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Vision article

Learning and forgetting in systems neuroscience: A control perspective ☆

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ABSTRACT

A longstanding open problem of systems neuroscience is to understand how the brain calibrates thousands of reflexes to achieve near instantaneous disturbance rejection. While reflexes typically act locally at the site of sensory measurements, the adaptation of reflex gains is the result of an ingenious architecture in which knowledge of disturbances is transferred from the cerebellum to the deep cerebellar nuclei or the brainstem. This paper investigates the use of control theory as the mathematical foundation to explain the mechanisms by which such forms of learning, as well as forgetting, manifest themselves in systems neuroscience. Particularly, we use adaptive control and averaging theory to model the computations performed in learning appropriate reflex gains. While forgetting is perceived as counter-productive to learning, we show that if incorporated correctly, it can endow the much needed robustness to train thousands of reflexes without interfering with their adaptation. This is accomplished using the μ-modification which achieves robustness of adaptive schemes through the estimation of exciting subspaces. Our techniques are combined in a comprehensive model, with simulations illustrating their effectiveness.

1. Introduction

This paper regards certain forms of learning and forgetting in the brain; however, given the current state of research this statement probably tells the reader very little. The term "learning", in particular, conjures a wealth of interpretations: *machine learning, unsupervised learning, supervised learning, reinforcement learning, associative learning, deep learning*, to name just a few. To be more precise, in systems neuroscience it has been proposed that unsupervised learning is the domain of the *cerebral cortex*; supervised learning is the domain of the *cerebellum*; and reinforcement learning is the domain of the *basal ganglia* (Caligiore et al., 2017; Doya, 1999). Using this anatomical classification, we can say that this paper regards a form of learning in the cerebellum that is more closely associated with *adaptive control* and *output regulation*, two mainstays of control theory. More particularly, we study a form of learning in which an adaptive brain process in the cerebellum trains another adaptive process (Broussard & Kassardjian, 2004; Lee et al., 2015; Shutoh, Ohki, Kitazawa, Itohara, & Nagao, 2006; Yamazaki, Nagao, Lennon, & Tanaka, 2015).

The term "forgetting" has fewer connotations, yet still arises in several contexts. It is commonly used in the *visuomotor adaptation* literature to refer to an adaptive process in which a re-alignment between what is seen and how to move is gradually forgotten after the removal of a visual perturbation (such as magnifying lenses or artificially rotated vision) (Smith, Ghazizadeh, & Shadmehr, 2006). The term also has connections to *neuroplasticity*, where the *Use It or Lose It Principle* says that neural circuits not actively engaged over an extended period begin to degrade (Kleim & Jones, 2008). Finally, in the control literature our interpretation of forgetting is most directly related to the σ-modification of adaptive control to robustify parameter adaptation laws (Ioannou & Sun, 2012). The σ-modification itself finds its analogue in neuroscience by way of the *Oja rule*, a modification of *Hebb's rule* of synaptic plasticity of a neuron to include a forgetting or leakage term with a variable (regressor-dependent) weight (Dayan & Abbott, 2001; Oja, 1982).

This paper is concerned with modeling adaptive processes associated with the cerebellum, so we begin in the next section to review its most salient facts.

1.1. The cerebellum

Cerebellum-like structures appeared in aquatic vertebrates such as lampreys roughly 500 million years ago, while the cerebellum proper was first present in sharks roughly 400 million years ago (Montgomery & Bodznick, 2016). Its role was likely to assist in tracking prey using the shark's electrosensory system, while also suppressing disturbances

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induced by the shark's own rhythmic movements, namely tail movement and breathing. Over 400 million years, all vertebrates possess a cerebellar design close to that of sharks (Bell, 2002; Montgomery & Bodznick, 2016), while the cerebellum is larger in size in many species, with the extreme case of the elephant cerebellum that contains 97.5 percent of the neurons in the elephant brain Herculano-Houzel et al. (2014).

Early anatomical work showed that the cerebellum is made up of relatively few neuron types (Eccles, Ito, & Szentagothai, 1967); its remarkable structure may be summarized thus (Ramnani, 2006):

(i) a regular and simple cellular organization repeated throughout the cerebellar cortex;
(ii) a purely feedforward architecture (in striking contrast with the recurrent architecture of the cerebral cortex);
(iv) global connectivity with the rest of the brain through highly organized cerebellar loops, including the *cerebello-thalamo-cortical* pathway and the *cortico-ponto-cerebellar* pathway on the output and input sides, respectively, of the cerebellum (analogous pathways for the oculomotor system connect with the *vestibular nuclei* in the brainstem);
(iv) extraordinary information processing capabilities provided by approximately 50 billion neurons — roughly half the total number of neurons in the brain.

Within an individual, the *cerebellar neural circuit* is duplicated many times over forming functionally equivalent *modules* (Apps et al., 2018; Oscarsson, 1979), while at the macroscopic level, cerebellar modules are organized into 10 cerebellar *lobules*. It is estimated there are 4000 functional modules in the mouse cerebellum (Hawkes, 1997), while in non-human primates and humans the total number is still not known. All of these matters are of extreme interest to control theorists because the cerebellum is responsible for regulation of all *behaviors of precision*.

The foundation of our work is the intriguing hypothesis, inspired by experimental evidence (Cerminara, Apps, & Marple-Horvat, 2009; Lisberger, 2009), that the cerebellum contains adaptive internal models to fulfill the *internal model principle* of control theory (Francis & Wonham, 1975, 1976). The question that must be asked is: how far can one go to model behaviors associated with the cerebellum using the internal model principle? Thus far, it has allowed us to distill in control theoretic terms a number of behaviors associated with the *flocculus* and *nodulus/uvula*, two cerebellar lobules responsible for eye movement regulation (Broucke, 2020, 2021a, 2022). Also we have made preliminary investigations to understand sensorimotor adaptation in terms of the internal model principle (Broucke, 2021b; Gawad & Broucke, 2020; Hafez, Uzeda, & Broucke, 2021). This article continues our pursuit of the question by examining how adaptive internal models sub-serving the internal model principle can be used to train reflexes of the body.

The hypothesis that the cerebellum contains internal models is not new. Particularly two types of internal models have been studied in neuroscience. A so-called *forward model* (in control terminology, an *observer*) uses an *efference copy* of the motor command as input and generates an estimate of the state of the part of the body being regulated (Jordan & Rumelhart, 1992). An *inverse model* reverses this process: it takes a desired reference signal and generates the ideal control input to track that reference signal. There is broad consensus among neuroscientists that either forward or inverse models or both reside in the cerebellum (Gomi & Kawato, 1992; Ito, 2005; Kawato, 1999; Kawato & Gomi, 1992; Miall & Wolpert, 1996; Porrill, Dean, & Anderson, 2013; Ramnani, 2006; Wolpert & Kawato, 1998; Wolpert, Miall, & Kawato, 1998).

While forward and inverse models may not directly reference the internal model principle, it is not uncommon to find statements in the neuroscience literature proposing "internal models of the environment" (Doya, 1999); see for example the first proposal on the function of the flocculus in Lisberger (2009). In a related vein, adaptive filters performing disturbance rejection have been proposed to reside in the cerebellum to suppress noise and compensate for time delays (Anderson et al., 2012; Dean, Porrill, & Stone, 2002; Porrill et al., 2013).

Despite the promise of the internal model principle and more generally internal models to explain the function of the cerebellum, there is still intense debate on what exactly the cerebellum is doing. The challenge is that the cerebellum is involved in such a diverse array of regulatory functions, it is difficult to see how all its functions can be fit into the confines of one theory (Caligiore et al., 2017; D'Angelo & Casali, 2013; Manto et al., 2012). The contribution of the cerebellum to the motor systems is well-known: sensorimotor adaptation, control of locomotion, arm movement, balance, etc. But it is also involved in speech regulation, emotion regulation, self-motion perception, parcellating endogenous and exogenous sensory signals, precise timing of sensory events, perception of the motion of objects, and other cognitive functions. Whether these cognitive functions can be explained using the internal model principle is of extraordinary interest.

In the next section we review some salient facts about the body's reflexes, which are known to be trained by the cerebellum.

1.2. Reflexes

A *reflex* is a rapid, involuntary response to environmental disturbances based on sensory measurements of the body. In control theoretic terms, reflexes are simply feedforward control inputs. The key advantage of reflexes is their low latency. *Short-latency reflexes* have delays in the range of 20–45 ms, while *long-latency reflexes* have delays in the range of 50–100 ms (Kurtzer, 2015), making reflexes the fastest acting responses of the body to environmental disturbances. The prevalence of *feedback* in control design belies the fact that instead *feedforward* reflexes appear to perform the bulk of the work of disturbance rejection. Understanding how reflexes work in concert to perform complex functions such as locomotion has the potential, in our view, to revolutionize robotics (Narkhede & Hazarika, 2018; Ramos & Kim, 2019; Tieck et al., 2020).

Familiar examples of basic reflexes are the *eye blink reflex*, present in most vertebrates; the *vestibuloocular reflex* (VOR) that drives the eye position opposite to head movement to stabilize the retinal image; the *optokinetic reflex* to drive the eye velocity in the same direction and speed as the entire visual field; and the *stretch reflexes* to maintain postural stability. The stretch reflexes provide a case study in the degree of redundancy of the reflex control architecture of the body. For stabilization of the head, there are at least four reflexes at play. The *cervico-collic reflex* (CCR), driven by neck proprioceptive measurements, maintains stability of the head with respect to disturbances from trunk movement. The *cervico-spinal reflex* (CSR), also driven by neck muscle priopriocеption, stabilizes the trunk relative to disturbances from head movement. The *vestibulo-collic reflex* (VCR) and the *vestibulo-spinal reflex* (VSR) perform the analogous functions as the CCR and CSR, but they are driven by sensory measurements of head movements provided by the ears.

To appreciate the necessity of reflexes for robust control, we can take the example of the VOR. Its purpose is to assist in stabilizing retinal images during head movement (Robinson, 1981). Head velocity is detected in the brainstem via signals from the *semicircular canals* of the ears. The VOR uses this measurement to generate an eye movement command that drives the eyes exactly opposite to the head movement, effectively canceling the resulting movement of images on the retina (such movement is called *retinal slip*). Human subjects with an impaired VOR are unable to read signs while walking because the impulsive disturbance induced by one's footfalls causes blurred vision when not compensated by the reflex (Leigh & Zee, 2015). There is no other control mechanism fast enough to cancel this brief disturbance except the reflex. The relevance of reflexes for robust control design in the context of systems biology is discussed in Nahahira (2021), Sarma et al. (2022), Stenberg et al. (2022).

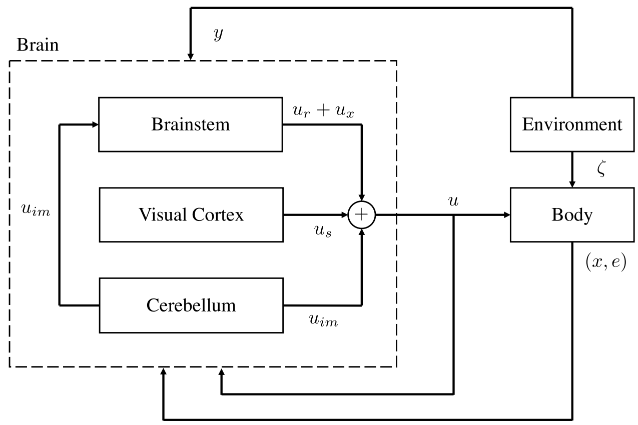


Fig. 1. Block diagram of the connections between various brain regions in the oculomotor system.

Of particular interest to us is the fact that all reflexes are subject to a process of *long-term adaptation* of the *reflex gain* (MacKay & Murphy, 1979). Moreover, this adaptive process ceases to function without the cerebellum, despite the fact that the cerebellum is not directly involved in the execution of reflexes (Anzai, Kitazawa, & Nagao, 2010; Boyden, Katoh, & Raymond, 2004; Broussard & Kassardjian, 2004; Herzfeld, Hall, Tringides, & Lisberger, 2020; Kassardjian et al., 2005; Lisberger & Pavelko, 1986; Miles & Lisberger, 1981; Nagao, Honda, & Yamazaki, 2013; Shutoh et al., 2006). Our interest is to develop a control architecture that can explain how the cerebellum trains reflexes (over a long timescale) to have appropriately modulated amplitudes.

1.3. Scope and aims

The connection between adaptive control and the cerebellum is well-documented; witness the research monograph (Barlow, 2002) reviewing this extensive subject. Similarly, it is well-established that the cerebellum is involved in adaptation of reflexes. This raises a fundamental question: is there anything new to be said about these subjects? The answer is simple: until very recently, neither the cerebellum nor its adaptation of reflexes has been examined under the lens of regulator theory. One factor is that the forward model theory of the cerebellum has dominated for the last 40 years, creating a research climate in which there is reduced incentive to explore outlying theories. Second, regulator theory has been overlooked both by control theorists and by neuroscientists as a suitable mathematical foundation to explain cerebellar function until recently (Huang et al., 2018). As such, one will not find any mathematical model of the cerebellum or of reflex adaptation that is built up from adaptive regulator theory, except in our work.

Whether one decides to take on the challenge of applying regulator theory in systems neuroscience is to some extent a question of timing. With regard to the cerebellum, there are two challenges. First, the experimental record for the cerebellum, while vast, is incomplete. Recording from neurons in the human cerebellum is practically infeasible, while similar work with non-human primates is difficult and costly. Non-invasive brain stimulation is an increasingly popular therapeutic and experimental method, but it is inaccurate and experimental results can be non-reproducible (Jalali, Miall, & Galea, 2017). To date, only the oculomotor system has a sufficiently complete experimental record to allow mathematical modeling to proceed (Leigh & Zee, 2015).

The second challenge is that control theory is not sufficiently developed to address modeling problems in systems neuroscience. Witness that the area of adaptive internal models is still under development with many control problems remaining to be solved. The aim of our research has been to contend with the second issue, with a focus on opportunities for new developments in regulator theory and adaptive control.

The paper is addressed to control theorists who are interested in control problems relevant to systems neuroscience. For graduate students looking for new research areas in control, Section 2 provides a rapid overview of the major theoretical tools. Such readers can then check our references for a deeper study of each of the sub-areas of control that we use. Those readers who are already experts in output regulation can skim Section 2 to become familiar with the notation, but the material will mostly be a tutorial.

We expect that most neuroscientists would not be familiar with recent methods of output regulation and adaptive control; however, a neuroscientist working on the cerebellum will immediately recognize that we tackle one of the central problems of the area: how the cerebellum trains the reflexes of the body. We have sprinkled remarks (with the (∗) symbol) about neuroscience connections throughout the paper for the benefit of such readers. Also, Section 6.5 summarizes the biological plausibility of the obtained model and areas of ambiguity for further investigations.

This paper may be read as a theoretically oriented consolidation of our earlier papers (Battle & Broucke, 2021; Broucke, 2020, 2021a, 2021b, 2022; Gawad & Broucke, 2020; Hafez et al., 2021) and as a companion to our concurrent work (Hafez, Uzeda, & Broucke, 2023; Mejia Uzeda & Broucke, 2023; Uzeda & Broucke, 2022, 2023). The major split between the earlier papers and the concurrent work regards two issues. First, the former papers developed ideas about cerebellar function using adaptive internal models without including any long-term adaptation of reflexes; namely all reflex gains in the earlier models were constants. Second, any regressors used in adaptation processes in the earlier models were always assumed to be persistently exciting to ease the analysis.

Our more recent papers (Uzeda & Broucke, 2022, 2023) develop the *μ-modification* which is the underpinning of our "forgetting" method, with (Uzeda & Broucke, 2022) particularly providing a more in depth analysis of subspace estimation methods. The papers (Hafez et al., 2023; Mejia Uzeda & Broucke, 2023) develop the learning problem presented here. Particularly, Hafez et al. (2023) explores the use of state feedback for disturbance rejection in the context of *optimal steady-state control* (Jokic, Lazar, & vanden Bosch, 2009). The paper (Mejia Uzeda & Broucke, 2023) uses linear regulator theory to introduce the idea of adapting reflexes to improve robustness and manage long-term energy expenditure, especially compared to standard adaptive control algorithms. The present paper provides a broader perspective on reflexes in the context of nonlinear regulator theory. By utilizing novel theoretical developments on exciting subspaces, we also obtain stronger closed-loop stability properties.

Our overarching goal is to translate brain architectures into a network of control modules with clear and distinct roles. By first considering the learning problem of training reflexes in Section 3, we elucidate a feedback loop acting over long timescales, thus explaining the role of the cerebellum in reflex adaptation. Learning processes are then supplemented through the use of forgetting in Section 5, with the help of subspace estimators, to provide the much needed robustness to perform accurate computations in noisy environments such as the brain. To keep our developments concrete, a running example of output regulation for single-input single-output (SISO) linear time-invariant (LTI) systems is presented in great detail in Sections Section 4, 6 with supporting simulations in Section 7. As a result, we build controllers that implement the functionality of architectural diagrams studied in systems neuroscience, such as Fig. 1 representing the brain architecture for the oculomotor system.

2. Analysis and design tools

This section reviews the most important theoretical tools for our development: notions of stability (Khalil, 2002); the basics of adaptive control (Ioannou & Sun, 2012; Narendra & Annaswamy, 1989; Sastry & Bodson, 1989); an adaptive internal model design for minimum-phase LTI systems (Isidori, 2017; Serrani, Isidori, & Marconi, 2001); and two timescale analysis (Sastry & Bodson, 1989; Teel, Moreau, & Nesic, 2003).

2.1. Notation

Given a function $f(t)$, we use the shorthands

$$d_t f(t) = \frac{d}{dt}\left[f(t)\right], \quad \partial_t f(t) = \frac{\partial}{\partial t}\left[f(t)\right]$$

to express derivatives. We use the notation $f(\cdot)$ to refer to a function of its (unspecified) arguments and let $\mathcal{B}(\delta) := \{ x : \|x\| \le \delta \}$ denote the closed ball of radius $\delta \ge 0$, where the underlying space in which the ball resides is inferred from context. Given a symmetric matrix P, we write $P > 0$ if it is positive definite and $P \ge 0$ if it is positive semi-definite. Analogous statements hold for negative (semi-) definite matrices. Moreover, we write $P \ge Q$ to say $P - Q \ge 0$. Unless otherwise specified, $\|\cdot\|$ denotes the 2-norm. For a subspace $\mathcal{W}$, we write $\mathcal{W}^\perp$ as its orthogonal complement under the standard inner product. For a set $\mathcal{S}$, we let $\mathrm{int}(\mathcal{S})$ denote its interior and $\mathrm{cl}(\mathcal{S})$ denote its closure. Lastly, useful canonical matrices (whose dimensions depend on the context) are:

$$A_\circ = \left[\begin{array}{c|c} 0 & \\ \vdots & I \\ 0 & \\ \hline 0 & \cdots \quad 0 \end{array}\right], \quad B_\circ = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right], \quad C_\circ = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \end{array}\right].$$

2.2. Preliminaries

We review some results on stability analysis; the reader is referred to Khalil (2002) for more background. Consider the nonautonomous system

$$\dot{x} = f(t, x), \tag{1}$$

satisfying $f(t, 0) = 0$ for all $t \ge 0$.

Definition 1. The equilibrium $x = 0$ of (1) is:

- *stable* if for each $\epsilon > 0$ and any $t_0 \ge 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \ge t_0 \ge 0$.
- *uniformly stable* if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \ge t_0 \ge 0$.
- *asymptotically stable* (AS) if it is stable and there exists $c = c(t_0)$ such that $\|x(t_0)\| < c$ implies $\lim_{t\to\infty} x(t) = 0$.
- *uniformly asymptotically stable* (UAS) if it is uniformly stable and there is a positive constant c, independent of t_0, such that $\|x(t_0)\| < c$ implies $\lim_{t\to\infty} x(t) = 0$.
- *globally uniformly asymptotically stable* (GUAS) if it is uniformly stable, $\delta(\epsilon) > 0$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$, and for each pair $\eta, c > 0$, there is $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \quad \forall t \ge t_0 + T(\eta, c), \quad \forall \|x(t_0)\| < c.$$

- *locally exponentially stable* (LES) if there exist positive constants δ, c, and λ such that

$$\|x(t)\| \le c\|x(t_0)\| e^{-\lambda(t-t_0)}, \qquad \forall t \ge t_0 \ge 0, \ \forall \|x(t_0)\| < \delta,$$

and *globally exponentially stable* (GES) if the above is satisfied for any initial state $x(t_0)$.
- *exponentially stable over a set* $\mathcal{S} \subseteq \mathbb{R}^n$ if there exist $c > 0$ and $\lambda > 0$ such that

$$\|x(t)\| \le c\|x(t_0)\| e^{-\lambda(t-t_0)}, \qquad \forall t \ge t_0 \ge 0, \ \forall x(t_0) \in \mathcal{S}.$$

Each of these stability definitions may be said to hold *uniformly* with respect to some parameter θ in a (compact) set if the associated constants are independent of the value of θ. Exponential stability over a set naturally extends to exponential stability over every ball, a notion sometimes called *semi-global exponential stability*. The following relates LES, GUAS, and exponential stability over every ball; see for example (Haidar, Chitour, Mason, & Sigalotti, 2022) with a proof provided in Section 9.

Proposition 1. *Consider the system (1). The equilibrium $x = 0$ is GUAS and LES if and only if it is ES over every ball. Moreover, the exponential rate can be selected independently of the ball considered.*

Below is a useful characterization of locally Lipschitz functions as a semi-global property.

Proposition 2. *Let $\mathcal{S}$ be a set. A function $f(\theta, x)$ is locally Lipschitz in x uniformly in $\theta \in \mathcal{S}$ if and only if it is Lipschitz over any compact set in x uniformly in $\theta \in \mathcal{S}$.*

2.3. Adaptive control

We highlight three important results of adaptive control: the notion of persistent excitation, and the exponential stability theorems for the two most common error models. The reader is referred to Ioannou and Sun (2012), Narendra and Annaswamy (1989), Sastry and Bodson (1989), Slotine and Li (1991) for more background.

Definition 2. The signal $w(t) \in \mathbb{R}^q$ is *persistently exciting* (PE) if there exist β_0, β_1, $T > 0$ such that

$$\beta_0 I \le \frac{1}{T}\int_t^{t+T} w(\tau)w^\top(\tau)\, d\tau \le \beta_1 I, \quad \forall\, t \ge 0.$$

We primarily deal with regressors $w(t)$ that are *stationary*, for which the following limit, called the *autocovariance* of w, exists uniformly in $t_0 \ge 0$:

$$R_w(0) := \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} w(\tau)w^\top(\tau) d\tau\,. \tag{2}$$

Lemma 1. *Let $w(t) \in \mathbb{R}^q$ be stationary. Then $w(t)$ is PE if and only if $R_w(0)$ is positive definite.*

The following gives some geometric intuition for persistent excitation.

