots, 0) \\ A_m &= \mathrm{circ}(A_{m-1} - I, I, 0, \ldots, 0), \quad m = 2, \ldots, L \end{aligned}$$

where A_m is composed of n_m blocks of dimension $N_{m-1} \times N_{m-1}$. The eigenvalues of A_m are given by

$$\mathrm{eigs}(A_m) = \bigcup_{r=1}^{n_m} \mathrm{eigs}\left(A_{m-1} + (\mathrm{e}^{2\pi j(r-1)/n_m} - 1)I\right).$$

The proof of this lemma will be omitted due to space constraints. In order to determine the rate of convergence of the agents to their centroid, the γ-value of $\mathrm{eigs}(A_L)$ must be determined. The process for determining this value is simply an extension of the process used for the two layer case. An example is shown in Fig. 5.

Theorem 5 *The γ-value of* $\mathrm{eigs}(A_L)$ *is given by*

$$\gamma := \mathrm{e}^{2\pi j/w} - 1 \qquad w = \max_m\{n_m\}, \quad m = 1, \ldots, L.$$

The increase in the rate of convergence of this scheme when compared to N_L agents in traditional cyclic pursuit is approximated by

$$R \approx \left(\frac{N_L}{\max_m\{n_m\}}\right)^2 =: R_L. \qquad (17)$$

As the number of agents becomes very large ($N_L \to \infty$), $R/R_L \to 1$.

In order to determine the highest rate of convergence for the generalized scheme we introduce the following definition.

Definition 6 *Given N_L agents in an L layer hierarchy, a distribution of n_m's satisfying* (15) *which yields the highest rate of convergence is an* optimal distribution.

Theorem 7 *In the case where $\sqrt[L]{N_L}$ is an integer, the uniform distribution of n_m's*

$$n_1 = n_2 = \cdots = n_L = \sqrt[L]{N_L}, \qquad (18)$$

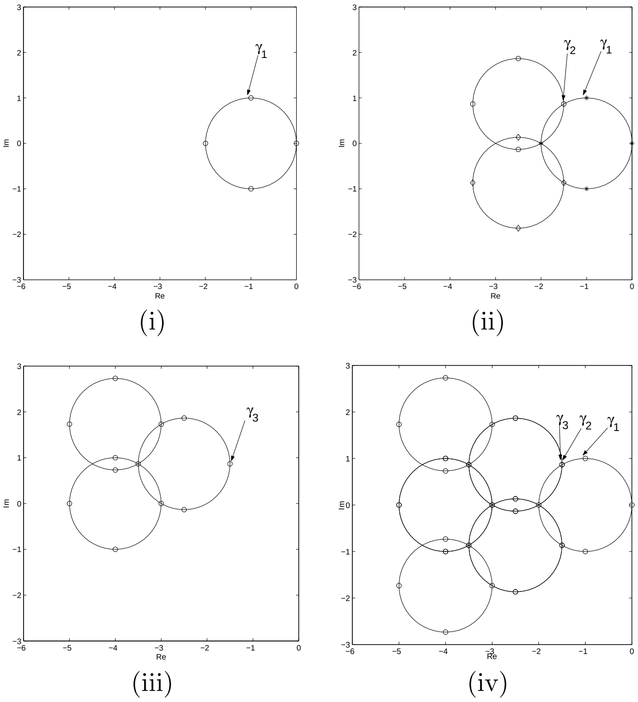

Fig. 5. Determining the possible γ-values for $\mathrm{eigs}(A_3)$. (i) The eigenvalues of A_1. (ii) The eigenvalues of A_2. (iii) The eigenvalues of A_2 shifted by $\sigma_3(2) = \mathrm{e}^{2\pi j/3} - 1$. (iv) The eigenvalues of A_3. For this distribution of agents, $n_3 = 3$ groups of $n_2 = 3$ subgroups of $n_1 = 4$ agents (total of 36 agents) the γ-value is γ_1.

which yields an increase in the rate of convergence of

$$R_L = N_L^{2(L-1)/L}, \qquad (19)$$

is an optimal distribution. Moreover, it is the only optimal distribution.