Lemma 2. *If $w(t) \in \mathbb{R}^q$ is PE and there exists $\alpha \in \mathbb{R}^q$ such that $\alpha^\top w(t) = 0$ for all $t \ge 0$, then $\alpha = 0$.*

Consider $y = \psi^\top w(t)$, a scalar measurement that depends linearly on unknown parameters $\psi \in \mathbb{R}^q$ and a known regressor $w(t) \in \mathbb{R}^q$. Let $\hat{\psi}(t)$ be an estimate of ψ and define the parameter error $\tilde{\psi} := \hat{\psi} - \psi$. The simplest error model of adaptive control, a static model, along with the gradient adaptation law are

$$e = \hat{\psi}^\top w(t) - y \tag{3a}$$

$$\dot{\hat{\psi}} = -\gamma e w(t)\,, \tag{3b}$$

where $\gamma > 0$ is the adaptation rate. Equivalently one can write a linear time-varying (LTV) differential equation

$$\dot{\tilde{\psi}} = -\gamma w(t) w^\top(t) \tilde{\psi}\,. \tag{4}$$

The main stability result is Ioannou and Sun (2012, Theorem 4.3.2), Narendra and Annaswamy (1989, Theorem 2.16), or Sastry and Bodson (1989, Theorem 2.5.1).

Theorem 1. *Suppose $w(t)$ is piecewise continuous and PE. Then the equilibrium $\tilde{\psi} = 0$ of (4) is GES.*

Remark 1 (∗)**.** The gradient adaptation law (3b) is ubiquitous in theoretical neuroscience, where it is the starting point for discussions on *synaptic plasticity*, *Hebb's law*, and unsupervised learning at the neuronal level (Dayan & Abbott, 2001, Ch. 8). Its biological plausibility specifically with regard to learning in the cerebellum is discussed in Sejnowski (1977); therein called the *covariance rule*. See also Dean and Porrill (2008, 2014) for a detailed discussion in terms of an adaptive filter model of the cerebellum. ◃

The second simplest error model is a dynamic model with error state $x \in \mathbb{R}^n$, presented along with the gradient adaptation law:

$$\dot{x} = Ax + Bw^{\intercal}(t)\tilde{\psi} \tag{5a}$$

$$\dot{\tilde{\psi}} = -\gamma(B^{\intercal}Px)w(t), \tag{5b}$$

where $\gamma > 0$, A is Hurwitz, (A, B) is controllable, and $P \succ 0$ solves the Lyapunov equation $A^{\intercal}P + PA = -I$. The main stability result is Ioannou and Sun (2012, Corollary 4.3.1), Narendra and Annaswamy (1989, Theorem 2.17), or Sastry and Bodson (1989, Theorem 2.6.5).

Theorem 2. *Suppose $w(t)$ is PE and $w, \dot{w}$ are bounded. Then the equilibrium $(x, \tilde{\psi}) = (0, 0)$ of (5) is GES.*

2.4. Adaptive internal models

Since the early papers (Nikiforov, 1996, 1997a, 1997b), adaptive internal model designs have proliferated in the control literature; for example (Basturk & Krstic, 2014, 2015; Gerasimov, Paramonov, & Nikiforov, 2020; Marino & Tomei, 2003, 2015; Marino & Tomei, 2021; Serrani et al., 2001; Yilmaz & Basturk, 2019); see Nikiforov and Gerasimov (2022) and the references therein. Here we use a representative design from Serrani et al. (2001), while our presentation follows Isidori (2017). Consider the SISO LTI system

$$\dot{x} = Ax + Bu + E\zeta \tag{6a}$$

$$\dot{\zeta} = S\zeta \tag{6b}$$

$$e = Cx + D\zeta \tag{6c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, $\zeta(t) \in \mathbb{R}^q$ is the exosystem state, and $e(t) \in \mathbb{R}$ is the error. Standard assumptions on the system are as follows.

Assumption 1. The open-loop system (6) satisfies:

(A1) (C, A, B) is a minimal realization and minimum phase;
(A2) S only has simple eigenvalues on the $j\omega$-axis;
(A3) the non-resonance condition holds:

$$\det \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} \neq 0, \quad \forall \lambda \in \sigma(S);$$

(A4) the dimension q is interpreted as a known upper bound on the order of the exosystem;
(A5) the relative degree r and the sign of the high frequency gain $b := CA^{r-1}B$ of (C, A, B) are known;
(A6) the measurement is e.

Using (A3), let (Π, Γ) uniquely solve the regulator equations

$$\Pi S = A\Pi + B\Gamma + E, \quad 0 = C\Pi + D. \tag{7}$$

Defining the error state $z := x - \Pi\zeta$, system (6) becomes

$$\dot{z} = Az + B(u - d) \qquad e = Cz \tag{8a}$$

$$\dot{\zeta} = S\zeta \qquad d = \Gamma\zeta. \tag{8b}$$

We consider a regulator of the form

$$u = u_s + u_{im},$$

where u_s is for stabilization and u_{im} is for disturbance rejection.

We first deal with the stabilizer for minimum phase LTI systems. Given that the system is SISO with no feedthrough term, the relative degree satisfies $r \geq 1$. According to Isidori (2017, Proposition 2.2), there exists a coordinate transformation $z \mapsto (z_0, \xi)$ mapping the system (8a) to its *strict normal form*

$$\dot{z}_0 = A_{00}z_0 + A_{01}\xi \tag{9a}$$

$$\dot{\xi} = (A_\circ + B_\circ a_{11}^{\intercal})\xi + B_\circ a_{10}^{\intercal}z_0 + bB_\circ(u - d) \tag{9b}$$

$$e = C_\circ\xi \tag{9c}$$

where $z_0(t) \in \mathbb{R}^{n-r}$, $\xi(t) \in \mathbb{R}^r$, A_{00}, A_{01}, a_{10}, a_{11} are some matrices, and the matrices $A_\circ$, $B_\circ$, $C_\circ$ are given in Section 2.1. If the relative degree is $r = 1$, then we choose the high-gain stabilizer

$$u_s = -\operatorname{sgn}(b)Ke. \tag{10}$$

If $r > 1$, we instead employ the reduction procedure described in Isidori (2017, Section 2.4) that effectively converts the system to relative degree 1 using a high-gain observer. To that end, select $a \in \mathbb{R}^{r-1}$ such that $A_\circ - B_\circ a^{\intercal}$ is Hurwitz. Define $\xi_\circ := (\xi_1, \ldots, \xi_{r-1})$ and $e_0 := \begin{bmatrix} a^{\intercal} & 1 \end{bmatrix}\xi$ to obtain

$$\begin{bmatrix} \dot{z}_0 \\ \dot{\xi}_\circ \end{bmatrix} = \begin{bmatrix} A_{00} & A'_{01} \\ 0 & A_\circ - B_\circ a^{\intercal} \end{bmatrix} \begin{bmatrix} z_0 \\ \xi_\circ \end{bmatrix} + B_{0r}e_0$$

$$\dot{e}_0 = a_{rr}e_0 + a_{r0}^{\intercal} \begin{bmatrix} z_0 \\ \xi_\circ \end{bmatrix} + b(u - d)$$

for some matrices A'_{01}, B_{0r}, a_{r0}, and a_{rr}. By (A1) the matrix A_{00} is Hurwitz and, therefore, so is

$$\begin{bmatrix} A_{00} & A'_{01} \\ 0 & A_\circ - B_\circ a^{\intercal} \end{bmatrix}.$$

Then we use the high-gain observer and high-gain stabilizer

$$\dot{\hat{\xi}} = A_\circ\hat{\xi} + D_\kappa L(e - C_\circ\hat{\xi}) \tag{11a}$$

$$\hat{e}_0 = \begin{bmatrix} a^{\intercal} & 1 \end{bmatrix}\hat{\xi} \tag{11b}$$

$$u_s = -\operatorname{sgn}(b)K\hat{e}_0, \tag{11c}$$

where $\hat{\xi}(t) \in \mathbb{R}^r$, K, $\kappa > 0$ are to be selected sufficiently large, $D_\kappa := \operatorname{diag}(\kappa, \kappa^2, \ldots, \kappa^r)$, and $L \in \mathbb{R}^r$ is selected so that $A_\circ - LC_\circ$ is Hurwitz. This completes the design of the stabilizer u_s.

Next we present the internal model design that determines u_{im}. From the perspective of the plant, it is wlog to assume the exosystem has the property that (Γ, S) is observable. Given any controllable pair (F, G) with $F \in \mathbb{R}^{q\times q}$ Hurwitz, there exists a coordinate transformation $w_0 := M\zeta$ (Nikiforov & Gerasimov, 2022, Lemma 2.1) such that

$$\dot{w}_0 = Fw_0 + Gd \tag{12a}$$

$$d = \psi_0^{\intercal}w_0 \tag{12b}$$

with $\psi_0^{\intercal} := \Gamma M^{-1}$. Then the adaptive internal model is

$$\dot{\hat{w}}_0 = F\hat{w}_0 + Gu \tag{13a}$$

$$\dot{\hat{\psi}}_0 = -\operatorname{sgn}(b)\gamma\hat{e}_0\hat{w}_0 \tag{13b}$$

$$u_{im} = \hat{\psi}_0^{\intercal}\hat{w}_0, \tag{13c}$$

where $\hat{w}_0(t)$, $\hat{\psi}_0(t) \in \mathbb{R}^q$ and $\gamma > 0$.

To analyze the resulting closed-loop system, define the observer errors $\tilde{w}_0 := \hat{w}_0 - w_0 - b^{-1}Ge_0$ and $\tilde{\xi} := \kappa^r D_\kappa^{-1}(\hat{\xi} - \xi)$. Also define the parameter error $\tilde{\psi}_0 := \hat{\psi}_0 - \psi_0$. Let $x_\circ := (z_0, \xi_\circ, \tilde{w}_0, e_0)$ be an aggregate state. The closed-loop dynamics are

$$\begin{bmatrix} \dot{x}_\circ \\ \dot{\tilde{\psi}}_0 \end{bmatrix} = \begin{bmatrix} A(K) & bB_\circ\hat{w}_0^{\intercal} \\ -\operatorname{sgn}(b)\gamma\hat{w}_0B_\circ^{\intercal} & 0 \end{bmatrix} \begin{bmatrix} x_\circ \\ \tilde{\psi}_0 \end{bmatrix} + B_1(K, \kappa, \hat{w}_0)\tilde{\xi} \tag{14a}$$

$$\dot{\tilde{\xi}} = \kappa(A_\circ - LC_\circ)\tilde{\xi} + B_2(K, \hat{w}_0)\begin{bmatrix} x_\circ \\ \tilde{\psi}_0 \end{bmatrix} + B_3(K, \kappa)\tilde{\xi} \tag{14b}$$

where $\hat{w}_0 = w_0(t) + M_w x_\circ$ for some M_w and

$$A(K) := \begin{bmatrix} \begin{bmatrix} A_{00} & A'_{01} \\ 0 & A_\circ - B_\circ a^\intercal \end{bmatrix} & 0 & B_{0r} \\ -b^{-1} G a_{r0}^\intercal & F & b^{-1}(FG - Ga_{rr}) \\ a_{r0}^\intercal & b\psi_0^\intercal & a_{rr} + \psi_0^\intercal G - |b| K \end{bmatrix}$$

$$B_1(K, \kappa, \hat{w}_0) := -\begin{bmatrix} |b| B_\circ K \\ \operatorname{sgn}(b)\gamma \hat{w}_0 \end{bmatrix} \begin{bmatrix} a^\intercal & 1 \end{bmatrix} \kappa^{-r} D_\kappa$$

$$B_3(K, \kappa) := |b| B_\circ K \begin{bmatrix} a^\intercal & 1 \end{bmatrix} \kappa^{-r} D_\kappa$$

with $B_2(\cdot)$ continuously differentiable. Stability analysis of (14) proceeds in two steps. First we analyze stability when $\tilde{\xi} = 0$, resulting in a system whose form is notably similar to (5):

$$\begin{bmatrix} \dot{x}_\circ \\ \dot{\tilde{\psi}}_0 \end{bmatrix} = \begin{bmatrix} A(K) & bB_\circ \hat{w}_0^\intercal \\ -\operatorname{sgn}(b)\gamma \hat{w}_0 B_\circ^\intercal & 0 \end{bmatrix} \begin{bmatrix} x_\circ \\ \tilde{\psi}_0 \end{bmatrix}. \tag{15}$$

Stability of the matrix $A(K)$ is dealt with using a Schur complement argument; see Isidori (2017, Section 2.3) for details.

Lemma 3. *There exists $K_\star$ such that for all $K \geq K_\star$, the matrix $A(K)$ is Hurwitz. Moreover, there exists a matrix $P_0 \succ 0$ and a scalar $\rho > 0$ such that $P := \operatorname{diag}(P_0, 1)$ satisfies the Lyapunov Linear Matrix Inequality (LMI)*

$$A^\intercal(K)P + PA(K) \preceq -\rho I$$

for all $K \geq K_\star$.

The main results on stability of (15) and then (14) are the following (Isidori, 2017; Serrani et al., 2001); proofs are provided in Section 9. First, we have a stability result under the assumption of a perfect state measurement when $\tilde{\xi} = 0$ (Isidori, 2017, Section 4.8).

Lemma 4. *Consider system (15). Suppose $w_0(t)$ is PE. Let $K_\star$ be defined as in Lemma 3. Then for every $K \geq K_\star$ the equilibrium $(x_\circ, \tilde{\psi}_0) = (0, 0)$ is GUAS and LES.*

Next we reintroduce the high gain observer. The following stability result depends on the nonlinear separation principle of Teel and Praly (1995).

Theorem 3. *Consider system (14). Suppose $w_0(t)$ is PE. For each $\delta_1 > 0$ there exists $K_\star > 0$ and $\kappa_\star(K, \delta_1) \geq 1$ such that for any fixed $K \geq K_\star$ and $\kappa \geq \kappa_\star$ the equilibrium $(x_\circ, \tilde{\psi}_0, \tilde{\xi}) = (0, 0, 0)$ is ES over $(x_\circ, \tilde{\psi}_0)(t_0) \in \mathcal{B}(\delta_1)$ and $\tilde{\xi}(t_0) \in \mathcal{B}(\delta_1 \kappa^{r-1})$.*

2.5. Two timescale analysis

Consider a two timescale system of the form

$$\dot{\chi} = F(t, \chi, \tilde{\alpha}) + \varepsilon g(t, \chi, \tilde{\alpha}) \tag{16a}$$

$$\dot{\tilde{\alpha}} = \varepsilon f(t, \chi, \tilde{\alpha}) \tag{16b}$$

where $\varepsilon > 0$ and $\chi(t) \in \mathbb{R}^{n_f}$, $\tilde{\alpha}(t) \in \mathbb{R}^{n_s}$ are the fast and slow states respectively. To study this system, it is useful to define the averaging operator $(\cdot)_{av}$ applied to $f(\cdot)$ as

$$f_{av}(\tilde{\alpha}_{av}) := \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, 0, \tilde{\alpha}_{av})\, d\tau. \tag{17}$$

Then we may identify fast and averaged systems

$$\dot{\chi}_f = F(t, \chi_f, \tilde{\alpha}), \quad \dot{\tilde{\alpha}} = 0 \tag{18a}$$

$$\dot{\tilde{\alpha}}_{av} = \varepsilon f_{av}(\tilde{\alpha}_{av}). \tag{18b}$$

The standard assumptions for two timescale averaging analysis may be found in Sastry and Bodson (1989); here they are presented in a more parsimonious (and slightly stronger) form; see Section 9.4.

Assumption 2. The system (16) and its associated systems (17)–(18) satisfy:

(C1) $F(\cdot)$, $f(\cdot)$, and $g(\cdot)$ are piecewise continuous in t and continuously differentiable in $(\chi, \tilde{\alpha})$ uniformly in $t \geq 0$;
(C2) $F(t, 0, 0) = g(t, 0, 0) = 0$ and $f(t, 0, 0) = 0$ for all $t \geq 0$;
(C3) $f_{av}(\cdot)$ and $(\partial_{\tilde{\alpha}} f)_{av}(\cdot)$ exist, and convergence to the latter is uniform in $t_0 \geq 0$ and in $\tilde{\alpha}_{av}$ over compact sets;
(C4) for some $\delta_f, \delta_\alpha > 0$, the equilibrium $\chi_f = 0$ of (18a) is ES over $\chi_f(t_0) \in \mathcal{B}(\delta_f + 1)$ uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$;
(C5) the equilibrium $\tilde{\alpha}_{av} = 0$ of (18b) is GES.

The main result concerning practical asymptotic stability of two timescale systems is a specialization of the result in Teel et al. (2003), here restated for a time-varying framework and restricted to exponential stability.

Theorem 4. *Consider the system (16) satisfying Assumption 2. Then for each residual $\epsilon > 0$ there exists $\delta_s(\delta_\alpha)$, $\varepsilon_\star(\delta_f, \delta_\alpha, \epsilon) > 0$ such that for any fixed $\varepsilon \in (0, \varepsilon_\star)$ the trajectories satisfy*

$$\|\chi(t)\| \leq c_f \|\chi(t_0)\| \mathrm{e}^{-\lambda_f (t-t_0)} + \epsilon$$

$$\|\tilde{\alpha}(t)\| \leq c_s \|\tilde{\alpha}(t_0)\| \mathrm{e}^{-\varepsilon\lambda_s (t-t_0)} + \epsilon$$

for all $t \geq t_0 \geq 0$ over $(\chi, \tilde{\alpha})(t_0) \in \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_s)$, where $c_f(\delta_f, \delta_\alpha)$, $\lambda_f(\delta_f, \delta_\alpha)$, c_s, $\lambda_s > 0$. More generally, one can always select $\delta_s(\delta_\alpha) = \delta_s(\delta_\alpha, f_{av}(\cdot))$ independent of $F(\cdot)$ and $f(\cdot)$ such that

$$\lim_{\delta_\alpha\to\infty} \delta_s(\delta_\alpha) = \infty.$$

By imposing an equilibrium in (C2) and exponential stability in (C4)–(C5), one may also recover exponential convergence to the equilibrium using the Lyapunov method in Sastry and Bodson (1989, Section 4.4).

Theorem 5. *Consider the system (16) satisfying Assumption 2. Then there exists $\delta_s(\delta_\alpha)$, $\varepsilon_\star(\delta_f, \delta_\alpha) > 0$ such that for any fixed $\varepsilon \in (0, \varepsilon_\star)$ the equilibrium $(\chi, \tilde{\alpha}) = (0, 0)$ is ES over $(\chi, \tilde{\alpha})(t_0) \in \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_s)$. More generally, one can always select $\delta_s(\delta_\alpha) = \delta_s(\delta_\alpha, f_{av}(\cdot))$ independent of $F(\cdot)$ and $f(\cdot)$ such that*

$$\lim_{\delta_\alpha\to\infty} \delta_s(\delta_\alpha) = \infty.$$

Proof of Theorems 4 and 5 are found in Section 9.

3. A learning problem: Training reflexes

This section presents the main learning problem we consider: the training of reflexes for the purpose of reducing the work of disturbance rejection by the cerebellum. We frame the problem in the context of nonlinear regulator theory; see Huang (2004), Isidori (1995, 2017) for more background. In the section following this one, the LTI case will be considered by invoking the regulator design from Section 2.4. Throughout this section we will assume that any regressors used in parameter adaptation are PE. This unrealistic assumption will be removed in Section 5, where lack of persistent excitation will be tied to a biological process of *forgetting*.

Consider the single-input autonomous nonlinear system

$$\dot{x} = f_\circ(x, \zeta, u) \tag{19a}$$

$$\dot{\zeta} = s(\zeta) \tag{19b}$$

$$e = h(x, \zeta) \tag{19c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, $\zeta(t) \in \mathbb{R}^q$ is the exosystem state, and $e(t) \in \mathbb{R}^m$ is the error. Also suppose we have a partial disturbance measurement $y(t) \in \mathbb{R}^p$ generated by a nonlinear exosystem

$$\dot{\zeta}_y = s_y(\zeta_y) \tag{20a}$$

$$y = y(\zeta, \zeta_y), \tag{20b}$$

where $\zeta_y(t) \in \mathbb{R}^{q_y}$ is the additional exosystem state that determines the measurement y. Notice that our choice of a separate exosystem for y allows for new frequency content not appearing in ζ. Additionally, the decoupling of ζ and ζ_y captures the conceptual fact that regulation is tasked only with rejecting ζ, whereas y is a supplementary measurement which may or may not be used towards this purpose.

Remark 2 (∗)**.** The possibility of distinct frequency content in y that is not present in the exosystem state ζ is most clearly captured in an oculomotor experiment called *VOR cancellation* (Leigh & Zee, 2015). In this experiment, a subject seated in a sinusoidally rotating chair is tasked with visually tracking a head fixed target at the central position. Since the target remains at the central position with respect to the head throughout the experiment, the reference signal for tracking is zero, so there is no exogenous disturbance acting on the retinal error. However, the measurement y, which corresponds to the head angular velocity, is a sinusoid due to the rotation of the chair. Therefore, y contains disturbance frequencies not present in the exogenous disturbances acting on the plant, which are independently modeled by the exosystem (19b). ◁

Assumption 3. The open-loop system (19) and the measurement (20) satisfy:

(R1) the functions $f_\circ(\cdot)$, $s(\cdot)$, $s_y(\cdot)$, $h(\cdot)$, and $y(\cdot)$ are continuously differentiable;
(R2) the exosystems (19b) and (20a) are neutrally stable;
(R3) for some set $\mathcal{Z} \subseteq \mathbb{R}^q$ compact and invariant under (19b) there exist unique functions $\pi, \gamma : \mathcal{Z} \to \mathbb{R}^n$ solving the regulator equations

$$\partial_\zeta \pi(\zeta) s(\zeta) = f_\circ(\pi(\zeta), \zeta, \gamma(\zeta)) \tag{21a}$$

$$0 = h(\pi(\zeta), \zeta)\,. \tag{21b}$$

Remark 3. The notion of *neutrally stable* is discussed in Isidori (1995, p. 388). It implies that the nonlinear exosystems (19b) and (20a) each generate solutions that are bounded and persist in time. This notion provides a suitable generalization to the nonlinear setting of an LTI exosystem with simple eigenvalues on the imaginary axis, used to generate step and sinusoidal signals. The generalization to nonlinear exosystems is relevant in neuroscience applications where disturbance and reference signals often cannot be modeled by LTI exosystems. ◁

We consider a disturbance rejection problem where our controller is of the form

$$\dot{x}_c = f_c(x_c, x, \zeta, \zeta_y) \tag{22a}$$

$$u_s = u_s(x_c, x, \zeta, \zeta_y) \tag{22b}$$

$$u_{im} = u_{im}(x_c, x, \zeta, \zeta_y) \tag{22c}$$

$$u_r = -\hat{\alpha}_r^\intercal y \tag{22d}$$

$$u_x = \hat{\alpha}_x^\intercal x \tag{22e}$$

$$u = u_s + u_{im} + u_r + u_x\,, \tag{22f}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the controller state; u_s is for stabilization; u_{im} is the output of the internal model; and u_r and u_x are the *reflexes*: one a feedforward input and the other a state feedback. The vectors $\hat{\alpha}_r$ and $\hat{\alpha}_x$ are called *reflex gains* (MacKay & Murphy, 1979).

Remark 4 (∗)**.** We will see in the sequel that mathematically there is little distinction between u_r and u_x since both are involved in disturbance rejection. Indeed, neuroscientists often do not draw a strict line between states and exogenous measurements for the reason that the distinction can be ambiguous. For example, the head velocity measurement in the semicircular canals of the ears utilized by the oculomotor system may be viewed as a measurement of a state in a combined oculomotor-head model, or as an exogenous input in an oculomotor model only. For a sufficiently flexible reflex control architecture, we must allow both interpretations. On the other hand, we make a distinction between the stabilizing term u_s and the disturbance rejection term u_x. The key qualitative difference is that u_s vanishes as regulation progresses (see (R7)), whereas u_x does not, so long as disturbances persist. ◁

Remark 5. We also make some additional comments on the measurement structure. We assume a full state measurement of x is available to u_x for the sake of simplicity, but the control architecture may be easily extended to deal with partial state measurements. Also, the form of (22) is meant to capture state feedback, error feedback, and other types of controllers. Thus, the dependence of the right hand sides of (22a)–(22c) on x, ζ, and ζ_y does not imply that direct measurements of those signals are available to the regulator. ◁

Our goal is to devise an adaptation scheme for the reflex gains

$\hat{\alpha} := (\hat{\alpha}_r, \hat{\alpha}_x)$

so that $u_r + u_x$ enhances the disturbance rejection properties of u_{im}. Formally, we aim to minimize the following cost function

$$\min_{\hat{\alpha}} \left(\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \frac{1}{2} \|u_{im}(\tau)\|^2 \, d\tau \right) \tag{23}$$

$$\text{subject to } \lim_{t\to\infty} e(t) = 0\,,$$

capturing the (persistent) average cost of operating the internal model, which must itself achieve error regulation.

In the spirit of modular design, we want the reflex adaptation scheme to be agnostic to the choice of the internal model and the stabilizing compensator. As such, we must identify the general properties of any regulator (that determines $u_s + u_{im}$) so that learning in $\hat{\alpha}$ reduces the work of the regulator. The following assumption imposed on the regulator is a variant of the *Nonlinear Output Regulation Problem with Exponential Stability* (Huang, 2004, p. 77).

Assumption 4. Suppose $\hat{\alpha}$ is constant. The closed-loop system (19), (20) and (22) satisfies:

(R4) the functions $f_c(\cdot)$, $u_s(\cdot)$, and $u_{im}(\cdot)$ are continuously differentiable;
(R5) for some set $\mathcal{Z}_y \subseteq \mathbb{R}^{q_y}$ compact and invariant under (20a) and constants δ_f, $\delta_\alpha > 0$ there exists a continuously differentiable function $\pi_c : \mathcal{Z} \times \mathcal{Z}_y \times \mathcal{B}(\delta_\alpha) \to \mathbb{R}^{n_c}$ and constants $c_f(\delta_f, \delta_\alpha, \mathcal{Z}, \mathcal{Z}_y)$, $\lambda_f(\delta_f, \delta_\alpha, \mathcal{Z}, \mathcal{Z}_y) > 0$ such that, defining the *error states*

$$z := x - \pi(\zeta)\,, \quad z_c := x_c - \pi_c(\zeta, \zeta_y, \hat{\alpha})\,, \tag{24}$$

we have

$$\|(z, z_c)(t)\| \le c_f \|(z, z_c)(t_0)\| \mathrm{e}^{-\lambda_f (t-t_0)}$$

for all $t \ge t_0 \ge 0$ over $(z, z_c)(t_0) \in \mathcal{B}(\delta_f + 1)$ uniformly in $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ and $\hat{\alpha} \in \mathcal{B}(\delta_\alpha)$.
(R6) there exists $\delta > 0$ such that $(x, x_c)(t_0) \in \mathcal{B}(\delta)$, $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$, and $\hat{\alpha} \in \mathcal{B}(\delta_\alpha)$ implies $(z, z_c)(t_0) \in \mathcal{B}(\delta_f)$;
(R7) $\lim_{t\to\infty} u_s(x_c(t), x(t), \zeta(t), \zeta_y(t)) = 0$.

Remark 6. The classical statement of the nonlinear output regulation problem includes asymptotic stability of the *unforced system* when $(\zeta, \zeta_y) = 0$, and regulation of $e(t)$ to zero (Isidori, 1995, p. 394). Here the regulation requirement has been replaced by the stronger requirement (R5), which says that for some set of initial conditions of (x, x_c) and for any constant $\hat{\alpha}$ in a ball, the plant and regulator states (x, x_c) will converge exponentially to their steady-states $(\pi(\zeta), \pi_c(\zeta, \zeta_y, \hat{\alpha}))$. This property in turn implies the classical requirement of error regulation. Namely, by continuity of $h(\cdot)$ and by the nonlinear regulator Eqs. (21) we have

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} h(x(t), \zeta(t)) = \lim_{t\to\infty} h(\pi(\zeta(t)), \zeta(t)) = 0.$$

The requirement of asymptotic stability of the unforced system in the classical regulator problem is removed in the adaptive problem because asymptotic stability generally cannot be achieved by parameter adaptation laws when the exosystem state is not PE. ◃

3.1. Reflex adaptation law

Given any regulator satisfying Assumption 4, we investigate how $u_r + u_x$ can be used to enhance the disturbance rejection process as is done by reflexes in biological systems. Consider the change of coordinates (24), where we assume $\hat{\alpha}$ is constant. Using (19), (20a), (21), and (22), we obtain the closed-loop system

$$\dot{z} = f_\circ(z + \pi(\zeta), \zeta, u) - f_\circ(\pi(\zeta), \zeta, \gamma(\zeta)) \tag{25a}$$

$$\dot{z}_c = f_c(z_c + \pi_c(\zeta, \zeta_y, \hat{\alpha}), z + \pi(\zeta), \zeta, \zeta_y) - \begin{bmatrix} \partial_\zeta \pi_c(\zeta, \zeta_y, \hat{\alpha}) & \partial_{\zeta_y} \pi_c(\zeta, \zeta_y, \hat{\alpha}) \end{bmatrix} \begin{bmatrix} s(\zeta) \\ s_y(\zeta_y) \end{bmatrix}. \tag{25b}$$

It follows from (R5) that $(z, z_c) = (0, 0)$ is an exponentially stable equilibrium of this system. Now we want to examine the system's steady-state; namely when $(z, z_c)(t) \equiv (0, 0)$. Let u_{ss} denote the steady-state of u (again assuming $\hat{\alpha}$ is a constant). From (25a) at steady-state and (R3) we have

$$\begin{aligned} 0 &= f_\circ(\pi(\zeta), \zeta, u_{ss}) - f_\circ(\pi(\zeta), \zeta, \gamma(\zeta)) \\ &= f_\circ(\pi(\zeta), \zeta, u_{ss}) - \partial_\zeta \pi(\zeta) s(\zeta). \end{aligned} \tag{26}$$

On the other hand, using (22f), (R7), and by continuity in (R4), we know that

$$u_{ss} = u_{im}(\pi_c(\zeta, \zeta_y, \hat{\alpha}), \pi(\zeta), \zeta, \zeta_y) + \hat{\alpha}_x^\intercal \pi(\zeta) - \hat{\alpha}_r^\intercal y(\zeta, \zeta_y).$$

Since u_{ss} is defined for $\zeta \in \mathcal{Z}$, and from (26) it solves the same regulator equations as $\gamma(\cdot)$, it must be that $u_{ss} = \gamma(\zeta)$ by uniqueness in (R3). As a result, we can express u_{im} in steady-state as

$$u_{im}(\pi_c(\zeta, \zeta_y, \hat{\alpha}), \pi(\zeta), \zeta, \zeta_y) = \gamma(\zeta) + \hat{\alpha}^\intercal \begin{bmatrix} y(\zeta, \zeta_y) \\ -\pi(\zeta) \end{bmatrix}. \tag{27}$$

This formula clarifies the overall strategy of a control architecture that exploits reflexes. Firstly, if we had $\hat{\alpha} = 0$ (no reflexes), then the internal model would be tasked with generating a compensatory input u_{im} to cancel the *effective disturbance* $\gamma(\zeta)$. Now suppose that some component of the signal $\gamma(\zeta)$ could be perfectly canceled by a signal generated as $\alpha^\intercal \begin{bmatrix} y(\zeta, \zeta_y)^\intercal & -\pi(\zeta)^\intercal \end{bmatrix}^\intercal$ for a suitable α. Then the work of the internal model could be reduced in steady-state with the help of properly adapted reflexes if the *residual disturbance*

$$\gamma'(\zeta) := \gamma(\zeta) + \alpha^\intercal \begin{bmatrix} y(\zeta, \zeta_y) \\ -\pi(\zeta) \end{bmatrix} \tag{28}$$

has smaller signal energy in a suitable norm.

To contain the complexity of our developments, we consider the case when the residual disturbance $\gamma'(\zeta)$ can be perfectly canceled by suitably adapted reflexes $u_r + u_x$ such that $\gamma'(\zeta) = 0$.

Assumption 5. The effective disturbance $\gamma(\zeta)$ and our measurement $v := (y, -\pi)$ in steady-state satisfy:

(R8) there exists $\alpha = (\alpha_r, \alpha_x)$ such that $\gamma(\zeta) = -\alpha^\intercal v(\zeta, \zeta_y)$;

(R9) the autocovariance matrix

$$R_v(0) := \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} v(\tau) v^\intercal(\tau)\, d\tau$$

of $v(t) := v(\zeta(t), \zeta_y(t))$ exists, where its convergence is uniform in $t_0 \geq 0$ and $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$.

Remark 7. We emphasize that (R8) is a somewhat artificial assumption that will not be valid for all reflexes. For example, the assumption fails for certain reflexes of the oculomotor system because there is no direct measurement in the brain of visual target positions that constitute a reference signal for oculomotor tracking. Instead, for the stretch reflexes, the assumption is satisfied because a full measurement of the torque disturbance on each joint is available for use by the reflex. This means the stretch reflexes can fully compensate for torque disturbances, without direct support from the cerebellum, so long as their reflex gains are properly adapted.

Fortunately, (R8) can be removed, and the stability analysis tools needed for the more general development will be the same as the ones presented here, especially as regards the use of the two timescale results of Section 2.5. The more general result will be presented in a separate paper. Finally, we mention that (R9) is a technical assumption to ensure that the averaging process is valid. ◃

Using (R8) we can re-express u_{im} in steady-state as

$$u_{im}(\pi_c(\zeta, \zeta_y, \hat{\alpha}), \pi(\zeta), \zeta, \zeta_y) = (\hat{\alpha} - \alpha)^\intercal v(\zeta, \zeta_y). \tag{29}$$

The goal of the reflex inputs is to minimize the steady-state work of the internal model, as stated in (23). As elaborated in Hafez et al. (2023), one can equivalently consider the instantaneous cost function

$$J_{im} = \frac{1}{2} u_{im}^2, \tag{30}$$

with u_{im} interpreted to be in steady-state. By the steady-state expression for u_{im} in (29), we arrive at the gradient law for adaptation of the reflex gains:

$$\dot{\hat{\alpha}} = \begin{bmatrix} \dot{\hat{\alpha}}_r \\ \dot{\hat{\alpha}}_x \end{bmatrix} = -\varepsilon u_{im} \begin{bmatrix} y \\ -x \end{bmatrix} \tag{31}$$

with $\varepsilon > 0$ to be selected sufficiently small.

We notice that ε must be selected to be sufficiently small in order that the internal model, which operates on a fast timescale, has sufficient time for u_{im} to reach a quasi-steady-state. Meanwhile the slow variation of $\hat{\alpha}$ makes it appear quasi-static to the internal model. The two rates associated with the fast adaptive process in (22a) and the slow adaptive process in (31) are what give rise to a two timescale control architecture, motivating the discussion in Section 2.5.

Remark 8. The design of the reflex adaptation law (31) started from an (R8) that the residual disturbance can be made zero by a suitable value of α. The gradient of the instantaneous cost (30) immediately leads to the reflex adaptation law. These ideas can more generally be cast in terms of *optimal steady-state control* (OSS), recently developed primarily for power system applications (Bianchin, Cortes, Poveda, & Dall'Anese, 2022; Colombino, Dall'Anese, & Bernstein, 2020; Dall'Anese, Dhople, & Giannakis, 2015; Hauswirth, Bolognani, Hug, & Dörfler, 2021; Jokic et al., 2009; Lawrence, Simpson-Porco, & Mallada, 2021). We further developed the connection with OSS control problems by formulating the *optimal steady-state regulation* (OSSR) problem in Hafez et al. (2023). The idea of OSSR is that regulation can be performed by several types of controllers: adaptive internal models, static internal models, state feedbacks, and feedforward controls, among others, but these controllers can have widely different steady-state costs associated with them. The goal of the OSSR problem is to find the right mix of controllers to minimize the overall steady-state cost of regulation.

The OSSR problem considered here can be interpreted as one where there are three types of controllers for regulation: a state feedback u_x, a feedforward control u_r, and an adaptive internal model with output u_{im}. The average cost (23) and the instantaneous cost (30) capture the fact that the (biological) cost of operation of the adaptive internal model far outweighs the cost of operating state feedbacks and feedforward controls. In the OSSR framework, rather than imposing the assumption (R8), one proves existence and uniqueness of solutions of the posed instantaneous and averaged optimization problems (Hafez et al., 2023). ◃

Remark 9 (∗)**.** Over 400 million years, it appears all vertebrates utilize a two timescale control architecture in which reflexes are trained by the cerebellum in a process of long-term adaptation of the reflex gains (MacKay & Murphy, 1979). From a control perspective, this striking fact begs the question of what makes this control architecture superior to other methods of disturbance rejection. At this point based on our models, we can make some speculations. First, it is clear that when the sensory signal y does not contain a full measurement of the disturbance acting on a particular sub-system being regulated (e.g. the oculomotor system), then the cerebellum is needed as a backup disturbance rejection module to cancel the residual disturbance (28). But if the cerebellum is available to perform disturbance rejection, why would reflexes be needed? The first argument is that adaptive internal models are unable to handle rapid, burst-like disturbances (such as the effect of footfalls on eye position), whereas they deal effectively with persistent disturbances. The second argument for retaining reflexes is that they offload the steady-state work of the cerebellum, which likely has a high biological cost of operation.

It becomes clear that an architecture that includes both the cerebellum containing adaptive internal models for rejecting persistent disturbances and reflexes for rejecting brief disturbances has benefits. But why are the adaptation processes in the cerebellum and of the reflex acting on two timescales? This issue is more subtle, and it has been more deeply explored in our companion paper (Mejia Uzeda & Broucke, 2023). In a nutshell, the reflex gains capture unmodeled physical pathways in the body whose values should remain close to ideal physical constants for proper function of the reflexes. Allowing these reflex gains to be modified on a short timescale can be shown to result in poor performance of the reflexes. ◁

3.2. Two timescale dynamics

The two timescale system is obtained by combining the (z, z_c) dynamics, naturally assuming $\hat{\alpha}$ is no longer constant, with the $\hat{\alpha}$ dynamics. We define

$$\chi := \begin{bmatrix} z \\ z_c \end{bmatrix}, \quad \tilde{\alpha} := \hat{\alpha} - \alpha, \quad v(t) := v(\zeta(t), \zeta_y(t)),$$

where (z, z_c) is defined in Section 3.1. Notice from (25b) that at equilibrium we have

$$f_c(\pi_c(\zeta, \zeta_y, \hat{\alpha}), \pi(\zeta), \zeta, \zeta_y) =$$
$$\begin{bmatrix} \partial_\zeta \pi_c(\zeta, \zeta_y, \hat{\alpha}) & \partial_{\zeta_y} \pi_c(\zeta, \zeta_y, \hat{\alpha}) \end{bmatrix} \begin{bmatrix} s(\zeta) \\ s_y(\zeta_y) \end{bmatrix}.$$

Invoking (25), (31), and the previous statement, we derive the closed-loop dynamics which take the form (16) with

$$F(t, \chi, \tilde{\alpha}) := \begin{bmatrix} f_o(x, \zeta, u) - f_o(\pi(\zeta), \zeta, \gamma(\zeta)) \\ f_c(x_c, x, \zeta, \zeta_y) - f_c(\pi_c(\zeta, \zeta_y, \hat{\alpha}), \pi(\zeta), \zeta, \zeta_y) \end{bmatrix}$$
$$g(t, \chi, \tilde{\alpha}) := \begin{bmatrix} 0 \\ -\partial_{\hat{\alpha}} \pi_c(\zeta, \zeta_y, \hat{\alpha}) f(t, \chi, \tilde{\alpha}) \end{bmatrix}$$
$$f(t, \chi, \tilde{\alpha}) := -u_{im}(x, x_c, \zeta, \zeta_y) \begin{bmatrix} y(\zeta, \zeta_y) \\ -x \end{bmatrix}$$

with u given in (22). Notice that the time t argument arises from the exogenous signals $(\zeta, \zeta_y)(t)$. Indeed, we obtain a family of time-varying systems, one for each $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$. The following is the main result of this section, proved in Section 3.3.

Theorem 6. *Consider the system (19)–(20) with the regulator (22), (31) satisfying Assumptions 3–5. Also, suppose that $v(t)$ is uniformly PE in $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ and suppose $\alpha \in \text{int}(\mathcal{B}(\delta_\alpha))$. Let (16) denote the resulting closed-loop dynamics. Then Assumption 2 holds uniformly in $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ and for each residual $\epsilon > 0$ the conclusions of Theorems 4–5 hold. In particular, the result holds for $(x, x_c)(t_0) \in \mathcal{B}(\delta)$, $\hat{\alpha}(t_0) \in \alpha + \mathcal{B}(\delta_s)$, and $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$.*

From Theorem 6 we may conclude the output regulation problem is solved since $\chi \to 0$ implies $e \to 0$.

Remark 10. The additional statement, found in Theorems 4–5, that $\delta_s \to \infty$ as $\delta_\alpha \to \infty$ regardless of the regulator is meant to highlight a (theoretical) design methodology. First, for a collection of nominal $\mathcal{Z}$, $\mathcal{Z}_y$ one selects a desired set of initial conditions for the slow state: $\hat{\alpha}(t_0) \in \mathcal{B}(\delta_s)$. This selection of δ_s in turn determines δ_α characterizing the ball over which the slow averaged states evolve. Next, one constructs a regulator satisfying Assumption 4 assuming a constant $\hat{\alpha}$ lying in the δ_α ball. ◁

Remark 11. To obtain a global version of Theorem 6 all functions should be globally Lipschitz uniformly in (ζ, ζ_y) over compact sets, and the regulator (22) should achieve global exponential stability in (R5). Also we note that continuous differentiability requirements can be relaxed but we do not do so for simplicity and because all considered regulator designs are sufficiently smooth. ◁

3.3. Proof of Theorem 6

Theorem 6 basically follows from Theorems 4–5, so we carry out the necessary set up. First we show that our regulator with reflex adaptation satisfies Assumption 2.

Proposition 3. *If $v(t)$ is uniformly PE in $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ and we have $\alpha \in \text{int}(\mathcal{B}(\delta_\alpha))$, then the resulting closed-loop system (16) satisfies Assumption 2 uniformly in $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$.*

Proof. We verify each condition holds uniformly for $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$.

(C1): Continuous differentiability follows from (R1) and (R4), and piecewise continuity as well as uniformity with respect to $t \geq 0$ follows from the fact that $(\zeta, \zeta_y)(t)$ is generated by an ODE and $\mathcal{Z}$, $\mathcal{Z}_y$ in (R3) are compact and invariant;

(C2): It follows from (R5) and (R8) (implying (29)) that $(\chi, \tilde{\alpha}) = (0, 0)$ is an equilibrium of the considered closed-loop system cast as (16) for all $\epsilon > 0$, implying that $F(\cdot)$, $g(\cdot)$, and $f(\cdot)$ vanish at $(\chi, \tilde{\alpha}) = (0, 0)$;

(C3): Convergence of the averages follows from the existence of the autocovariance matrix in (R9) and linearity of the function $f(t, 0, \tilde{\alpha}) = -v(t)v^\intercal(t)\tilde{\alpha}$ in $\tilde{\alpha}$;

(C4): Since $\alpha \in \text{int}(\mathcal{B}(\delta_\alpha))$, let $\delta'_\alpha(\delta_\alpha) > 0$ define the largest ball such that the shifted ball $\alpha + \mathcal{B}(\delta'_\alpha) \subseteq \mathcal{B}(\delta_\alpha)$. Then $\tilde{\alpha} \in \mathcal{B}(\delta'_\alpha)$ and the relevant stability requirement for the χ dynamics holds for δ_f, $\delta'_\alpha > 0$ by (R5);

(C5): Given that $v(t)$ is uniformly PE for $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ and that the autocovariance matrix $R_v(0)$ exists by (R9), we can use Lemma 1 to conclude $R_v(0)$ is uniformly positive definite and bounded. Then $-R_v(0)$ is uniformly Hurwitz and thus the relevant stability requirement holds because $f_{av}(\tilde{\alpha}_{av}) = -R_v(0)\tilde{\alpha}_{av}$. □

Immediately applying Theorem 5 for each time-varying system obtained by substituting $(\zeta, \zeta_y) \mapsto (\zeta, \zeta_y)(t)$, for each initial condition $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ we have that there exists $\delta_s(\delta'_\alpha, \zeta(t_0), \zeta_y(t_0))$, $\epsilon_\star(\delta, \delta'_\alpha, \zeta(t_0), \zeta_y(t_0)) > 0$ such that for any fixed $\epsilon \in (0, \epsilon_\star)$ the equilibrium $(\chi, \tilde{\alpha}) = (0, 0)$ is ES over $(\chi, \tilde{\alpha})(t_0) \in \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_s)$. But since Assumption 2 holds true uniformly in $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ by Proposition 3, all constants obtained can be selected independent of $(\zeta, \zeta_y)(t_0)$. That is, for the family of time-varying two timescale systems (16) obtained by substituting $(\zeta, \zeta_y) \mapsto (\zeta, \zeta_y)(t)$, there exists $\delta_s(\delta'_\alpha, \mathcal{Z}, \mathcal{Z}_y)$, $\epsilon_\star(\delta, \delta'_\alpha, \mathcal{Z}, \mathcal{Z}_y) > 0$ such that the conclusions of Theorem 5 hold true uniformly for any of the time-varying systems associated to some $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$. An identical reasoning is used when applying Theorem 4.