PROOF. First we show that distribution (18) is optimal. Since the numerator of (17) is a constant, the highest rate of convergence is obtained when the denominator is minimized. Therefore, an optimal distribution is one which minimizes the maximum n_m. Suppose there exists a distribution which yields a rate of convergence greater than (19). This implies there exists a distribution $\{n_m\}$, $m = 1, \ldots, L$, such that

$$\max_m\{n_m\} =: M < \sqrt[L]{N_L} \quad \text{and} \quad \prod_{m=1}^{L} n_m = N_L.$$

Thus, $M \geq n_m$ for all m. But then $M^L \geq \prod_{m=1}^{L} n_m = N_L$ which is a contradiction, since we assumed $M < \sqrt[L]{N_L}$. Therefore $\max_m\{n_m\} \geq \sqrt[L]{N_L}$ and $R_L \leq N_L^{2(L-1)/L}$ for all distributions, and thus (18) is an optimal distribution.

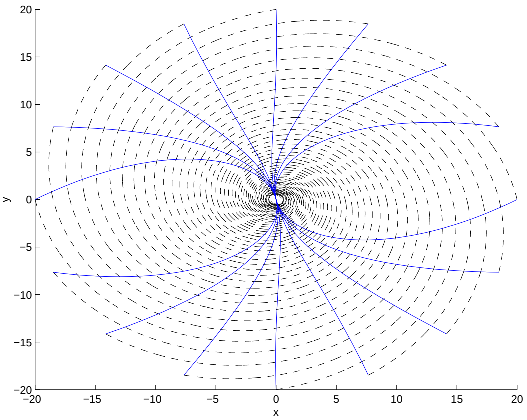

Fig. 6. Trajectories for 16 agents in traditional cyclic pursuit (dashed line) and 16 agents with 4 layers of hierarchy (solid line).

Now suppose a distribution which is not identical to (18) is also optimal. This implies there exists a distribution, $\{n_m\}$, $m = 1, \dots, L$, where $n_{m_1} > n_{m_2}$ for some $m_1, m_2 \in \{1, \dots, L\}$, such that

$$\max_m \{n_m\} = M = \sqrt[L]{N_L} \quad \text{and} \quad \prod_{m=1}^{L} n_m = N_L.$$

Thus, $M \geq n_m$ for all m, with $M > n_{m_2}$. But then $M^L > \prod_{m=1}^{L} n_m = N_L$, which is a contradiction since we assumed $M \leq \sqrt[L]{N_L}$. Therefore, for any distribution not identical to (18), $\max_m\{n_m\} > \sqrt[L]{N_L}$, and $R_L < N_L^{2(L-1)/L}$, which is not optimal. Therefore, (18) is the optimal distribution. □

When $\sqrt[L]{N_L}$ is not an integer there may be multiple optimal distributions. For example, if $N_L = 12$ and $L = 2$, there are two optimal distributions, $\{n_1, n_2\} = \{3, 4\}$ and $\{n_1, n_2\} = \{4, 3\}$, since they both yield the highest rate of convergence.

In Fig. 6 trajectories are shown for 16 agents in traditional cyclic pursuit and 16 agents with $L = 4$ ($n_1 = n_2 = n_3 = n_4 = 2$).

6 A new comparison

We have obtained a significant increase in the rate of convergence of a group of agents to their centroid when comparing hierarchical cyclic pursuit to the traditional cyclic pursuit scheme. However, the hierarchical scheme has more communication; each agent sees more than one other agent, whereas in the traditional scheme each agent only sees one other agent. Because of this, a rate of convergence comparison will now be performed between the hierarchy scheme and a scheme in which each agent chases the centroid of a group of agents.