Returning to Theorem 6, we need to show that if $(x, x_c)(t_0) \in \mathcal{B}(\delta)$ and $\hat{\alpha}(t_0) \in \alpha + \mathcal{B}(\delta_s)$, where δ is provided by (R6), then the exponential stability results above for $(\chi, \tilde{\alpha})$ are valid. To this end, it suffices to show that if we take $(x, x_c)(t_0) \in \mathcal{B}(\delta)$, $(\zeta, \zeta_y)(t_0) \in \mathcal{Z}\times\mathcal{Z}_y$, and $\hat{\alpha}(t_0) \in \alpha+\mathcal{B}(\delta_s)$, then $\chi(t_0) \in \mathcal{B}(\delta_f)$ and $\tilde{\alpha}(t_0) \in \mathcal{B}(\delta_s)$. First, note that $\delta_s < \delta'_\alpha$ by the proof of Theorem 4. Therefore

$$\hat{\alpha}(t_0) \in \alpha + \mathcal{B}(\delta_s) \subseteq \alpha + \mathcal{B}(\delta'_\alpha) \subseteq \mathcal{B}(\delta_\alpha).$$

Then $\chi(t_0) \in \mathcal{B}(\delta_f)$ follows by (R6), and $\tilde{\alpha}(t_0) \in \mathcal{B}(\delta_s)$ is immediate. At last, we point out that if $\delta_\alpha \to \infty$, then $\delta'_\alpha \to \infty$ and so $\delta_s \to \infty$, as desired. Note that this property is independent of the regulator used because the family of $f_{av}(\cdot)$ considered depends solely on $\mathcal{Z}$ and $\mathcal{Z}_y$; that is, each $f_{av}(\cdot)$ is fully characterized by its associated regressor $\upsilon(t)$ which only depends on $\pi(\cdot)$ given by the regulator equations for the open-loop system in (R3) and $(\zeta, \zeta_y)(t_0) \in \mathcal{Z}\times\mathcal{Z}_y$. This concludes the proof.

4. Training reflexes in the LTI case

In this section we illustrate the ideas of Section 3 by invoking the regulator design reviewed in Section 2.4. Consider the SISO LTI system (6). Suppose we have a partial disturbance measurement $y(t) \in \mathbb{R}^p$ whose components are generated by the LTI exosystems

$$\dot{\zeta}_i = S_i\zeta_i \tag{32a}$$

$$y_i = \Gamma_i\zeta_i \tag{32b}$$

where $\zeta_i(t) \in \mathbb{R}^q$ wlog. To relate (32) with (20), we define $\zeta_y := (\zeta_1, \dots, \zeta_p)$ and identify $s_y(\cdot)$ with the matrix $\mathrm{diag}(S_i)$. Using a similar coordinate transformation as for (12) we may write

$$\dot{w}_i = F w_i + G y_i$$
$$y_i = \psi_i^\top w_i$$

for appropriate $\psi_i \in \mathbb{R}^q$.

Assumption 6. The open-loop system (6) satisfies Assumption 1. Additionally, it and the measurement (32) satisfy:

(A7) the matrices S_i only have simple eigenvalues on the $j\omega$-axis;
(A8) the dimension q is interpreted as a known upper bound on the order of each exosystem needed to generate y_i;
(A9) the measurements are e, x, and y.

The next assumption for the LTI case is analogous to (R8) in the nonlinear setting.

Assumption 7. There exists $\alpha = (\alpha_r, \alpha_x)$ such that

$$d = \alpha_x^\top \Pi\zeta - \alpha_r^\top y.$$

We utilize a controller of the form (22f), where u_s is given by (10) or (11). Define $\upsilon := (y, -\Pi\zeta)$. From (27) we know that u_{im} is tasked with rejecting the *residual disturbance* given by (28). For the LTI case considered here, the residual disturbance is given by

$$\begin{aligned} d + \hat{\alpha}^\top\upsilon = \tilde{\alpha}^\top\upsilon &= (\hat{\alpha}_r - \alpha_r)^\top y - (\hat{\alpha}_x - \alpha_x)^\top \Pi\zeta \\ &= \tilde{\alpha}^\top \begin{bmatrix} 0 & \mathrm{diag}(\psi_1^\top, \dots, \psi_p^\top) \\ -\Pi M^{-1} & 0 \end{bmatrix} w \\ &=: \psi^\top(\tilde{\alpha}) w\,, \end{aligned} \tag{33}$$

where $w(t) := (w_0, \dots, w_p)(t) \in \mathbb{R}^{q(p+1)}$. As a result, we choose the adaptive internal model to be

$$\dot{\hat{w}}_0 = F\hat{w}_0 + Gu \tag{34a}$$

$$\dot{\hat{w}}_i = F\hat{w}_i + Gy_i \tag{34b}$$

$$\hat{w} := (\hat{w}_0, \dots, \hat{w}_p) \tag{34c}$$

$$\dot{\hat{\psi}} = -\,\mathrm{sgn}(b)\gamma\hat{e}_0\hat{w} \tag{34d}$$

$$u_{im} = \hat{\psi}^\top\hat{w} \tag{34e}$$

where $\hat{\psi}(t) \in \mathbb{R}^{q(p+1)}$ and $\gamma > 0$. Finally, in accordance with the requirement that the reflex adaptation law is agnostic to the choice of regulator, the reflex adaptation law for $\hat{\alpha}$ remains the one given in (31).

Remark 12. We have elected to use the same filters for $\hat{w}_0$ and $\hat{w}_i$ to keep the notation simple. More generally, the pair (F_i, G_i) could be different for each filter, modulo the controllability and stability requirements discussed for (12). It may seem curious that we include the filters (34b), in the style of Kreisselmeier filters, since afterall the signals y_i are directly available. The key architectural issue at play is that the adaptive internal model should only process persistent components of disturbances measurable through y. In the same vein, only the reflexes should react to brief impulsive disturbances, because only they have the properly adapted reflex gains to respond appropriately. For this reason, we include extra filtering on y before it can be utilized by the adaptive internal model. ◁

4.1. Stability with persistent excitation

The goal of this section is to show that the regulator defined in the previous section with $\hat{\alpha}$ constant satisfies the exponential stability requirement of Assumption 4. First we write the closed-loop dynamics. Define the observer errors $\tilde{w}_0 := \hat{w}_0 - w_0 - b^{-1}Ge_0$, $\tilde{w}_i := \hat{w}_i - w_i$ for $i \in \{1, \dots, p\}$, $\tilde{w} = (\tilde{w}_0, \dots, \tilde{w}_p)$, and $\tilde{\xi} := \kappa^r D_\kappa^{-1}(\hat{\xi} - \xi)$. Also define the parameter error $\tilde{\psi} := \hat{\psi} - \psi(\tilde{\alpha})$, and note that $\psi(\tilde{\alpha}) = \Psi\tilde{\alpha}$ for an appropriate matrix Ψ. Fix $\tilde{\alpha}$ to be constant and let $x_\circ := (z_0, \xi_\circ, \tilde{w}, e_0)$. Then using (9), (11), (22d)–(22f), (33), and (34), we can derive the closed-loop dynamics

$$\begin{bmatrix} \dot{x}_\circ \\ \dot{\tilde{\psi}} \end{bmatrix} = \begin{bmatrix} A(K,\tilde{\alpha}) & bB_\circ\hat{w}^\top \\ -\,\mathrm{sgn}(b)\gamma\hat{w}B_\circ^\top & 0 \end{bmatrix} \begin{bmatrix} x_\circ \\ \tilde{\psi} \end{bmatrix} + B_1(K,\kappa,\hat{w})\tilde{\xi} \tag{35a}$$

$$\dot{\tilde{\xi}} = \kappa(A_\circ - LC_\circ)\tilde{\xi} + B_2(K,\tilde{\alpha},\hat{w}) \begin{bmatrix} x_\circ \\ \tilde{\psi} \end{bmatrix} + B_3(K,\kappa)\tilde{\xi} \tag{35b}$$

where (for some appropriate $\bar{a}_{r0}(\alpha)$, $\bar{a}_{rr}(\alpha)$, T_0, T_r, M_w)

$$A(K,\tilde{\alpha}) := \begin{bmatrix} \begin{bmatrix} A_{00} & A'_{01} \\ 0 & A_\circ - B_\circ a^\top \end{bmatrix} & 0 & B_{0r} \\ \begin{bmatrix} -b^{-1}Ga_{r0}^\top \\ 0 \end{bmatrix} & \mathrm{diag}(F) & \begin{bmatrix} b^{-1}(FG - Ga_{rr}) \\ 0 \end{bmatrix} \\ \bar{a}_{r0}^\top + \tilde{\alpha}^\top T_0 & b\tilde{\alpha}^\top\Psi^\top & \bar{a}_{rr} + \tilde{\alpha}^\top T_r - |b|K \end{bmatrix}$$

and

$$B_1(K,\kappa,\hat{w}) := -\begin{bmatrix} |b|B_\circ K \\ \mathrm{sgn}(b)\gamma\hat{w} \end{bmatrix} \begin{bmatrix} a^\top & 1 \end{bmatrix} \kappa^{-r} D_\kappa$$

$$B_3(K,\kappa) := |b|B_\circ K \begin{bmatrix} a^\top & 1 \end{bmatrix} \kappa^{-r} D_\kappa$$

$$\hat{w} = w(t) + M_w x_\circ\,,$$

and $B_2(K,\tilde{\alpha},\hat{w})$ is continuous uniformly in $t \geq t_0 \geq 0$. We proceed by fixing some $(\zeta, \zeta_y)(t_0)$ (implying a fixed $w(t_0)$) and assuming that the associated exogenous regressor $w(t)$ is PE.

The next result, related to Lemma 4, shows that the system (35a) without the high-gain observer is exponentially stable in every closed ball of initial conditions; see Proposition 1.

Lemma 5. *Consider system (35a) with $\tilde{\xi} = 0$. Also, suppose $w(t)$ is PE and $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$ for some $\delta_\alpha > 0$. There exists $K_\star(\delta_\alpha) > 0$ such that for every $K \geq K_\star$, the equilibrium $(x_\circ, \tilde{\psi}) = (0, 0)$ is GUAS and LES uniformly in $\tilde{\alpha}$.*

Proof. The proof is a minor variation of that of Lemma 4; see Section 9. We select $K_\star(\delta_\alpha)$ according to Lemma 3 such that for all $K \geq K_\star$, the matrix $A(K, \tilde{\alpha})$ is Hurwitz uniformly in $\tilde{\alpha}$. Using the same stability argument, one deduces that $(x_\circ, \tilde{\psi}) = (0, 0)$ is GUAS uniformly in $\tilde{\alpha}$. For the exponential stability argument, we notice that the linearization is GES uniformly in $\tilde{\alpha}$ because P and ρ in Lemma 3 are independent of $\tilde{\alpha}$. □

The main result on stability of (35) is an extension of Theorem 3 stating that for any initial conditions, one can build an observer with sufficiently high gain.

Theorem 7. *Consider system (35). Also, suppose $w(t)$ is PE and $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$ for some $\delta_\alpha > 0$. For each $\delta_1 > 0$ there exists $K_\star(\delta_\alpha) > 0$ and $\kappa_\star(K, \delta_1, \delta_\alpha) \geq 1$ such that for any fixed $K \geq K_\star$ and $\kappa \geq \kappa_\star$ the equilibrium $(x_\circ, \tilde{\psi}, \tilde{\xi}) = (0,0,0)$ is ES over $(x_\circ, \tilde{\psi})(t_0) \in \mathcal{B}(\delta_1)$ and $\tilde{\xi}(t_0) \in \mathcal{B}(\delta_1 \kappa^{r-1})$ uniformly in $\tilde{\alpha}$.*

Proof. The proof of Theorem 3 in Section 9 may be applied, with a few added notes. First, when applying Teel and Praly (1995, Lemma 2.4), the uniform Lyapunov property (Teel & Praly, 1995, Assumption ULP) is satisfied by $V_1(x_\circ, \tilde{\psi})$ (Lin, Sontag, & Wang, 1996, Theorem 2.9) because the equilibrium $(x_\circ, \tilde{\psi}) = (0,0)$ is GUAS uniformly in $\tilde{\alpha}$ by Lemma 5 and since the dynamics (35a) with $\tilde{\xi} = 0$ are locally Lipschitz in $(x_\circ, \tilde{\psi})$ uniformly in $(w, \tilde{\alpha}) \in \mathcal{W} \times \mathcal{B}(\delta_\alpha)$.

Second, because of Lemma 5, we can invoke a converse Lyapunov function $V_2(t, x_\circ, \tilde{\psi}; \tilde{\alpha}) \;:\; [0,\infty) \times \mathcal{B}(\delta_2) \times \mathcal{B}(\delta_\alpha) \to \mathbb{R}_+$ satisfying the conclusions of Khalil (2002, Theorem 4.14) with constants $c_i > 0$ that hold uniformly for all $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$. As such, the remaining arguments are independent of $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$. □

4.2. Proof of correctness

We can finally show that our LTI design solves the reflex adaptation problem (23) of offloading the internal model through appropriate adaptation of the reflex gain $\hat{\alpha}$. The main work, in addition to verifying Assumptions 3 and 5, is to show that the internal model design (34) as well as the stabilizer (11) satisfy all the conditions of Assumption 4. Then we invoke Theorem 6 to conclude the result. Note that in the following statement we take the sets $\mathcal{Z}$ and $\mathcal{Z}_y$ to be orbits of the exosystems (19b) and (20a) in order to guarantee a uniform PE assumption.

Theorem 8. *Consider system (6) with the regulator (11) (or (10)), (22d)–(22f), (31), (34) satisfying Assumption 1, 6–7. Also, fix orbits $\mathcal{Z}$, $\mathcal{Z}_y$ of the exosystems and suppose that $w(t)$ and $v(t)$ are each PE. Define $x_c := (\hat{w}, \hat{\xi}, \hat{\psi})$. Then for each $\delta > 0$ one may instantiate one such regulator so that Assumptions 3–5 hold and the coordinate transformation (24) yields a closed-loop system of the form (16). Therefore, for each regulator built the conclusions of Theorem 6 hold.*

Proof. The result essentially follows from Theorem 6, so we need to verify the relevant assumptions hold. We begin with Assumption 3.

(R1): Continuous differentiability follows from linearity of the plant and exosystems;

(R2): Neutral stability follows from the fact that S and each S_i only have simple eigenvalues on the $j\omega$-axis by (A2), (A7);

(R3): Uniqueness of $\pi(\zeta) := \Pi\zeta$ and $\gamma(\zeta) := \Gamma\zeta$ (defined for all $\zeta \in \mathbb{R}^q$) follows from the non-resonance List (A3) (Saberi, Stoorvogel, & Sannuti, 2000, Ch. 2.5). Note that since $\mathcal{Z}$ and $\mathcal{Z}_y$ are orbits of a neutrally stable exosystem, they are compact and invariant.

Next we verify Assumption 5.

(R8): Equivalent to Assumption 7;

(R9): Since the output of an LTI exosystem is almost periodic in t, Hale (1980, Appendix, Theorem 6) gives us that the autocovariance matrix of $v(t)$ exists with convergence uniform $t_0 \in \mathbb{R}$. Uniform convergence with respect to $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ follows from the fact that $\mathcal{Z}$ and $\mathcal{Z}_y$ are orbits and our exosystems are time-invariant. That is, any two trajectories of, e.g., (19b) in $\mathcal{Z}$ must be related by a time shift, which is dealt with by uniformity with respect to $t_0 \in \mathbb{R}$ of the convergence of the autocovariance matrix.

Before verifying Assumption 4, for a fixed $\mathcal{Z} \times \mathcal{Z}_y$ and $\delta > 0$, we need to construct a regulator with sufficiently high gains K and κ. To this end, we must first determine sufficiently large constants δ_f, $\delta_\alpha > 0$ such that (R6) holds. To begin, note that $v(t)$ is uniformly PE with respect to $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$ because the PE property is time shift invariant and we have selected $\mathcal{Z}$, $\mathcal{Z}_y$ to be orbits. Additionally, $v(t)$ is uniformly bounded by compactness of $\mathcal{Z}$ and $\mathcal{Z}_y$. As a result, using the final part of Theorem 6 which is independent of the regulator $u_s + u_{im}$ we are to build, there exists $\delta_s(\cdot)$ such that

$$\lim_{\delta_\alpha \to \infty} \delta_s(\delta_\alpha, \mathcal{Z}, \mathcal{Z}_y) = \infty.$$

Next, remark that our initial conditions satisfy

$$(x, x_c, \hat{\alpha})(t_0) \in \mathcal{B}(\delta) \implies (x, x_c)(t_0) \in \mathcal{B}(\delta)\,, \;\; \hat{\alpha}(t_0) \in \mathcal{B}(\delta).$$

As such, we pick $\delta_\alpha(\delta, \mathcal{Z}, \mathcal{Z}_y) > 0$ sufficiently large such that

$$\mathcal{B}(\delta) \subseteq \alpha + \mathcal{B}(\delta_s)\,, \quad \alpha \in \mathrm{int}(\mathcal{B}(\delta_\alpha)).$$

Given that (R6) uses the (z, z_c) error states, we need to construct the continuously differentiable function $\pi_c(\cdot)$ capturing the steady-state behavior of the regulator for any constant $\hat{\alpha}$. As suggested by Theorem 7, if $\tilde{\alpha}$ is constant then we could design a regulator such that $(x_\circ, \tilde{\psi}, \tilde{\xi}) \to (0,0,0)$. It is quite direct to see that this implies

$$\begin{aligned} &\tilde{\xi} \to 0 \implies \hat{\xi} \to \xi \\ &\tilde{\psi} \to 0 \implies \hat{\psi} \to \psi(\tilde{\alpha}) \\ &x_\circ \to 0 \implies \xi_\circ \to 0 \text{ and } e_0 \to 0 \implies \xi \to 0\,. \end{aligned}$$

Furthermore, we have that

$$x_\circ \to 0 \implies \tilde{w} \to 0 \text{ and } e_0 \to 0 \implies \hat{w} \to w = \bar{M}\begin{bmatrix}\zeta \\ \zeta_y\end{bmatrix}$$

for some invertible $\bar{M}$. Therefore, we define

$$\pi_c(\zeta, \zeta_y, \hat{\alpha}) = (\bar{M}\begin{bmatrix}\zeta \\ \zeta_y\end{bmatrix}, 0, \psi(\tilde{\alpha})).$$

Since $\mathcal{Z}$, $\mathcal{Z}_y$ are compact and $\pi_c(\cdot)$ is (at least) continuous, we conclude that there exists $\delta_f(\delta, \delta_\alpha, \mathcal{Z}, \mathcal{Z}_y) > 0$ such that (R6) holds, where $z_c := x_c - \pi_c(\zeta, \zeta_y, \alpha)$.

To satisfy (R5), for which we have already determined δ_f, δ_α, and $\pi_c(\cdot)$, we need to build a regulator whose basin of attraction (with exponential stability) contains $(z, z_c)(t_0) \in \mathcal{B}(\delta_f + 1)$. For this, note that the change of coordinates

$$(z, z_c, \zeta, \zeta_y, \hat{\alpha}) \mapsto (x_\circ, \tilde{\psi}, \tilde{\xi}, \zeta, \zeta_y, \hat{\alpha})$$

is in fact a diffeomorphism. As such, and because $\|\kappa^r D_\kappa^{-1}\| \leq \kappa^{r-1}$, there exists a $\delta_1(\delta_f, \delta_\alpha, \mathcal{Z}, \mathcal{Z}_y) > 0$ such that if $(z, z_c)(t_0) \in \mathcal{B}(\delta_f + 1)$, $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$, and $\hat{\alpha} \in \mathcal{B}(\delta_\alpha)$, then

$$(x_\circ, \tilde{\psi})(t_0) \in \mathcal{B}(\delta_1)\,, \quad \tilde{\xi}(t_0) \in \mathcal{B}(\delta_1 \kappa^{r-1})$$

for all $\kappa \geq 1$. We are now in a position to apply Theorem 7. Similar to the reasoning presented for $v(t)$ earlier, we have that $w(t)$ is uniformly PE and uniformly bounded. Therefore, the proof of Theorem 7 proceeds as before with the addition that all relevant properties are uniform in $\tilde{\alpha}$ and $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$. Hence there exists

$$\begin{aligned} K_\star(\delta, \mathcal{Z}, \mathcal{Z}_y) &:= K_\star(\delta_\alpha) > 0 \\ \kappa_\star(K, \delta, \mathcal{Z}, \mathcal{Z}_y) &:= \kappa_\star(K, \delta_1, \delta_\alpha) \geq 1 \end{aligned}$$

such that (R5) is met for any $K \geq K_\star$ and $\kappa \geq \kappa_\star$. Then for any appropriate fixed K, κ we verify Assumption 4.

(R4): Clearly the controller consisting of (11) (or (10)), (22f), and (34) is continuously differentiable;

(R5): Since $\psi(\tilde{\alpha})$ is a linear function of $\tilde{\alpha}$, it is easy to see that $\pi_c(\cdot)$ is continuously differentiable. With a bit of algebra, one can show that there exists an invertible matrix M_κ such that

$$\begin{bmatrix} z \\ z_c \end{bmatrix} = M_\kappa \begin{bmatrix} x_\circ \\ \tilde{\xi} \\ \tilde{\psi} \end{bmatrix}.$$

Therefore the required exponential stability follows from the particular choices we made for δ_α, δ_f, and δ_1 in conjunction with Theorem 7;

(R6): The fact that δ satisfies the desired property is due to the choice of δ_f;

(R7): Again by the steady-state analysis used to construct $\pi_c(\cdot)$, we have $\hat{\xi} \to 0$ and so $u_s \to 0$.

With all the assumptions met, we invoke Theorem 6 to conclude that for each $K \geq K_\star$ and $\kappa \geq \kappa_\star$ (which characterize the regulator being employed) the result holds. □

Remark 13. Since z is a component of χ, one deduces that $e = Cz \to 0$. Therefore, we have error regulation. ◁

In the previous two sections we developed a two timescale control architecture in which an adaptive internal model trains feedforward and state feedback reflex gains in a process of long-term adaptation. Whenever needed, we assumed regressors appearing in parameter adaptation laws are PE. This unrealistic assumption is arguably the most egregious departure from a real biological setting. The next section develops biologically inspired methods to remove the PE assumption, yet still retain robustness of the adaptation laws.

5. The value of forgetting: A robustness mechanism

The idea that forgetting can be used as a means for robustness is not new. This can be seen in the literature through the use of the σ-modification in adaptive control and leaky integrators for PID control. What has eluded researchers for quite some time is the possibility of using forgetting to introduce robustness without hindering the learning process, at least without introducing additional assumptions. In our recent works (Uzeda & Broucke, 2022, 2023), we emphasized that if one adopts the *Use it or Lose it Principle* from neuroplasticity (Kleim & Jones, 2008), then one may obtain robustness without sacrificing error regulation via a suitable forgetting term. Intuitively, the principle states that:

> "Adaptation dynamics not excited by the regressor driving the learning process should be gradually forgotten, as they are ultimately not needed".

In the upcoming developments, we demonstrate how one goes about systematically accounting for the lack of persistent excitation of regressors. The proposed method is called the μ-modification, and it does not require a priori information about unknown parameters nor does it resort to weaker notions of excitation.

Remark 14. While our developments will focus on persistent excitation, the ideas to be presented can be adapted to work with other notions of excitation, such as initial excitation (Roy & Bhasin, 2019). Since we are primarily interested in learning from the environment using exogenous signals, the PE condition is suitable for our purposes. For tasks such as adaptive stabilization, where the regressor is a state being driven to the origin, initial excitation appears to be the more appropriate notion of excitation to leverage. This is because persistent excitation of the state would be in direct conflict with the control objective of stabilizing the origin. ◁

5.1. Illustrative example

To put our ideas into context, consider the simple example of linear regression. Suppose we have a measurement $r(t) = w^\intercal(t)\psi$ where $w(t)$ is a measured regressor and ψ is the unknown parameter we would like to identify. A common approach to identify ψ, prototypical of the myriad adaptive algorithms in the literature (Ioannou & Sun, 2012), is to employ the *standard gradient algorithm*

$$\dot{\hat{\psi}} = -\gamma e w(t)\,, \tag{36}$$

where $\hat{\psi}(t)$ is the parameter estimate, $e = w^\intercal(t)\hat{\psi} - r(t)$ is an error signal that needs to be regulated to zero, and $\gamma > 0$ is the adaptation gain. When $w(t)$ is PE, Theorem 1 tells us that the equilibrium $\hat{\psi} = \psi$ of (36) is GES. In turn, this implies that we have robustness with respect to additive disturbances (Khalil, 2002, Lemma 4.6); namely, bounded noise does not cause unbounded growth of $\hat{\psi}$.

More often than not, the regressor $w(t)$ will not be PE. This issue is further exacerbated by the fact that the PE condition is difficult to verify given that it must hold for the entire time horizon. When one does not have persistent excitation, even small disturbances can cause unbounded growth of $\hat{\psi}$. To see this, suppose we have a non-PE regressor

$$w(t) = \sin(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{W} := \operatorname{Im}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and consider the coordinate transformation

$$\begin{bmatrix} \hat{\psi}_{pe} \\ \hat{\psi}_\perp \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\hat{\psi}\,, \tag{37}$$

with an analogous coordinate transformation for the unknown parameter ψ. Observe that we have selected (37) such that the first component $\hat{\psi}_{pe}$ is the projection of $\hat{\psi}$ along the subspace $\mathcal{W}$ spanned (or excited) by the regressor $w(t)$, and $\hat{\psi}_\perp$ is the (unexcited) component along its orthogonal complement $\mathcal{W}^\perp$. As a result, (36) becomes

$$\dot{\hat{\psi}}_{pe} = -\gamma e w_{pe}(t) \tag{38a}$$

$$\dot{\hat{\psi}}_\perp = 0 \tag{38b}$$

where $w_{pe}(t) = \sqrt{2}\sin(t)$ is PE and

$$e = w^\intercal(t)(\hat{\psi} - \psi) = w_{pe}^\intercal(t)(\hat{\psi}_{pe} - \psi_{pe}),$$

where $\psi_{pe} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\psi$. Again by Theorem 1, the equilibrium $\hat{\psi}_{pe} = \psi_{pe}$ of (38a) is GES and so $e \to 0$. In contrast, the $\hat{\psi}_\perp$ dynamics are stable but not asymptotically stable. To emphasize the resulting lack of robustness, note that if we introduce a bounded disturbance $n(t)$ then (38b) becomes the open-loop integrator dynamics

$$\dot{\hat{\psi}}_\perp = n(t),$$

which can cause unbounded growth of $\hat{\psi}$. For example, picking $n(t) = n_0 > 0$ implies that

$$\lim_{t\to\infty} \hat{\psi}_\perp(t) = \hat{\psi}_\perp(t_0) + \lim_{t\to\infty} n_0(t - t_0) = \infty.$$

The form of (38) and the error signal e indicates rather clearly what should be done for robustness. Referring back to the Use it or Lose it Principle: only the PE dynamics $\hat{\psi}_{pe}$ are used by the learning process, implying that the non-PE dynamics $\hat{\psi}_\perp$ should be gradually forgotten. In particular, forgetting $\hat{\psi}_\perp$ can be accomplished through the introduction of a leakage term in (38b). Let $\Omega\Omega^\intercal$ be the orthogonal projection matrix onto $\mathcal{W}^\perp$, given by

$$\Omega\Omega^\intercal = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We modify (36) along the subspace $\mathcal{W}^\perp$ to obtain

$$\dot{\hat{\psi}} = -\gamma e w(t) - \mu\Omega\Omega^\intercal\hat{\psi}\,, \tag{39}$$

with $\mu > 0$. Applying the coordinate transformation (38), the system becomes

$$\dot{\hat{\psi}}_{pe} = -\gamma e w_{pe}(t) \tag{40a}$$

$$\dot{\hat{\psi}}_{\perp} = -\mu \hat{\psi}_{\perp} \tag{40b}$$

Since the $\hat{\psi}_{pe}$ and $\hat{\psi}_{\perp}$ dynamics are decoupled, we see that the equilibrium $(\hat{\psi}_{pe}, \hat{\psi}_{\perp}) = (\psi_{pe}, 0)$ of (40) is GES and moreover $e \to 0$. Hence we have managed to render the adaptation scheme (36) robust through a targeted forgetting mechanism so that bounded disturbances do not induce unbounded growth of the state $\hat{\psi}$.

Suppose we did not already have in hand the orthogonal projection matrix $\Omega\Omega^{\intercal}$, but rather some arbitrary matrix $\Omega_{\circ}$ with the property that $\mathrm{Im}(\Omega_{\circ}) = \mathcal{W}^{\perp}$. Then one can construct the orthogonal projection onto $\mathcal{W}^{\perp}$ using the following known property of the Moore–Penrose inverse (matrix pseudoinverse), denoted †.

Proposition 4. *Let $\Omega_{\circ}$ be a matrix. Then $\Omega_{\circ}\Omega_{\circ}^{\dagger}$ is the orthogonal projection matrix onto $\mathrm{Im}(\Omega_{\circ})$.*

Finally, there is the more realistic case when only an estimate of $\Omega_{\circ}$ is available. Since the pseudoinverse does not play well with vanishing perturbations, it is unfortunately not adequate to compute the orthogonal projection. To illustrate, consider the example

$$\lim_{t\to\infty}\left(\begin{bmatrix}1 & 0\\ 0 & t^{-1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & t^{-1}\end{bmatrix}^{\dagger}\right) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

where the limit of the pseudoinverse does not coincide with the pseudoinverse of the limit

$$\left(\lim_{t\to\infty}\begin{bmatrix}1 & 0\\ 0 & t^{-1}\end{bmatrix}\right)\left(\lim_{t\to\infty}\begin{bmatrix}1 & 0\\ 0 & t^{-1}\end{bmatrix}\right)^{\dagger} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

In the sequel, we present a more in-depth treatment providing the relevant tools allowing one to extend these ideas to more sophisticated adaptive systems, such as those presented in the prior section.

5.2. PE decomposition

In Section 5.1 the coordinate transformation (37) enabled a clear interpretation of how the Use it or Lose it Principle could be applied to obtain robust parameter adaptation via the μ-modification. This coordinate transformation was based on explicit and geometric excitation properties of the regressor $w(t)$. The first order of business of this section will be to extract an analogous geometric characterization for an arbitrary regressor, resulting in a classification of regressor excitation that goes beyond simply PE or not.

Assumption 8. The regressor $w(t) \in \mathbb{R}^q$ is bounded, piecewise continuous, and its autocovariance matrix (2) exists with convergence uniform in $t_0 \geq 0$.

Definition 3. The *PE subspace* $\mathcal{W}$ of the regressor $w(t)$ is

$$\mathcal{W} := \mathrm{Im}(R_w(0))$$

and its *non-PE subspace* $\mathcal{W}^{\perp}$ is

$$\mathcal{W}^{\perp} := \mathrm{Ker}(R_w^{\intercal}(0)) = \mathrm{Ker}(R_w(0)).$$

We denote $q_{pe} := \dim(\mathcal{W})$ as its *degree of persistent excitation*.

When $q_{pe} = q$, $w(t)$ is PE by Lemma 1. When $q_{pe} = 0$, we say $w(t)$ has *no persistent excitation*. In the general case, we can perform a *PE decomposition* of the regressor into a PE component and a component with no persistent excitation.

Proposition 5. *Suppose Assumption 8 holds. If $1 \leq q_{pe} < q$, let $\begin{bmatrix}W & W_{\perp}\end{bmatrix} \in \mathbb{R}^{q\times q}$ be any orthogonal matrix such that*

$$\mathcal{W} = \mathrm{Im}(W), \quad \mathcal{W}^{\perp} = \mathrm{Im}(W_{\perp}).$$

Then the PE decomposition

$$w = WW^{\intercal}w + W_{\perp}W_{\perp}^{\intercal}w =: Ww_{pe} + W_{\perp}w_{\perp}$$

exists, where $w_{pe}(t) \in \mathbb{R}^{q_{pe}}$ is PE and $w_{\perp}(t) \in \mathbb{R}^{q-q_{pe}}$ has no persistent excitation. Moreover, we can select $W = I$ and $W_{\perp} = 0$ if $w(t)$ is PE ($q_{pe} = q$), and $W = 0$ and $W_{\perp} = I$ if $w(t)$ has no persistent excitation ($q_{pe} = 0$).

Proof. By orthogonality, we have $I = WW^{\intercal} + W_{\perp}W_{\perp}^{\intercal}$ and so we let $w_{pe} := W^{\intercal}w$ and $w_{\perp} := W_{\perp}^{\intercal}w$. Computing the autocovariance matrices, we have

$$R_{w_{pe}}(0) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} w_{pe}(\tau)w_{pe}^{\intercal}(\tau)\,d\tau = W^{\intercal}R_w(0)W$$

$$R_{w_{\perp}}(0) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} w_{\perp}(\tau)w_{\perp}^{\intercal}(\tau)\,d\tau = W_{\perp}^{\intercal}R_w(0)W_{\perp}$$

where convergence of the averages is uniform in $t_0 \geq 0$. Since $\mathrm{Ker}(R_w(0)) = \mathcal{W}^{\perp} = \mathrm{Im}(W_{\perp})$ by definition, it is immediate that $R_{w_{\perp}}(0) = 0$. Therefore, $w_{\perp}(t)$ has no persistent excitation.

To show that $w_{pe}(t)$ is PE, by Sastry and Bodson (1989, Proposition 2.7.1) it suffices to show $R_{w_{pe}}(0) \succ 0$. Given that $R_{w_{pe}}(0)$ is always positive semi-definite, we only need to show it has full rank. By symmetry of $R_w(0)$, we have that

$$R_w(0) = IR_w(0)I = WW^{\intercal}R_w(0)WW^{\intercal} = WR_{w_{pe}}(0)W^{\intercal}.$$

As a result, we obtain the inequality

$$\begin{aligned} q_{pe} = \mathrm{rank}(R_w(0)) &\leq \min\{\,\mathrm{rank}(W),\, \mathrm{rank}(R_{w_{pe}}(0)),\, \mathrm{rank}(W^{\intercal})\,\} \\ &= \min\{\,q_{pe},\, \mathrm{rank}(R_{w_{pe}}(0))\,\}. \end{aligned}$$

Hence $R_{w_{pe}}(0) \in \mathbb{R}^{q_{pe}\times q_{pe}}$ must satisfy $\mathrm{rank}(R_{w_{pe}}(0)) \geq q_{pe}$, implying that it is full rank and so $w_{pe}(t)$ is PE. □

In general, the component $w_{\perp}$ does not vanish. A more in depth discussion is found in Uzeda and Broucke (2022, Section II). To keep our developments concise, we introduce the following regularity assumption, which holds for many classes of signals of interest to control theorists and neuroscientists.

Assumption 9. The component $w_{\perp}$ of the regressor $w(t)$ along its non-PE subspace satisfies $w_{\perp} = 0$.

5.3. Error model

We are ready to set up the μ-modification based on the PE decomposition of the previous section. However, we want to consider a more general error model than the simple linear regression in Section 5.1. To that end, we must identify an error model that captures salient features of those in the adaptive control literature (Narendra & Annaswamy, 1989). There are two key observations. First, we consider the classical case when the dynamics can be parameterized linearly in the unknown parameters (Annaswamy & Fradkov, 2021). Second, despite many different methods to implement adaptive controllers, ultimately, the form of the closed-loop system determines the structure of interest. An example of the latter point is that the open-loop plant may be presented with the unknown parameters in matched or unmatched form. After applying a suitable control technique, say backstepping, one arrives at a closed-loop error model in which the parameter estimate is matched with the unknown parameter (Krstic, Kanellakopoulos, & Kokotovic, 1995).

Based on our two key observations, we consider an error model denoted

$$e = \mathscr{E}[\hat{\psi}, w(t), v]$$

where $w(t)$ is an exogenous regressor and v is a vanishing perturbation. The closed-loop system with adaptation can be written as a time-varying system with error dynamics

$$\dot{\xi} = A(t,\xi) + B(t)(\hat{w}^\intercal \hat{\psi} - w^\intercal \psi) \tag{41a}$$

$$e = C(t,\xi) + D(t)(\hat{w}^\intercal \hat{\psi} - w^\intercal \psi) \tag{41b}$$

using the adaptation law

$$\dot{\hat{\psi}} = -\gamma e \hat{w} \tag{42}$$

and regressor estimate

$$\dot{v} = \Delta(t,v) \tag{43a}$$

$$\hat{w} = w(t) + \tilde{w}(t,v) \tag{43b}$$

where $\xi(t) \in \mathbb{R}^n$ is the error state, $e(t) \in \mathbb{R}$ is a scalar error signal, $\hat{\psi}(t) \in \mathbb{R}^q$ is an estimate of the true parameter ψ, and $v(t) \in \mathbb{R}^v$ is the perturbation state. The main structural feature of the error model is how the parameters always appear multiplied with a regressor. As we will see shortly, in conjunction with the PE decomposition, this will enable a partition between excited and non-robust dynamics reminiscent of (38).

Remark 15. In scenarios when adaptive backstepping is used or in robotics applications, the regressor w is generally a matrix rather than a vector. Consequently, the associated error signal for adaptation is generally a vector rather than a scalar. We do not consider this scenario because special care is needed when associating the excitation of matrix regressors with non-robust adaptation, which is the subject of future work. ◃

Remark 16. We will restrict our attention to scalar adaptation gains $\gamma > 0$. Positive definite time-varying gain matrices could equally be considered, but we do not do so to keep the exposition direct. ◃

Remark 17. A more general error model is considered in Uzeda and Broucke (2022) which replaces w with a new regressor $w_\circ$ in the ξ dynamics. The new regressor $w_\circ$ is meant to capture the fact that the regressor interacting with ψ in the ξ dynamics may not coincide with the regressor w used for adaptation. One example is the adaptation of LTI systems using output feedback without the SPR condition, also known as *error model 4* in Narendra and Annaswamy (1989, Ch. 7.5). ◃

Next we state technical assumptions on the model to appropriately constrain its structure and nominal stability properties.

Assumption 10. The system (41)–(43) satisfies:

(E1) the regressor $w(t)$ satisfies Assumptions 8–9;
(E2) the functions $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are piecewise continuous in t and continuously differentiable in ξ uniformly in $t \geq t_0 \geq 0$. Moreover, $A(\cdot)$ and $C(\cdot)$ are globally Lipschitz in ξ uniformly in $t \geq t_0 \geq 0$, and $B(\cdot)$ and $D(\cdot)$ are bounded;
(E3) the functions $\Delta(\cdot)$ and $\tilde{w}(\cdot)$ are piecewise continuous in t and continuously differentiable in v uniformly in $t \geq t_0 \geq 0$.
(E4) the functions satisfy $A(t,0) = 0$, $C(t,0) = 0$, $\Delta(t,0) = 0$, and $\tilde{w}(t,0) = 0$ for all $t \geq t_0 \geq 0$;
(E5) the equilibrium $v = 0$ of (43a) is GUAS and LES;
(E6) the equilibrium $\xi = 0$ of $\dot{\xi} = A(t,\xi)$ is GES;
(E7) given any dimension $q \in \mathbb{N}$, if $w(t) \in \mathbb{R}^q$ is PE and $v = 0$ then the equilibrium $(\xi,\hat{\psi}) = (0,\psi)$ of

$$\dot{\xi} = A(t,\xi) + B(t)w^\intercal(t)(\hat{\psi} - \psi)$$
$$e = C(t,\xi) + D(t)w^\intercal(t)(\hat{\psi} - \psi)$$
$$\dot{\hat{\psi}} = -\gamma e w(t)$$

is GES.

An important feature of Assumption 10 is that the only stability properties we ask for in (E6)–(E7) deal with the two extreme cases most amenable to analysis: the unforced closed-loop system and a PE regressor. To see how the main ideas from the μ-modification presented in Section 5.1 carry over to the proposed error model, consider the *unperturbed* case when $v = 0$, and suppose $w(t)$ has a non-zero degree of persistent excitation. By (E1) we can invoke the PE decomposition to obtain

$$w = W w_{pe}, \quad \mathcal{W} = \mathrm{Im}(W).$$

Applying the coordinate transformation

$$\begin{bmatrix} \hat{\psi}_{pe} \\ \hat{\psi}_\perp \end{bmatrix} = \begin{bmatrix} W^\intercal \\ W_\perp^\intercal \end{bmatrix} \hat{\psi}$$

to (41)–(42), one has the error dynamics

$$\dot{\xi} = A(t,\xi) + B(t)w_{pe}^\intercal(t)(\hat{\psi}_{pe} - W^\intercal \psi)$$
$$e = C(t,\xi) + D(t)w_{pe}^\intercal(t)(\hat{\psi}_{pe} - W^\intercal \psi)$$

and adaptation dynamics

$$\dot{\hat{\psi}}_{pe} = -\gamma e w_{pe}(t)$$
$$\dot{\hat{\psi}}_\perp = 0\,.$$

By (E7) we have that the equilibrium $(\xi,\hat{\psi}_{pe}) = (0, W^\intercal \psi)$ is GES. Similar to (38), we also have that the $\hat{\psi}_\perp$ dynamics are decoupled from the learning process and must be gradually forgotten if one is to achieve robust adaptation. According to the μ-modification, we will accomplish this through the construction of a subspace estimator for the non-PE subspace $\mathcal{W}^\perp$ and an appropriate leakage term.

5.4. Subspace estimation

We have seen in Section 5.1 that to apply the μ-modification to achieve robust parameter adaptation, we require information about the non-PE subspace $\mathcal{W}^\perp$, particularly its orthogonal projection matrix. In Section 5.2 we saw that this subspace is defined in terms of the auto-covariance of the regressor; namely $\mathcal{W}^\perp = \mathrm{Ker}(R_w(0))$. In this section we address the fact that $R_w(0)$ is not known and must be estimated. Further, the regressor $w(t)$ is generally not directly measurable. Thus, we suppose we have a regressor estimate

$$\hat{w} = w(t) + \tilde{w}, \quad w(t) \in \mathbb{R}^q$$

satisfying the following.

Assumption 11. The transient satisfies $\lim_{t\to\infty} \tilde{w}(t) = 0$.

To obtain an estimate of $R_w(0)$ and ultimately $\mathcal{W}^\perp$, it makes sense from an averaging perspective (see Section 2.5) to consider the filter

$$\dot{\hat{\Sigma}} = -\varepsilon \hat{\Sigma} + \varepsilon \hat{w}\hat{w}^\intercal \tag{44}$$

with $\varepsilon > 0$. The motivation for this choice of filter is that the averaged dynamics of (44) are

$$\dot{\Sigma}_{av} = -\varepsilon \Sigma_{av} + \varepsilon R_w(0)\,. \tag{45}$$

The filter (44) has also be utilized in Kreisselmeier (1977) and Tomei and Marino (2022) with slightly different motivations. We want to get an estimate of the mismatch between $\hat{\Sigma}$ and Σ_{av}. First we notice that the equilibrium $\Sigma_{av} = R_w(0)$ of (45) is GES. However, $\hat{\Sigma} = R_w(0)$ is generally not an equilibrium of (44) since $w(t)$ is time-varying. Therefore, (44) will at best provide an approximation of $R_w(0)$. To gain some insight about the quality of that approximation, we compare $\hat{\Sigma}$ to the following exogenous system

$$\dot{\Sigma} = -\varepsilon \Sigma + \varepsilon w(t)w^\intercal(t)\,, \quad \Sigma(t_0) = R_w(0)\,. \tag{46}$$

First, we establish the main properties of Σ.