In an L layer hierarchy scheme, each agent has a communication link to L other agents. If there is a total of N_L agents, then the entire system consists of LN_L communication links. Now, consider another scheme involving a group of N_L agents. The strategy in this scheme is that agent i chases the centroid of agents $i+1$ to $i+L$ modulo N_L. This can be written as

$$\dot{z}_i = \frac{1}{L} \sum_{m=1}^{L} z_{i+m \ (\text{mod}\ N_L)} - z_i \quad i = 1, \dots, N_L. \qquad (20)$$

This scheme has the same number of total communication links as an L layer hierarchy scheme (i.e., there is a total of LN_L communication links).The system (20) can be written in the vector form as

$$\dot{z} = Az,$$

where A is the circulant matrix given by

$$A = \frac{1}{L} \text{circ}(-L, \underbrace{1, \dots, 1}_{L \text{ ones}}, 0, \dots, 0).$$

Matrix A can be written in terms of the matrix P and the polynomial

$$q_C(s) = \frac{1}{L}s^L + \frac{1}{L}s^{L-1} + \dots + \frac{1}{L}s - s^0,$$

as $A = q_C(P)$. The eigenvalues of A are given by

$$\text{eigs}(A) = \{q_C(1), q_C(\omega), q_C(\omega^2), \dots, q_C(\omega^{N_L-1})\},$$

where $\omega = \mathrm{e}^{2\pi j/N_L}$.

Lemma 8 *The matrix A has one eigenvalue at zero, and all others lie in the open left half-plane. If N_L is sufficiently large when compared to L, the γ-value of* eigs(A) *is given by $\gamma := q_C(\omega)$.*

The reason for the condition that N_L be sufficiently large in comparison to L can be better understood by looking at Fig. 7. In (i) and (ii) the rightmost nonzero eigenvalue is clearly given by γ, which is the first eigenvalue arrived upon when following the curve counterclockwise from the zero eigenvalue. However, when looking at (iii) all four nonzero eigenvalues have same real part, and in (iv) γ is no longer the rightmost nonzero eigenvalue. Therefore, only when N_L is sufficiently large in comparison to L, will γ be the γ-value. However, when performing the comparison between this scheme and the hierarchical scheme it is the limiting case when $N_L \to \infty$ that is of interest and in this case it is clear that the γ-value of A is given by γ. The real part of γ can be written as

$$\Re(\gamma) = \frac{1}{L} \sum_{m=1}^{L} (\cos(2m\pi/N_L)) - 1,$$

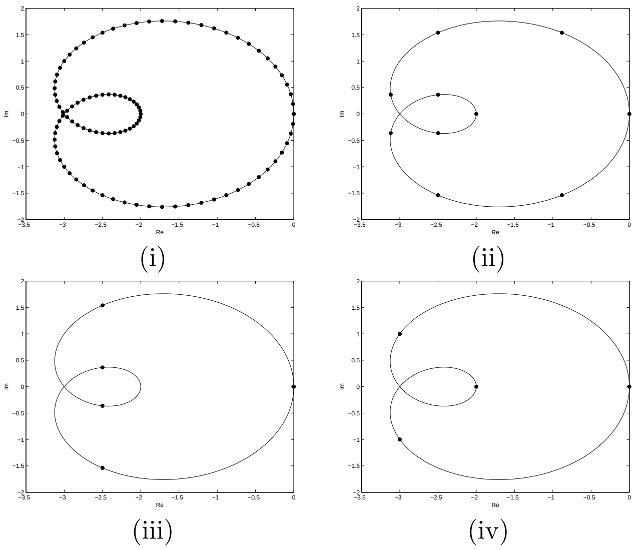

Fig. 7. (i) Eigenvalues of A with $N_L = 100$ and $L = 2$. (ii) Eigenvalues of A with $N_L = 10$ and $L = 2$. (iii) Eigenvalues of A with $N_L = 5$ and $L = 2$. (iv) Eigenvalues of A with $N_L = 4$ and $L = 2$. The eigenvalue $q_C(\omega)$ is no longer the γ-value, it is now given by $q_C(\omega^2)$.

Table 1
Comparing the rate of convergence of an $L = 3$ hierarchy scheme to the traditional and 3-link schemes.