Lemma 6. *Consider system (46) satisfying Assumptions 8–9. Let $q_{pe} = \dim(\mathcal{W}) \geq 1$, $w_{pe}(t)$ and $\mathcal{W} = \mathrm{Im}(W)$ result from the PE decomposition of $w(t)$, and let $\beta_0 > 0$ be the lower PE bound in Definition 2 for $w_{pe}(t)$. Then there exists a bounded symmetric matrix $\Lambda_{pe}(t) \in \mathbb{R}^{q_{pe}\times q_{pe}}$ such that*

$$\Sigma(t) = W \Lambda_{pe}(t) W^\intercal.$$

Moreover, there exists a class-$\mathcal{K}$ function $\delta(\cdot)$ and a constant $\varepsilon_\star > 0$ such that

$$\|\Sigma(t) - R_w(0)\| \leq \delta(\varepsilon)\,, \quad \Lambda_{pe}(t) \geq \left(\beta_0 - \delta(\varepsilon)\right) I$$

for all $\varepsilon \in (0, \varepsilon_\star]$ and $t \geq t_0 \geq 0$. Otherwise, if $q_{pe} = 0$ then $R_w(0) = \Sigma = 0$.

Proof. The case when $q_{pe} = 0$ is trivial and thus omitted. Let W and $W_\perp$ result from the PE decomposition of $w(t)$. By pre- and post-multiplying (46) by $W_\perp$, we have

$$W_\perp^\intercal \dot{\Sigma} = -\varepsilon W_\perp^\intercal \Sigma\,, \quad W_\perp^\intercal \Sigma(t_0) = 0$$
$$\dot{\Sigma} W_\perp = -\varepsilon \Sigma W_\perp\,, \quad \Sigma(t_0) W_\perp = 0$$

since $w_\perp = 0$ by Assumption 9 and $\mathrm{Im}(W_\perp) = \mathrm{Ker}(R_w(0)) = \mathrm{Ker}(R_w^\intercal(0))$. Recalling that $I = WW^\intercal + W_\perp W_\perp^\intercal$, it follows that

$$\Sigma(t) = WW^\intercal \Sigma(t) WW^\intercal.$$

Then $\Lambda_{pe} := W^\intercal \Sigma W$ is bounded because $\varepsilon > 0$ and $w(t)$ is bounded by Assumption 8. Note that Λ_{pe} is symmetric given that $\Sigma(t_0)$ is symmetric.

Next, consider the GES averaged dynamics (45) with initial condition $\Sigma_{av}(t_0) = R_w(0)$. By the Hovering Theorem (Sanders, Verhulst, & Murdock, 2007, Theorem 5.5.1), there exists a class-$\mathcal{K}$ function $\delta(\cdot)$ and a constant $\varepsilon_\star > 0$ such that

$$\delta(\varepsilon) \geq \|\Sigma(t) - \Sigma_{av}(t)\|_{\mathcal{L}_\infty} = \|\Sigma(t) - R_w(0)\|_{\mathcal{L}_\infty}$$

for all $\varepsilon \in (0, \varepsilon_\star]$. As a result, we have

$$\begin{aligned}\Lambda_{pe}(t) &= W^\intercal R_w(0) W + W^\intercal \left(\Sigma(t) - R_w(0)\right) W \\ &\geq R_{w_{pe}}(0) - \|\Sigma(t) - R_w(0)\|_{\mathcal{L}_\infty} I \\ &\geq \left(\beta_0 - \delta(\varepsilon)\right) I\end{aligned}$$

for all $t \geq t_0 \geq 0$. ☐

Remark 18. We would like to emphasize that restricting $\varepsilon > 0$ small is not needed in any part of the proposed subspace estimator design. That being said, it enables the intuitive interpretation of the filter matrix $\hat{\Sigma}$ as an approximation of $R_w(0)$. When ε is not small, one loses the ability to approximate $R_w(0)$ but retains the ability to recover $\mathcal{W}$ asymptotically. We defer the details to our paper (Uzeda & Broucke, 2022, Section III). ◃

Returning to the filter (44), we want to show that $\hat{\Sigma}$ well approximates Σ. To that end, consider the estimation error $\tilde{\Sigma} := \hat{\Sigma} - \Sigma$ with dynamics

$$\dot{\tilde{\Sigma}} = -\varepsilon \tilde{\Sigma} + \varepsilon(\tilde{w} w^\intercal(t) + w(t)\tilde{w}^\intercal + \tilde{w}\tilde{w}^\intercal).$$

By Assumption 11, the above is a stable LTI system forced by a vanishing term, implying that $\tilde{\Sigma} \to 0$ and so

$$\lim_{t\to\infty} \|\hat{\Sigma}(t) - W\Lambda_{pe}(t) W^\intercal\| = 0$$
$$\limsup_{t\to\infty} \|\hat{\Sigma}(t) - R_w(0)\| \leq \delta(\varepsilon)\,.$$

Therefore $\hat{\Sigma}$ is an approximation of $R_w(0)$ that recovers the PE subspace $\mathcal{W}$ of $w(t)$ asymptotically. In principle, if we had $\hat{\Sigma} = \Sigma$, then using Proposition 4 one could recover the orthogonal projection onto $\mathcal{W}^\perp$ by the formula $\Omega = I - \hat{\Sigma}\hat{\Sigma}^\dagger$. Unfortunately, the pseudoinverse operation is not robust, as discussed in Section 5.1, and so the above fact does not hold (not even asymptotically). Instead we choose

$$\Omega = I - \hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})\,, \tag{47}$$

where $\mathrm{pinv}(\cdot; \sigma_{tol})$ denotes the *robust pseudoinverse* for some tolerance $\sigma_{tol} \geq 0$. That is, $\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})$ performs the pseudoinverse of $\hat{\Sigma}$ by treating all of its singular values less than σ_{tol} as zero. To analyze the correctness of (47), we must characterize how the difference $\hat{\Sigma} - \Sigma$ relates to the orthogonal projection matrices recovered by $\hat{\Sigma}$ (thresholded by σ_{tol}) and Σ. First we need a result concerning the singular values of $\hat{\Sigma}$.

Lemma 7. *Consider system (46) satisfying Assumptions 8–9. Let $q_{pe} \geq 1$ be the degree of persistent excitation of $w(t)$, let $\beta_0 > 0$ be as defined in Lemma 6, and let $\sigma_i(\cdot)$ denote the ith largest singular value. Then there exists a constant $\varepsilon_\star > 0$ such that*

$$\sigma_{q_{pe}}(\hat{\Sigma}) \geq \beta_0 - \delta(\varepsilon) - \|\hat{\Sigma} - \Sigma\|\,, \quad \sigma_{q_{pe}+1}(\hat{\Sigma}) \leq \|\hat{\Sigma} - \Sigma\|$$

for each $\varepsilon \in (0, \varepsilon_\star]$, where $\sigma_{q_{pe}+1}(\hat{\Sigma}) := 0$ if $q_{pe} = q$.

Proof. Writing $\hat{\Sigma} = \Sigma + \tilde{\Sigma}$, one can apply Weyl's Theorem (Weyl's Inequality) to conclude

$$|\sigma_{q_{pe}}(\hat{\Sigma}) - \sigma_{q_{pe}}(\Sigma)| \leq \|\tilde{\Sigma}\| \tag{48a}$$
$$|\sigma_{q_{pe}+1}(\hat{\Sigma}) - \sigma_{q_{pe}+1}(\Sigma)| \leq \|\tilde{\Sigma}\|\,. \tag{48b}$$

By Lemma 6, we know that $\mathrm{rank}(\Sigma) = q_{pe}$ and so $\sigma_{q_{pe}+1}(\Sigma) = 0$. Thus (48b) becomes $\sigma_{q_{pe}+1}(\hat{\Sigma}) \leq \|\tilde{\Sigma}\|$. To show the first inequality, we use the fact that the columns of $W \in \mathbb{R}^{q\times q_{pe}}$ are orthonormal to obtain

$$\sigma_{q_{pe}}(\Sigma) = \sigma_{q_{pe}}(W\Lambda_{pe}W^\intercal) = \sigma_{q_{pe}}(\Lambda_{pe}) \geq \beta_0 - \delta(\varepsilon)$$

for $\varepsilon \in (0, \varepsilon_\star]$, where $\varepsilon_\star > 0$ is given by Lemma 6. The reverse triangle inequality applied to (48a) proves the result. ☐

Next, we establish the desired error bound and a range of admissible values for the tolerance σ_{tol}. The proof is in Section 9.

Lemma 8. *Consider system (46) satisfying Assumptions 8–9. Let $\mathcal{W} = \mathrm{Im}(W)$ result from the PE decomposition of $w(t)$, and let $\beta_0 > 0$ be as defined in Lemma 6. If $q_{pe} = \dim(\mathcal{W}) = 0$, set $\beta_0 = \infty$. Then for every $\sigma_{tol} \in (0, \beta_0)$ there exists constants $\varepsilon_\star(\sigma_{tol})$, $c_\circ(\sigma_{tol}) > 0$ such that*

$$\|\hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol}) - WW^\intercal\| \leq \min\{\, c_\circ \|\hat{\Sigma} - \Sigma\|, 1\,\}$$

for each $\varepsilon \in (0, \varepsilon_\star]$.

From the proof of Lemma 8 we saw that $\hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})$ is an orthogonal projection matrix. By symmetry and idempotence of orthogonal projections, it immediately follows that

$$\begin{aligned}\Omega\Omega^\intercal &= \left(I - \hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})\right)\left(I - \hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})\right)^\intercal \\ &= I - \hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol}) = \Omega\,.\end{aligned}$$

Then using the fact $I = WW^\intercal + W_\perp W_\perp^\intercal$, we have

$$\begin{aligned}\Omega\Omega^\intercal - W_\perp W_\perp^\intercal &= I - \hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol}) - I + WW^\intercal \\ &= WW^\intercal - \hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})\,.\end{aligned}$$

This formula allows us to relate closeness of $\Omega\Omega^\intercal$ and $W_\perp W_\perp^\intercal$ to closeness of $WW^\intercal$ and $\hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})$. But the latter can be made arbitrarily close by way of Lemma 8. As such, we arrive at our final statement concerning the correctness of the subspace estimator Ω.

Theorem 9. *Consider the system (46), (47) satisfying Assumptions 8–9. Let $w_{pe}(t)$ and $\mathcal{W}^\perp = \mathrm{Im}(W_\perp)$ result from the PE decomposition of $w(t)$, and let $\beta_0 > 0$ be the lower PE bound in Definition 2 for $w_{pe}(t)$. If $q_{pe} = \dim(\mathcal{W}) = 0$, set $\beta_0 = \infty$. Then for every $\sigma_{tol} \in (0, \beta_0)$ there exists constants $\varepsilon_\star(\sigma_{tol})$, $c_\circ(\sigma_{tol}) > 0$ such that*

$$\|\Omega\Omega^\intercal - W_\perp W_\perp^\intercal\| \leq \min\{\, c_\circ \|\hat{\Sigma} - \Sigma\|, 1\,\}$$

for each $\varepsilon \in (0, \varepsilon_\star]$. Moreover, if $\hat{\Sigma}$ evolves according to (44) and also satisfies Assumption 11, then

$$\lim_{t\to\infty} \|\Omega(t)\Omega^\intercal(t) - W_\perp W_\perp^\intercal\| = 0$$

for each $\varepsilon \in (0, \varepsilon_\star]$.

Proof. The first inequality follows from Lemma 8. The limit is obtained from the fact that Assumption 11 (with boundedness of $w(t)$ in Assumption 8) implies $\hat{\Sigma} - \Sigma \to 0$. □

Remark 19. To build the subspace estimator (44), (47) one needs to know of some $\sigma_{tol} \in (0, \beta_0)$. This requirement can be viewed as the assumption of knowing a strict lower bound on the excitation of the regressor $w(t)$. A more meaningful perspective is to view $\sigma_{tol} > 0$ as a design tolerance that a control designer can set. To elaborate on this idea, recall that $R_w(0) = W R_{w_{pe}}(0) W^\intercal$ with $\sigma_{\min}(R_{w_{pe}}(0)) \geq \beta_0$. Therefore, β_0 is a lower bound on the average excitation expected of the PE component $w_{pe}(t)$. Since $\hat{\Sigma}$ is an approximation of $R_w(0)$, we can view the computation $\mathrm{pinv}(\hat{\Sigma}; \sigma_{tol})$ as a mechanism to discard components of $w(t)$ that have an excitation level lower than σ_{tol}. In other words, $\sigma_{tol} > 0$ sets a soft excitation threshold for $w(t)$. Consequently, any component of $w(t)$ whose excitation is less than σ_{tol} will be discarded for the recovery of $\mathcal{W}$ and treated as noise irrelevant to the adaptation process. ◁

5.5. The μ-modification

We are now in a position to combine all our prior developments together to see the Use it or Lose it Principle in action. The μ-modification aims to render adaptation (i.e., the learning process) robust by modifying the gradient law (42) to

$$\dot{\hat{\psi}} = -\gamma e \hat{w} - \mu \Omega \Omega^\intercal \hat{\psi} \tag{49}$$

where $\mu > 0$ and Ω is the output of an appropriate subspace estimator. Below is our main result on using forgetting as a mechanism for robustness through the μ-modification. In particular, robustness is synonymous to exponential stability.

Theorem 10. *Consider the error model with adaptation law (41)–(43) satisfying Assumption 10. Let $\Sigma(t) \in \mathbb{R}^{q\times q}$ denote the exogenous matrix generated by*

$$\dot{\Sigma} = -\varepsilon \Sigma + \varepsilon w(t) w^\intercal(t), \quad \Sigma(t_0) = R_w(0).$$

Also, let $w_{pe}(t) \in \mathbb{R}^{q_{pe}}$ and $\mathcal{W} = \mathrm{Im}(W)$ result from the PE decomposition of $w(t)$, and let $\beta_0 > 0$ be the lower PE bound in Definition 2 for $w_{pe}(t)$. If $q_{pe} = \dim(\mathcal{W}) = 0$, set $\beta_0 = \infty$. Then for every $\sigma_{tol} \in (0, \beta_0)$ there exists a constant $\varepsilon_\star(\sigma_{tol}) > 0$ such that for each $\varepsilon \in (0, \varepsilon_\star]$ the μ-modification, which replaces (42) with (44), (47), and (49), guarantees:

1. *if $q_{pe} = 0$, then $(\xi, \hat{\psi}, \nu, \tilde{\Sigma}) = (0,0,0,0)$ is GUAS and LES;*
2. *if $q_{pe} = q$, then $(\xi, \hat{\psi}, \nu, \tilde{\Sigma}) = (0,\psi,0,0)$ is GUAS and LES;*
3. *otherwise $(\xi, \hat{\psi}, \nu, \tilde{\Sigma}) = (0, W W^\intercal \psi, 0, 0)$ is GUAS and LES;*

where we have defined $\tilde{\Sigma} := \hat{\Sigma} - \Sigma$.

Proof. We only prove the third case, as the other cases follow by specialization of the proof. Let $\varepsilon_\star(\sigma_{tol}) > 0$ be given by Theorem 9 due to (E1). Also by (E1) we have the PE decomposition

$$w = W w_{pe}, \quad \mathcal{W} = \mathrm{Im}(W),$$

so that we may define the coordinate transformation

$$\begin{bmatrix} \tilde{\psi}_{pe} \\ \hat{\psi}_\perp \end{bmatrix} = \begin{bmatrix} W^\intercal \\ W_\perp^\intercal \end{bmatrix} \left(\hat{\psi} - W W^\intercal \psi \right).$$

As a result, we may write

$$\hat{w}^\intercal \hat{\psi} - w^\intercal \psi = w_{pe}^\intercal \tilde{\psi}_{pe} + \tilde{w}^\intercal (W \tilde{\psi}_{pe} + W_\perp \hat{\psi}_\perp + W W^\intercal \psi)$$

resulting in an error signal of the form

$$e = e_\circ + D(t) \tilde{w}^\intercal (W \tilde{\psi}_{pe} + W_\perp \hat{\psi}_\perp + W W^\intercal \psi)$$
$$e_\circ = C(t,\xi) + D(t) w_{pe}^\intercal \tilde{\psi}_{pe}.$$

Next define the error variable $\tilde{\Sigma} := \hat{\Sigma} - \Sigma$, yielding the dynamics

$$\dot{\tilde{\Sigma}} = -\varepsilon \tilde{\Sigma} + \varepsilon (\tilde{w} w^\intercal + w \tilde{w}^\intercal + \tilde{w}\tilde{w}^\intercal).$$

Then we have $\Omega\Omega^\intercal = W_\perp W_\perp^\intercal + \tilde{\Omega}$, where

$$\tilde{\Omega}(t, \tilde{\Sigma}; \sigma_{tol}) = W W^\intercal - (\Sigma(t) + \tilde{\Sigma})\,\mathrm{pinv}(\Sigma(t) + \tilde{\Sigma}; \sigma_{tol}).$$

By Lemma 8 we have $\tilde{\Omega}(t, 0; \sigma_{tol}) = 0$. Altogether, we obtain the closed-loop dynamics

$$\dot{\xi} = A(t,\xi) + B(t) w_{pe}^\intercal(t) \tilde{\psi}_{pe} + B(t) \tilde{w}^\intercal (W \tilde{\psi}_{pe} + W_\perp \hat{\psi}_\perp + W W^\intercal \psi) \tag{50a}$$

$$\dot{\tilde{\psi}}_{pe} = -\gamma e_\circ w_{pe}(t) - \mu W^\intercal \tilde{\Omega} (W \tilde{\psi}_{pe} + W_\perp \hat{\psi}_\perp + W W^\intercal \psi) - \gamma e W^\intercal \tilde{w} - \gamma D(t) \tilde{w}^\intercal (W \tilde{\psi}_{pe} + W_\perp \hat{\psi}_\perp + W W^\intercal \psi) w_{pe}(t) \tag{50b}$$

$$\dot{\hat{\psi}}_\perp = -\mu \hat{\psi}_\perp - \mu W_\perp^\intercal \tilde{\Omega} (W \tilde{\psi}_{pe} + W_\perp \hat{\psi}_\perp + W W^\intercal \psi) - \gamma e W_\perp^\intercal \tilde{w} \tag{50c}$$

$$\dot{\nu} = \Delta(t, \nu) \tag{50d}$$

$$\dot{\tilde{\Sigma}} = -\varepsilon \tilde{\Sigma} + \varepsilon (\tilde{w} w^\intercal(t) + w(t) \tilde{w}^\intercal + \tilde{w}\tilde{w}^\intercal) \tag{50e}$$

which has the equilibrium $(\xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp, \nu, \tilde{\Sigma}) = (0,0,0,0,0)$. To conclude the result, we need to show that the origin is a GUAS and LES equilibrium of (50). The ensuing stability analysis will proceed in three stages. The first two steps consist of constructing appropriate Lyapunov functions for specific subsystems, and the last step combines them to prove stability.

First, consider the nominal unperturbed dynamics obtained by setting $(\nu, \tilde{\Sigma}) = (0,0)$:

$$\dot{\xi} = A(t,\xi) + B(t) w_{pe}^\intercal(t) \tilde{\psi}_{pe} \tag{51a}$$
$$\dot{\tilde{\psi}}_{pe} = -\gamma e_\circ w_{pe}(t) \tag{51b}$$
$$\dot{\hat{\psi}}_\perp = -\mu \hat{\psi}_\perp. \tag{51c}$$

By (E7) and the fact that $\mu > 0$ we have that the equilibrium $(\xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp) = (0,0,0)$ of (51) is GES. Note that if $q_{pe} = 0$, then we use (E6) rather than (E7). As a result, in conjunction with (E2), there exists a converse Lyapunov function $V_n(t, \xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp)$ for (51) satisfying the conclusions of Khalil (2002, Theorem 4.14) globally with constants $a_i > 0$.

Second, notice that the perturbation dynamics

$$\dot{\nu} = \Delta(t, \nu) \tag{52a}$$
$$\dot{\tilde{\Sigma}} = -\varepsilon \tilde{\Sigma} + \varepsilon (\tilde{w} w^\intercal(t) + w(t) \tilde{w}^\intercal + \tilde{w}\tilde{w}^\intercal) \tag{52b}$$

are decoupled from the $(\xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp)$ dynamics of (50). By (E4) we know that $(\nu, \tilde{\Sigma}) = (0,0)$ is the equilibrium of (52), which we want to show is GUAS and LES. By Proposition 1, it suffices to show that for every $\delta > 0$, we have ES over $(\nu, \tilde{\Sigma})(t_0) \in \mathcal{B}(\delta)$. Suppose $(\nu, \tilde{\Sigma})(t_0) \in \mathcal{B}(\delta)$, then $\nu(t_0) \in \mathcal{B}(\delta)$ and by (E5) there exists an $\delta_\nu(\delta) > 0$ such that $\nu(t) \in \mathcal{B}(\delta_\nu)$ for all $t \geq t_0 \geq 0$ since the ν dynamics are decoupled from all the rest. In conjunction with (E3), there exists a converse Lyapunov function $V_1(t, \nu)$ for (52a) satisfying the conclusions of Khalil (2002, Theorem 4.14) over $\mathbb{R}_+ \times \mathcal{B}(\delta_\nu)$ with constants $b_i > 0$. Let $\|\cdot\|_F$ denote the Frobenius norm and consider the candidate Lyapunov function

$$V_2(t, \nu, \tilde{\Sigma}) = V_1(t, \nu) + \gamma_1 \|\tilde{\Sigma}\|_F^2$$

for some $\gamma_1 > 0$ to be selected shortly. Taking its time derivative with respect to trajectories of (52) for $\nu(t_0) \in \mathcal{B}(\delta)$, we have

$$\begin{aligned} \dot{V}_2(t, \nu, \tilde{\Sigma}) &= \partial_t V_1(t, \nu) + \partial_\nu V_1(t, \nu) \Delta(t, \nu) + 2\gamma_1 \,\mathrm{trace}(\tilde{\Sigma}^\intercal \dot{\tilde{\Sigma}}) \\ &\leq -b_3 \|\nu\|^2 - \gamma_1 \varepsilon \|\tilde{\Sigma}\|_F^2 \end{aligned}$$

$$+\gamma_1 \varepsilon \left(2\|w(t)\|_{\mathcal{L}_\infty} + \|\tilde{w}(t, v(t))\|_{\mathcal{L}_\infty}\right)^2 \|\tilde{w}\|^2 ,$$

where the time argument t has been omitted in some places for clarity. By (E3), (E4), and since $v(t) \in \mathcal{B}(\delta_v)$, there exists a constant $w_\circ(\delta_v) > 0$ such that

$$\|\tilde{w}(t, v(t))\| \leq w_\circ \|v(t)\|.$$

This implies that for some $\gamma_1 > 0$ sufficiently small, we can obtain the bound

$$\dot{V}_2(t, v(t), \tilde{\Sigma}(t)) \leq -\gamma_2 V_2(t, v(t), \tilde{\Sigma}(t))$$

for some $\gamma_2 > 0$ and all $t \geq t_0 \geq 0$. The Comparison Lemma and the bounds $b_1\|v\|^2 \leq V_1(t, v) \leq b_2\|v\|^2$ gives us ES over $(v, \tilde{\Sigma})(t_0) \in \mathcal{B}(\delta)$. Consequently for every $\delta > 0$ there exists a converse Lyapunov function $V_p(t, v, \tilde{\Sigma})$ for (52) satisfying the conclusions of Khalil (2002, Theorem 4.14) over $\mathbb{R}_+ \times \mathcal{B}(\delta)$ with constants $c_i > 0$.

At last, we return to the study of (50). Again by Proposition 1 we will show ES over every ball. Let $\delta > 0$ and suppose $(\xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp, v, \tilde{\Sigma})(t_0) \in \mathcal{B}(\delta)$. Since we have shown $(v, \tilde{\Sigma}) = (0, 0)$ is GUAS, there exists $\delta_p(\delta) > 0$ such that $(v, \tilde{\Sigma})(t) \in \mathcal{B}(\delta_p)$ for all $t \geq t_0 \geq 0$. Now let $V_n(\cdot)$ defined globally and $V_p(\cdot)$ defined over $\mathbb{R}_+ \times \mathcal{B}(\delta_p)$ be the converse Lyapunov functions constructed earlier in the proof. Consider the candidate Lyapunov function

$$V(t, \xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp, v, \tilde{\Sigma}) = V_n(t, \xi, \tilde{\psi}_{pe}, \hat{\psi}_\perp) + \gamma_p V_p(t, v, \tilde{\Sigma})$$

where $\gamma_p > 0$ is to be selected sufficiently large. To keep our developments concise, we will skip most of the verbose algebra involved in computing the time derivative of $V(\cdot)$ with respect to the trajectories of (50) since they follow standard arguments (see Young's inequality and the Peter–Paul inequality). Instead, we remind the reader that since $v(t) \in \mathcal{B}(\delta_p)$, there exists $w_\circ(\delta_p) > 0$ such that

$$\|\tilde{w}(t, v(t))\| \leq w_\circ \|v(t)\| \leq w_\circ \|(v, \tilde{\Sigma})(t)\|$$

for all $t \geq t_0 \geq 0$. Similarly, if we let $\varepsilon_\star(\sigma_{tol}), c_\circ(\sigma_{tol}) > 0$ be the constants obtained from Theorem 9, then for each $\varepsilon \in (0, \varepsilon_\star]$ we have

$$\|\tilde{\Omega}(t, \tilde{\Sigma}; \sigma_{tol})\| \leq c_\circ \|\tilde{\Sigma}\| \leq c_\circ \|(v, \tilde{\Sigma})\|.$$

Letting $V(t)$ denote a shorthand for $V(\cdot)$ with the time-varying trajectories of (50) substituted in, one can show

$$\dot{V}(t) \leq -\left(\gamma_3 - \gamma_4 \|(v, \tilde{\Sigma})(t)\|\right) V(t)$$

for some constants $\gamma_3(\delta, \sigma_{tol}), \gamma_4(\delta, \sigma_{tol}) > 0$. Given that we know $(v, \tilde{\Sigma}) = (0, 0)$ is GUAS, we have that $\|(v, \tilde{\Sigma})\| \to 0$ uniformly in $(v, \tilde{\Sigma})(t_0) \in \mathcal{B}(\delta)$. Hence we have an asymptotically stable almost time-invariant linear system for $V(t)$ (Narendra & Annaswamy, 1989, Section 2.3.2) and the Comparison Lemma proves the result. □

Remark 20. If we let $W = 0$ when $q_{pe} = 0$ and $W = I$ when $q_{pe} = q$, as suggested by the PE decomposition (see Proposition 5), then the conclusions 1, 2, and 3 of Theorem 10 can be succinctly replaced by the statement that the equilibrium $(\xi, \hat{\psi}, v, \tilde{\Sigma}) = (0, WW^\intercal\psi, 0, 0)$ is GUAS and LES. That is, conclusion 3 summarizes Theorem 10. ◁

Remark 21. The one technicality not addressed in the proof of Theorem 10 is whether or not there exists (unique) solutions to the closed-loop system with the μ-modification. This is a valid question since it is known that the pseudoinverse is not a continuous operation. Yet we opt to ignore such details since one generally implements Ω by performing the pseudoinverse at discrete times instead of continuously. That is, given an increasing sequence of times $\{t_i\}_{i=0}^\infty$ satisfying $\lim_{i\to\infty} t_i = \infty$, one builds

$$\Omega(t, \hat{\Sigma}(\cdot); \sigma_{tol}) = \begin{cases} I - \hat{\Sigma}(t)\,\mathrm{pinv}(\hat{\Sigma}(t); \sigma_{tol}), & t = t_i \\ \Omega(t_i), & t \in (t_i, t_{i+1}) \end{cases}$$

which is guaranteed to be piecewise continuous in time. ◁

Remark 22. Sometimes we would like our parameter estimate to default to a known nominal value $\psi_\star$. In this case, one can update the μ-modification to

$$\dot{\hat{\psi}} = -\gamma e \hat{w} - \mu \Omega \Omega^\intercal (\hat{\psi} - \psi_\star).$$

The analysis for such a variation is analogous to the presented developments, with a suitable modification to the coordinate transformation for the non-PE dynamics considered. ◁

6. Learning and forgetting

In Section 3, we showed that learning through the training of reflexes serves as a mechanism employed by biological systems to reduce the work done by internal models, housed in energy hungry modules of the brain. The analysis proceeded for the case when associated regressors are PE. When regressors are not PE, a PE decomposition can be employed to highlight the lack of robustness of adaptive schemes. In fact, Section 5 demonstrates that exponential stability (and thus robustness) can be restored for an adaptive system through the use of forgetting. Here we extend the results of Section 4 for LTI systems by combining both the learning and forgetting techniques presented. The appealing property of our methods is that the forgetting mechanism, the μ-modification, does not interfere with the learning process, but only enhances it.

6.1. Final design and architecture

We begin by reiterating the full design presented in Section 4 with the addition of the μ-modification. For the regulator tasked with performing error regulation, we have: the *high-gain observer and stabilizer*

$$\dot{\hat{\xi}} = A_\circ \hat{\xi} + D_\kappa L(e - C_\circ \hat{\xi}) \tag{53a}$$

$$\hat{e}_0 = \begin{bmatrix} a^\intercal & 1 \end{bmatrix} \hat{\xi} \tag{53b}$$

$$u_s = -\,\mathrm{sgn}(b) K \hat{e}_0 , \tag{53c}$$

where $K, \kappa > 0$ are to be selected sufficiently large, r is the relative degree, $D_\kappa := \mathrm{diag}(\kappa, \kappa^2, \ldots, \kappa^r)$, $L \in \mathbb{R}^r$ is selected so that $A_\circ - LC_\circ$ is Hurwitz, and $a \in \mathbb{R}^{r-1}$ is selected so that $A_\circ - B_\circ a^\intercal$ is Hurwitz; and the *adaptive internal model*

$$\dot{\hat{w}}_0 = F \hat{w}_0 + G u \tag{54a}$$

$$\dot{\hat{w}}_i = F \hat{w}_i + G y_i \tag{54b}$$

$$\hat{w} = (\hat{w}_0, \ldots, \hat{w}_p) \tag{54c}$$

$$\dot{\hat{\Sigma}}_w = -\varepsilon_w \hat{\Sigma}_w + \varepsilon_w \hat{w} \hat{w}^\intercal \tag{54d}$$

$$\Omega_w = I - \hat{\Sigma}_w\,\mathrm{pinv}(\hat{\Sigma}_w; \sigma_w) \tag{54e}$$

$$\dot{\hat{\psi}} = -\,\mathrm{sgn}(b) \gamma \hat{e}_0 \hat{w} - \mu_w \Omega_w \Omega_w^\intercal \hat{\psi} \tag{54f}$$

$$u_{im} = \hat{\psi}^\intercal \hat{w} \tag{54g}$$

where γ, $\mu_w > 0$, ε_w, $\sigma_w > 0$ are to be selected sufficiently small, and (F, G) is controllable with $F \in \mathbb{R}^{q\times q}$ Hurwitz. Tasked with offloading the steady-state work of the internal model, we have the *reflexes*

$$\dot{\hat{\Sigma}}_v = -\varepsilon_v \hat{\Sigma}_v + \varepsilon_v \hat{v} \hat{v}^\intercal \tag{55a}$$

$$\Omega_v = I - \hat{\Sigma}_v\,\mathrm{pinv}(\hat{\Sigma}_v; \sigma_v) \tag{55b}$$

$$\dot{\hat{\alpha}} = -\varepsilon u_{im} \hat{v} - \varepsilon \mu_v \Omega_v \Omega_v^\intercal \hat{\alpha} \tag{55c}$$

$$u_r + u_x = -\hat{\alpha}^\intercal \hat{v} \tag{55d}$$

where $\mu_v > 0$, ε, ε_v, $\sigma_v > 0$ are to be selected sufficiently small, $\hat{\alpha} := (\hat{\alpha}_r, \hat{\alpha}_x)$, and $\hat{v} := (y, -x)$. Altogether, the controller is

$$u = u_s + u_{im} + u_r + u_x . \tag{56}$$

A block diagram of the overall controller architecture is given in Fig. 2. The reader may compare this architecture with the neural architecture we set out to model in Fig. 1.

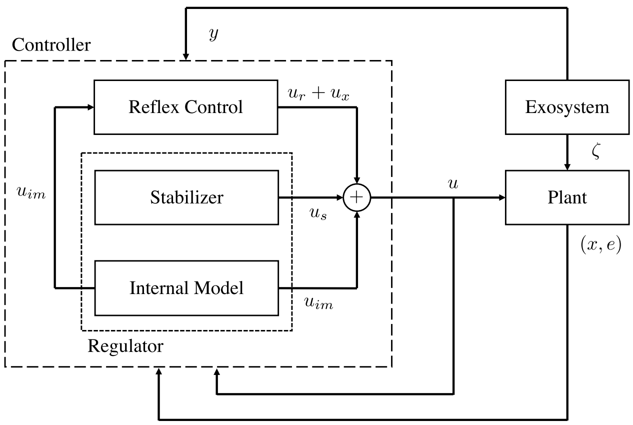


Fig. 2. Block diagram of the regulator with reflexes.

6.2. Stability without persistent excitation

We begin by once again showing that the regulator (53)–(54) with $\hat{\alpha}$ constant satisfies the exponential stability requirements of Assumption 4. From the statement of Lemmas 5 and 7, it is clear that our prior results need to be revisited when the exogenous regressor $w(t)$ is not PE. Considering the same coordinates defined in Section 4.1 with $\tilde{\alpha}$ constant, we let $z_\circ := (z_0, \xi_\circ, e_0)$. Also, define the exogenous matrix $\Sigma_w(t)$ generated by

$$\dot{\Sigma}_w = -\varepsilon_w \Sigma_w + \varepsilon_w w(t) w^\mathsf{T}(t), \quad \Sigma_w(t_0) = R_w(0).$$

Therefore, letting $\tilde{\Sigma}_w := \hat{\Sigma}_w - \Sigma_w$, we can express the closed-loop dynamics as

$$\dot{z}_\circ = A_\circ(K,\tilde{\alpha}) z_\circ + b B_\circ(\hat{w}^\mathsf{T}\hat{\psi} - w^\mathsf{T}\psi) + B_{11}(K,\kappa,\hat{w})\tilde{\xi} \tag{57a}$$

$$\dot{\hat{\psi}} = -\operatorname{sgn}(b)\gamma(B_\circ^\mathsf{T} z_\circ)\hat{w} - \mu_w \Omega_w \Omega_w^\mathsf{T}\hat{\psi} + B_{12}(K,\kappa,\hat{w})\tilde{\xi} \tag{57b}$$

$$\dot{\tilde{\Sigma}}_w = -\varepsilon_w \tilde{\Sigma}_w + \varepsilon_w(\tilde{w} w^\mathsf{T} + w\tilde{w}^\mathsf{T} + \tilde{w}\tilde{w}^\mathsf{T}) \tag{57c}$$

with disturbance observer estimation dynamics

$$\dot{\tilde{w}} = \operatorname{diag}(F)\tilde{w} + b^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix}\begin{bmatrix} -Ga_{r0}^\mathsf{T} & (FG - Ga_{rr}) \end{bmatrix} z_\circ \tag{58}$$

and state observer estimation dynamics

$$\dot{\tilde{\xi}} = \kappa(A_\circ - LC_\circ)\tilde{\xi} + B_2(K,\tilde{\alpha})z_\circ - bB_\circ(\hat{w}^\mathsf{T}\hat{\psi} - w^\mathsf{T}\psi) + B_3(K,\kappa)\tilde{\xi}\,. \tag{59}$$

In particular, we have the quantities

$$A_\circ(K,\tilde{\alpha}) := \begin{bmatrix} \begin{bmatrix} A_{00} & A_{01}' \\ 0 & A_\circ - B_\circ a^\mathsf{T} \end{bmatrix} & B_{0r} \\ \tilde{a}_{r0}^\mathsf{T} + \tilde{\alpha}^\mathsf{T} T_0 & \tilde{a}_{rr} + \tilde{\alpha}^\mathsf{T} T_r - |b| K \end{bmatrix}$$

$$B_1(K,\kappa,\hat{w}) =: \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}(K,\kappa,\hat{w})$$

where the expressions for $B_1(\cdot)$ and $B_3(\cdot)$ are given in Section 4.1 and $B_2(\cdot)$ is continuous in $\tilde{\alpha}$. To formally apply our results from Section 5 concerning the μ-modification, we need to make a simplifying assumption that removes the coupling between $\tilde{w}$ and $z_\circ$ in (58). We show later in simulation that indeed the μ-modification can be applied with the coupling present.

Assumption 12. The transient $\tilde{w} = \hat{w} - w(t)$ evolves according to the decoupled dynamics $\dot{\tilde{w}} = \operatorname{diag}(F)\tilde{w}$.

Lemma 9. *Consider system (57)–(58) with $\tilde{\xi} = 0$ satisfying Assumption 12. Let $w_{pe}(t) \in \mathbb{R}^{q_{pe}}$ and $\mathcal{W} = \operatorname{Im}(W)$ result from the PE decomposition of $w(t)$, and let $\beta_0^w > 0$ be the lower PE bound in Definition 2 for $w_{pe}(t)$. If $q_{pe} = \dim(\mathcal{W}) = 0$, set $\beta_0^w = \infty$. Also, define $\tilde{\psi} := \hat{\psi} - WW^\mathsf{T}\psi$. Then for every $\sigma_w \in (0, \beta_0^w)$ and $\delta_\alpha > 0$ there exists constants $\varepsilon_w^\star(\sigma_w) > 0$ and $K_\star(\delta_\alpha) > 0$ such that for each $\varepsilon_w \in (0, \varepsilon_w^\star]$ and $K \geq K_\star$ the equilibrium $(z_\circ, \tilde{\psi}, \tilde{w}, \tilde{\Sigma}_w) = (0,0,0,0)$ is GUAS and LES uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$.*

Proof. The result follows from Theorem 10, so we need to show that the system considered matches the error model of Section 5.3. First, by setting $\mu_w = 0$, $\tilde{\xi} = 0$, and invoking Assumption 12, the dynamics (57)–(58) become

$$\dot{z}_\circ = A_\circ(K,\tilde{\alpha}) z_\circ + bB_\circ(\hat{w}^\mathsf{T}\hat{\psi} - w^\mathsf{T}\psi)$$
$$e_\circ := bB_\circ^\mathsf{T} z_\circ$$

matching (41),

$$\dot{\hat{\psi}} = -|b|^{-1}\gamma e_\circ \hat{w}$$

matching (42), and

$$\dot{\tilde{w}} = \operatorname{diag}(F)\tilde{w}$$
$$\hat{w} = w(t) + \tilde{w}$$

matching (43). Therefore, we verify Assumption 10 is satisfied.

(E1): Given that $w(t)$ is the output of an LTI exosystem, it is an almost periodic signal. Then Assumptions 8–9 follow from Uzeda and Broucke (2022, Prop 4);
(E2): The appropriate continuity properties are immediate;
(E3): The appropriate continuity properties are immediate;
(E4): The appropriate functions vanish at the origin since they are linear in the states;
(E5): The equilibrium $\tilde{w} = 0$, whose dynamics are independent of $\tilde{\alpha}$, is GES uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$ since F is Hurwitz;
(E6): By the same proof technique as in Lemma 3, there exists $K_\star(\delta_\alpha) > 0$, $P_0 \succ 0$, and $\rho > 0$ such that $P := \operatorname{diag}(P_0, 1)$ satisfies the Lyapunov LMI

$$A_\circ^\mathsf{T}(K,\tilde{\alpha})P + PA_\circ(K,\tilde{\alpha}) \preceq -\rho I$$

for all $K \geq K_\star$ and $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$. Hence, for each $K \geq K_\star$, the equilibrium $z_\circ = 0$ of $\dot{z}_\circ = A_\circ(K,\tilde{\alpha})z_\circ$ is GES uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$;
(E7): Let $P \succ 0$ be defined as above. Setting $\tilde{w} = 0$, we have the nominal system (without the μ-modification)

$$\dot{z}_\circ = A_\circ(K,\tilde{\alpha})z_\circ + bB_\circ w^\mathsf{T}(t)(\hat{\psi} - \psi)$$
$$\dot{\hat{\psi}} = -|b|^{-1}\gamma(bB_\circ^\mathsf{T} P z_\circ)w(t)$$

which matches the dynamic error model in Section 2.3. Theorem 2 then implies that the equilibrium $(z_\circ, \hat{\psi}) = (0, \psi)$ is GES uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$ when $w(t)$ is PE. Note that w and $\dot{w}$ are bounded because they are generated by an LTI exosystem.

Since the system (57)–(58) with $\tilde{\xi} = 0$ satisfying Assumption 12 is the result of applying the μ-modification to the error model of Section 5.3, Theorem 10 proves the result. Note the fact that ψ is bounded for $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$ by continuity of $\psi = \psi(\tilde{\alpha})$ is used in the associated Lyapunov argument. □

Theorem 11. *Consider system (57)–(59) satisfying Assumption 12. Let $w_{pe}(t) \in \mathbb{R}^{q_{pe}}$ and $\mathcal{W} = \operatorname{Im}(W)$ result from the PE decomposition of $w(t)$, and let $\beta_0^w > 0$ be the lower PE bound in Definition 2 for $w_{pe}(t)$. If $q_{pe} = \dim(\mathcal{W}) = 0$, set $\beta_0^w = \infty$. Also, define $\tilde{\psi} := \hat{\psi} - WW^\mathsf{T}\psi$. Then for every $\sigma_w \in (0, \beta_0^w)$, $\delta_\alpha > 0$, and $\delta_1 > 0$ there exists constants $\varepsilon_w^\star(\sigma_w) > 0$, $K_\star(\delta_\alpha) > 0$, and $\kappa_\star(K, \varepsilon_w, \delta_1, \delta_\alpha, \sigma_w) \geq 1$ such that for each $\varepsilon_w \in (0, \varepsilon_w^\star]$, $K \geq K_\star$, and $\kappa \geq \kappa_\star$ the equilibrium $(z_\circ, \tilde{\psi}, \tilde{\xi}, \tilde{w}, \tilde{\Sigma}_w) = (0,0,0,0,0)$ is ES over $(z_\circ, \tilde{\psi}, \tilde{w}, \tilde{\Sigma}_w)(t_0) \in \mathcal{B}(\delta_1)$ and $\tilde{\xi}(t_0) \in \mathcal{B}(\delta_1 \kappa^{r-1})$ uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$.*

Proof. If we ignore the lack of continuous differentiability of Ω_w with respect to the relevant states, then the result follows from the proof of Theorem 7. In particular, we have augmented the state $(x_\circ, \tilde{\psi})$ to $(z_\circ, \tilde{\psi}, \tilde{w}, \tilde{\Sigma}_w)$ and replaced the use of Lemma 5 with Lemma 9. □

6.3. Reflexes without persistent excitation

The developments in this section can be equally applied to the nonlinear reflex adaptation problem in Section 3, so we opt to use its notation. In Theorem 6 we showed that the proposed reflex adaptation scheme is correct when the exogenous regressor $v(t) = (y, -\pi)(t) \in \mathbb{R}^k$ (for an appropriate $k \in \mathbb{N}$) is PE. The key property required from reflex adaptation is that its averaged dynamics be GES, which requires $v(t)$ to be PE. Here we show that GES can be recovered for the non-PE case with the help of the μ-modification. Similar to Section 6.2, define the exogenous matrix $\Sigma_v(t)$ generated by

$$\dot{\Sigma}_v = -\epsilon_v \Sigma_v + \epsilon_v v(t) v^\intercal(t), \quad \Sigma_v(t_0) = R_v(0),$$

resulting in the error dynamics

$$\dot{\tilde{\Sigma}}_v = -\epsilon_v \tilde{\Sigma}_v + \epsilon_v(\tilde{v} v^\intercal + v \tilde{v}^\intercal + \tilde{v}\tilde{v}^\intercal) \tag{60}$$

where $\tilde{\Sigma}_v := \hat{\Sigma}_v - \Sigma_v$ and $\tilde{v} := (0, -z)$ with $z = x - \pi(\zeta)$ being the error coordinate obtained from the regulator equations. Given that the state $\tilde{\Sigma}_v$ does not appear in (57)–(59), the following is immediate.