Number of agents	hierarchy/trad.	hierarchy/3-link
$3^3 = 27$	56	12
$4^3 = 64$	208	45
$10^3 = 1000$	9675	2075

from which the following result is obtained:

Theorem 9 *In comparing the L layer hierarchy scheme, which has a total of LN_L communication links, to the L-link scheme which has an equal amount of communication, the increase in the rate of convergence is approximated by*

$$\frac{hierarchical}{L - link} \approx \left(\frac{L}{\sum_{m=1}^{L} m^2} \right) \frac{N_L^2}{w^2},$$

where $w = \max\{n_1, n_2, \ldots, n_L\}$. As $N_L \to \infty$ this approximation approaches an equality.

Table 1 shows some comparisons between the different schemes for $L = 3$. The L layer hierarchical scheme has a much greater rate of convergence than the L-link scheme. The trajectories of 16 agents in 4-link pursuit and in hierarchical pursuit with $L = 4$ ($n_1 = n_2 = n_3 = n_4 = 2$) are shown in Fig. 8.

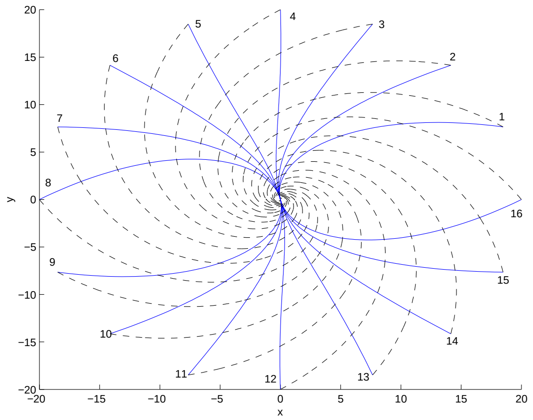

Fig. 8. Trajectories for 16 agents in 4-link cyclic pursuit (dashed line) and 16 agents with 4 layers of hierarchy (solid line).

7 Conclusions

In this brief paper a hierarchical cyclic pursuit scheme has been introduced. It has been shown that this scheme yields a higher rate of convergence of a group of vehicles to their centroid than either the traditional cyclic pursuit or the equal communication scheme. There are several areas for future work including extending this scheme to the three dimensional case, or to networks of wheeled vehicles. In addition, it would be nice to consider a more robust version of this scheme which accounts for errors in measuring the distances between agents.

References

[1] A. M. Bruckstein, N. Cohen, and A. Efrat. Ants, crickets and frogs in cyclic pursuit. *CIS Report No. 9105, Technion, IIT, Haifa, Israel*, July 1991.

[2] P. J. Davis. *Circulant Matrices*. Chelsea Publishing, New York, 2nd edition, 1979.

[3] Z. Lin, M. Broucke, and B. Francis. Local control strategies for groups of mobile autonomous agents. *IEEE Trans. on Automatic Control*, 49(4):622–629, 2004.

[4] J.A. Marshall, M.E. Broucke, and B.A. Francis. A pursuit strategy for wheeled-vehicle formations. In *Proc. of the 42nd IEEE Conference on Decision and Control*, volume 3, pages 2555–2560, December 2003.

[5] L. Moreau. Leaderless coordination via bidirectional and unidirectional time-dependent communication. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 3070–3075, December 2003.

[6] R. Olfati-Saber and R. Murray. Agreement problems in networks with directed graphs and switching topology. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 4126–4132, December 2003.

[7] R. Olfati-Saber and R. Murray. Consensus protocols for networks of dynamic agents. In *Proc. of the American Control Conference*, pages 951–956, July 2003.

[8] W. Ren and R. Beard. Consensus of information under dynamically changing interaction topologies. In *Proc. of the American Control Conference*, pages 4939–4944, July 2004.

[9] H. Tanner, A. Jadbabaie, and G. Pappas. Stable flocking of mobile agents, part i: Fixed topology. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 2010–2015, December 2003.

[10] H. Tanner, A. Jadbabaie, and G. Pappas. Stable flocking of mobile agents, part ii: Dynamic topology. In *Proc. of the 42nd IEEE Conference on Decision and Control*, pages 2016–2021, December 2003.

[11] A. Williams, S. Glavaski, and Tariq Samad. Formations of formations: Hierarchy and stability. In *Proc. of the American Control Conference*, pages 2992–2997, July 2004.