Corollary 1. *Let $v_{pe}(t) \in \mathbb{R}^{k_{pe}}$ and $\mathcal{V} = \mathrm{Im}(V)$ result from the PE decomposition of $v(t)$, and let $\beta_0^v > 0$ be the lower PE bound in Definition 2 for $v_{pe}(t)$. If $k_{pe} = \dim(\mathcal{V}) = 0$, set $\beta_0^v = \infty$. Then, in addition to the results of Theorem 11, for every $\sigma_v \in (0, \beta_0^v)$ there exists a constant $\epsilon_v^\star > 0$ such that for each $\epsilon_v \in (0, \epsilon_v^\star]$ the equilibrium $(z_\circ, \tilde{\psi}, \tilde{\xi}, \tilde{w}, \tilde{\Sigma}_w, \tilde{\Sigma}_v) = (0,0,0,0,0,0)$ is ES over $(z_\circ, \tilde{\psi}, \tilde{w}, \tilde{\Sigma}_w, \tilde{\Sigma}_v)(t_0) \in \mathcal{B}(\delta_1)$ and $\tilde{\xi}(t_0) \in \mathcal{B}(\delta_1 \kappa^{r-1})$ uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$.*

Proof. The choice of $\epsilon_v^\star(\sigma_v) > 0$ follows from Theorem 9. The combined system is a cascade interconnection from (57)–(59), which is GUAS and LES by Theorem 7 (ignoring the lack of continuous differentiability of Ω_w), to (60). The system (60) is input-to-state stable with respect to $\tilde{v}$, and its unforced dynamics $\dot{\tilde{\Sigma}}_v = -\epsilon_v \tilde{\Sigma}_v$ is GES. Therefore, the combined system is GUAS and LES by a standard Lyapunov argument. □

Letting $\chi := (z_\circ, \tilde{\psi}, \tilde{\xi}, \tilde{w}, \tilde{\Sigma}_w, \tilde{\Sigma}_v)$, we note that $\chi = 0$ implies $\Omega_v = V_\perp V_\perp^\intercal$ by Lemma 6. Therefore, the averaged dynamics become

$$\dot{\hat{\alpha}}_{av} = -\epsilon R_v(0)(\hat{\alpha}_{av} - \alpha) - \epsilon \mu_v V_\perp V_\perp^\intercal \hat{\alpha}_{av} \tag{61}$$

by (R8).

Theorem 12. *The equilibrium $\hat{\alpha}_{av} = V V^\intercal \alpha$ of (61) is GES.*

Proof. Since $V^\intercal V_\perp = 0$ and $R_v(0) = V R_{v_{pe}}(0) V^\intercal$, we have

$$\dot{\hat{\alpha}}_{av} = -\epsilon \left(V R_{v_{pe}}(0) V^\intercal + \mu_v V_\perp V_\perp^\intercal \right) (\hat{\alpha}_{av} - V V^\intercal \alpha).$$

Therefore $\hat{\alpha}_{av} = V V^\intercal \alpha$ is an equilibrium of (61). Given that $R_{v_{pe}}(0) \succ 0$, $\mu_v > 0$, and $I = V V^\intercal + V_\perp V_\perp^\intercal$, it is clear that $V R_{v_{pe}}(0) V^\intercal + \mu_v V_\perp V_\perp^\intercal \succ 0$. Hence (61) is a linear ODE with a Hurwitz state matrix and one concludes GES. □

6.4. Proof of correctness

We may now state our final result, which extends Theorem 8 to the case of non-PE regressors.

Theorem 13. *Consider system (6) with the regulator (53)–(55) satisfying Assumptions 1, 6–7, 12. Also, fix orbits $\mathcal{Z}$, $\mathcal{Z}_y$ of the exosystems. Define $x_c := (\hat{w}, \hat{\psi}, \hat{\xi}, \tilde{\Sigma}_w, \tilde{\Sigma}_v)$. Then for each $\delta > 0$ one may instantiate one such regulator so that Assumptions 3–5[1] hold and the coordinate transformation (24) yields a closed-loop system of the form (16). Therefore, for each regulator built the conclusions of Theorem 6 hold.*

[1] The assumption of continuous differentiability of Ω_w and Ω_v must be relaxed, since they are generally not continuously differentiable.

Proof. If we ignore the lack of continuous differentiability of Ω_w and Ω_v with respect to the relevant states, then the result follows from the proof of Theorem 8. Useful facts needed to account for the additional states are that $(\tilde{\Sigma}_w, \tilde{\Sigma}_v) \to (0,0)$ and that for every $\delta > 0$ the regulator is constructed using Corollary 1 rather than Theorem 7. Moreover, since the PE condition is time shift invariant there exists uniform PE lower bounds β_0^w, $\beta_0^v > 0$ and unique PE subspaces $\mathcal{W}$, $\mathcal{V}$ that apply for all $(\zeta, \zeta_y)(t_0) \in \mathcal{Z} \times \mathcal{Z}_y$. □

Remark 23. The choice of $(\tilde{\Sigma}_w, \tilde{\Sigma}_v)$ instead of $(\hat{\Sigma}_w, \hat{\Sigma}_v)$ as components of x_c is to circumvent any discussion about the exogenous filters $(\Sigma_w, \Sigma_v)(t)$ and their unique initial conditions $(R_w(0), R_v(0))$ in the statement of Theorem 11. Furthermore, this enables the construction of a steady-state map $\pi_c(\cdot)$ that is once again solely dependent on ζ, ζ_y, and $\hat{\alpha}$. ◁

Remark 24. To aid the reader on the ordering on how to select all constants involved, first one picks appropriate excitation tolerances $\sigma_w, \sigma_v > 0$. Then sufficiently small gains $\epsilon_w, \epsilon_v > 0$ for subspace estimation can be selected. In parallel, one may choose the high gain $K > 0$ for stability. Once the aforementioned constants have been selected, a sufficiently large high gain $\kappa \geq 1$ for the observer can be chosen. At last, the slow rate $\epsilon > 0$ must be selected sufficiently small. All other constants not mentioned can be selected a priori without any regard for the above constants. ◁

6.5. Biological plausibility

Our proposed model of long-term adaptation of reflexes by the cerebellum starts from a hypothesis that the cerebellum regulates certain error signals of the body. For several modules of the cerebellum, the error signals have been identified with reasonable confidence, particularly the flocculus which regulates the retinal error, and the nodulus/uvula, which regulates the retinal slip velocity with respect to the visual field.

Associated to each error signal regulated by the cerebellum we assume there is a linear plant model (6) or a nonlinear plant model (19). For the flocculus and nodulus/uvula, the first and second order linear oculomotor plant models are well known (Robinson, 1981). For the collic reflexes, plant models are also available (Peng, 1996; Peng, Hain, & Peterson, 1996). For regulation of the arm, the Euler–Lagrange framework can be used to model the arm, so long as one carefully models the actuation of the arm by the muscles. A limitation of the presented framework is the restriction to continuous-time plant models. We have overlooked reflexes that appear to operate in discrete-time such as the eyeblink reflex. More generally, we have not captured the role of timing in reflexes. A broader class of plant models will be necessary to achieve a comprehensive, biologically accurate framework of reflex adaptation.

Once an error signal to be regulated and a continuous-time plant model are identified, then the cerebellum is modeled as the adaptive internal model given in (54). Our prior work has discussed the plausibility of this internal model design in terms of the high-level neural architecture of the cerebellum (Broucke, 2021a, 2022). The input signals to the filters (54a)–(54b) correspond to the *mossy fiber inputs* of the cerebellum; the fast adaptation process (54f) takes place at the synapses between *parallel fibers* and *Purkinje cells*. The error signal appearing in (54f) is intended to represent the *climbing fiber input* arriving from the *inferior olive*. The Purkinje cell output of the cerebellum is modeled by u_{im} in (54g). The appearance of u_{im} as a mossy fiber input in (54a) corresponds to the *nucleo-cortical pathway* (Houck & Person, 2014, 2015), essential to ensure that excitation due to disturbances can be sustained by the purely feedforward neural architecture of the cerebellum. We note that the placement of the filters in (54b) is debatable; here they have been incorporated with the cerebellum.

The stabilizer (53c) using a high gain observer in (53a) is driven by the requirements of the internal model design in Serrani et al. (2001), and is not tied to any biological imperative. We consider this part of the model to be the least biologically plausible. On the other hand, the brain likely does handle higher relative degree systems in some form. For example, the oculomotor plant is often modeled as a first order system for simplicity, but a more detailed model includes a second pole, making it second-order and relative degree 2. The brain is believed to utilize observers (called forward models in the neuroscience literature); for instance the oculomotor system utilizes a *brainstem neural integrator*, as we will see in Section 7. The motor systems for the limbs also likely utilize observers, possibly housed in the motor cortex M1 and communicating with the cerebellum through the cerebello-thalamo-cortical pathway and the cortico-ponto-cerebellar pathway.

Generally, there is little neuroscience research on dedicated mechanisms of stabilization in the brain (aside from the reflexes themselves). Some clues on the possibility of high-gain error feedback in the cerebral cortex for stabilization of the oculomotor system is reported in Lee, Yang, and Lisberger (2013), Nuding et al. (2009). Recent results in Tomei and Marino (2023) on stabilization of uncertain minimum phase systems are promising for removing the high-gain observer in order to simplify our design.

Regarding the μ-modification, we hypothesize that the filtering in (54d) and (55a) takes place at the neuronal level at the site of plasticity. Evidence that neurons act as principal component analyzers is discussed in Oja (1982). Nevertheless, further investigation is need to understand how Hebbian learning can be modified to perform SVD-like computations as would be required for the μ-modification.

7. Simulations

This section presents simulations for a pedagogical example and for a systems neuroscience application. Unless otherwise specified, the update time of subspace estimators is every 2 seconds and all initial conditions for the simulations are zero.

7.1. Pedagogical example

We apply our design to a second-order unstable minimum phase LTI plant of relative degree 2 with an exosystem that generates a biased sinusoid. Consider the LTI plant model (6) with plant parameters

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The disturbance measurement is $y(t) = \begin{bmatrix} 10 & 10\sin(2t) \end{bmatrix}^\top$, and the exosystem parameters are

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}, \quad D = 0,$$

with $\zeta(0) = \begin{bmatrix} 5 & 0 & 5 \end{bmatrix}^\top$. One may verify with some algebra that the solution of the regulator Eqs. (7), the exosystem state $\zeta(t)$, and the disturbance measurement $y(t)$ satisfy Assumption 7; namely

$$\Gamma\zeta(t) = \alpha_x^\top \Pi \zeta(t) - \alpha_r^\top y(t),$$

with $\alpha_r = \begin{bmatrix} 0.5 & 4.5 \end{bmatrix}^\top$, $\alpha_x = \begin{bmatrix} \alpha_{x,1} & -2 \end{bmatrix}^\top$, $\alpha_{x,1} \in \mathbb{R}$. We note that $\alpha_{x,1}$ does not have a unique solution, corresponding to the fact it is the non-PE component of α_x.

Next we apply the regulator (53)–(55) with parameters selected as $K = 50$, $L = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$, $a = 2$, $\kappa = 100$, $\gamma = \mu_w = \varepsilon_w = \varepsilon_v = \sigma_w = \sigma_v = 1$, $\mu_v = 20$, $\varepsilon = 1 \times 10^{-5}$, and

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The simulation results are illustrated in Fig. 3. The initial conditions of the reflex gains are $\hat{\alpha}_r(0) = \hat{\alpha}_x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$. Fig. 3(a) shows that the error is regulated to zero on a short timescale. Figs. 3(b)–(c) show that the reflex gains $\hat{\alpha}_r$ and $\hat{\alpha}_x$ converge on a long timescale. Fig. 3(c) shows that because of the μ-modification, $\hat{\alpha}_{x,1}$ goes to zero since it is not excited by the regressor $v = (y, -\Pi\zeta)$. Fig. 3(d) shows that the reflexes are able to fully offload the work of disturbance rejection by the adaptive internal model on a long timescale. Moreover, the μ-modification ensures that $\hat{\psi}$ tends to zero, as seen in Fig. 3(e).

7.2. Oculomotor system

The oculomotor system is widely regarded by neuroscientists as the blueprint for all other motor systems of the body (Leigh & Zee, 2015). It well exemplifies the fact that control theory requires further development to address modeling problems of systems neuroscience. Here we apply our two timescale regulator design to realize a model of *long-term adaptation* of the *brainstem neural integrator* motor command and the *VOR gain*.

The model of the oculomotor system, taken from Broucke (2021a), is given by

$$\dot{x} = -Ax + u \tag{62a}$$

$$\dot{\hat{x}} = -\hat{A}\hat{x} + u \tag{62b}$$

$$u_b = \hat{\alpha}_x \hat{x} - \hat{\alpha}_r \dot{x}_h \tag{62c}$$

$$u = u_s + u_{im} + u_b\,. \tag{62d}$$

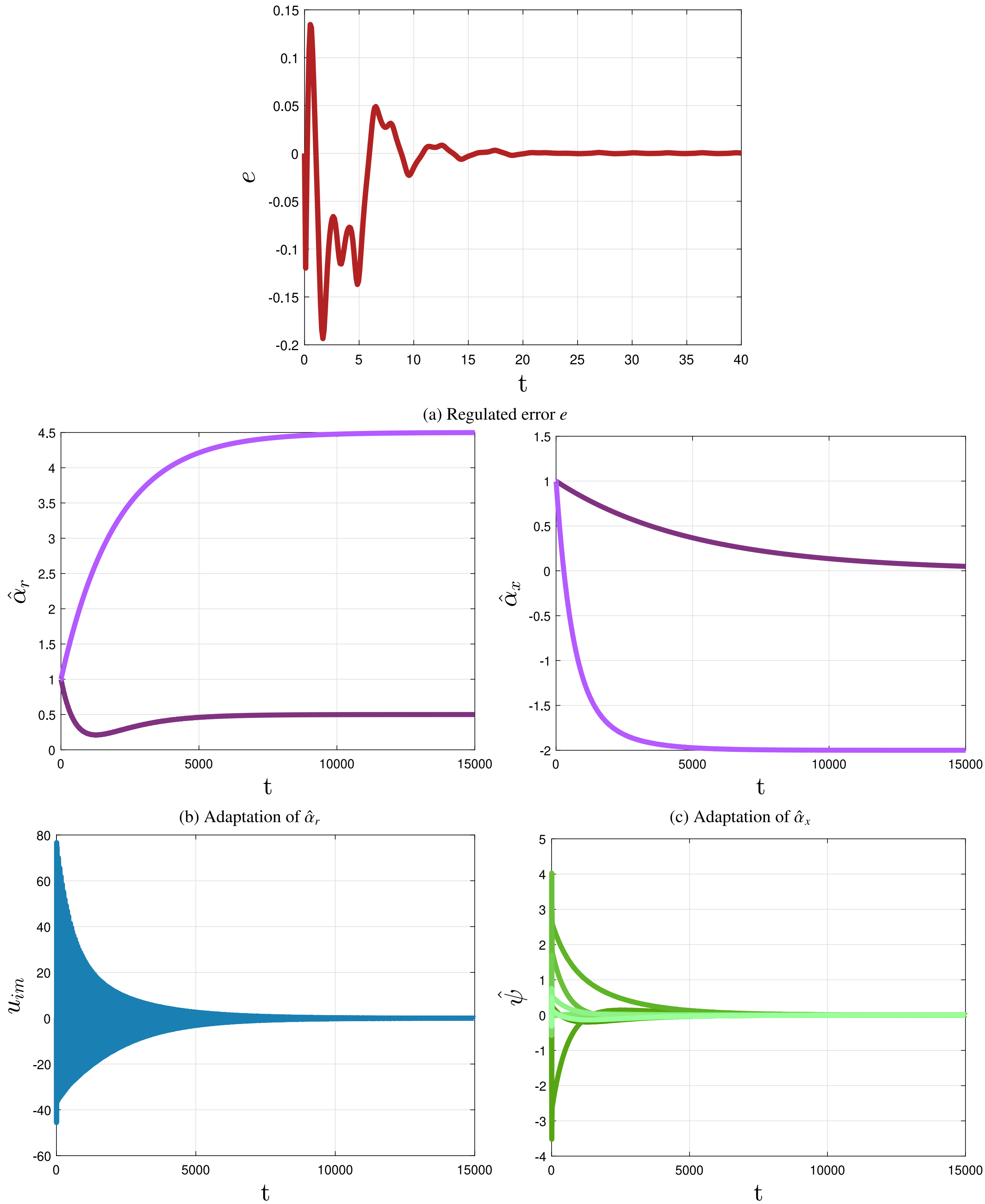
Eq. (62a) is a first-order model of the horizontal movement of a single human eye, where $x(t) \in \mathbb{R}$ is the horizontal eye angle in a head-fixed frame, $u(t) \in \mathbb{R}$ is the net horizontal torque on the eyeball, and $A > 0$ is a parameter that determines the time constant of the eye (Leigh & Zee, 2015; Robinson, 1981). The oculomotor system is supported by a *brainstem neural integrator* (62b), which is an adaptive observer of the oculomotor plant, where $\hat{x}(t) \in \mathbb{R}$ is an estimate of the eye position and $\hat{A}(t) \in \mathbb{R}$ is an estimate of A. The signal u_b is generated in the brainstem and includes two reflex signals, the state feedback $u_x = \hat{\alpha}_x \hat{x}$ generated by the brainstem neural integrator, and the vestibuloocular reflex (VOR) $u_r = -\hat{\alpha}_r \dot{x}_h$, where x_h is the horizontal angular position of the head in a world-fixed frame. The reflex gains are $\hat{\alpha}_x$ and $\hat{\alpha}_r$.

Despite the fact that the plant is a trivial first-order LTI system, regulation of eye position cannot be solved by standard methods of adaptive control. There are already three parameters in (62) to be adapted, those with "hats", but the oculomotor system by design must maintain the eye position in a highly regulated state such as gaze holding or tracking constant speed targets. Such reference signals do not provide enough excitation to adapt the parameters in the model and reflexes, not to mention the adaptive parameters needed by an adaptive internal model. The fundamental modeling issue is the need for more sophisticated methods to manage excitation while achieving near perfect regulation. Our hypothesis is that the adaptation of $\hat{A}$ is an independent process, so it will be treated in a separate paper. We assume $\hat{A}(t) \equiv A$. Notice that since $A > 0$, $\hat{x}(t)$ converges to $x(t)$ asymptotically.

We consider an oculomotor task to track a target in the visual field while the subject wears magnifying lenses. Thus, we define the (horizontal) *retinal error*

$$e := \alpha_m(r - x_h) - x,$$

where $r(t) \in \mathbb{R}$ is the horizontal angular position of a target, and α_m is the magnification factor. The goal of the oculomotor system is to drive e to zero. The cerebellum provides the top up signal u_{im} generated by an adaptive internal model to ensure complete regulation of the retinal error. The overall *motor command* is (62d), where $u_s(t) \in \mathbb{R}$ represents a signal that carries visual information only and improves closed-loop stability of the system. We select $u_s = Ke$.



(a) Regulated error e

(b) Adaptation of $\hat{\alpha}_r$

(c) Adaptation of $\hat{\alpha}_x$

(d) Internal Model Output: u_{im}

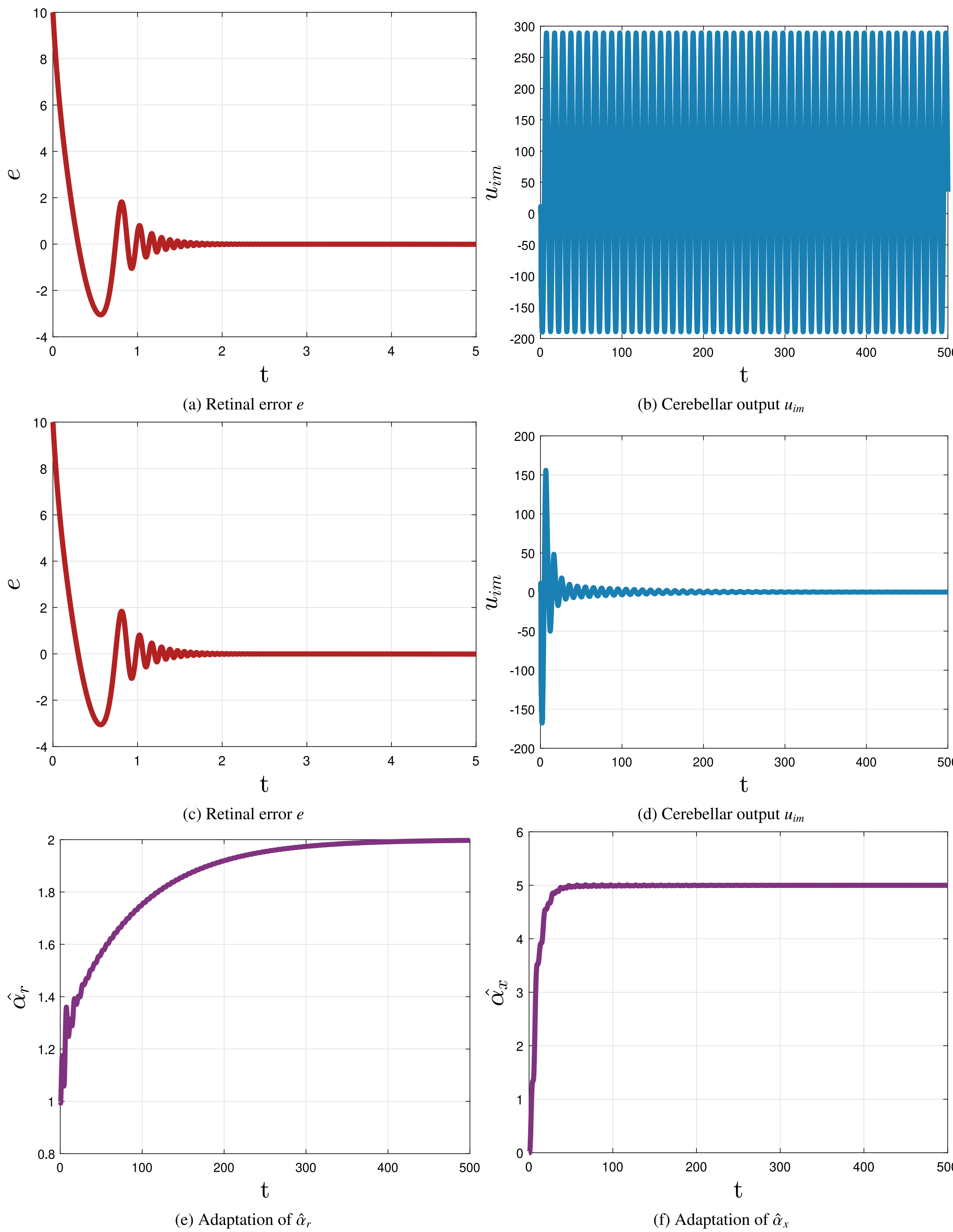
(e) Internal Model Parameter: $\hat{\psi}$

Fig. 3. Simulation results for a second-order pedagogical example.

We apply our design (53)–(55) with parameter values are $A = 5$, $\alpha_m = 2$, $K = 5$, $\gamma = \mu_w = \varepsilon_w = \varepsilon_v = 1$, $\mu_v = 50$, $\sigma_w = \sigma_v = 0.1$, $\varepsilon = 1 \times 10^{-5}$ and

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

First, we consider the case of a constant reference ($r = 5$) and a sinusoidal head velocity ($\dot{x}_h = 15\cos(0.2\pi t)$). The results are illustrated in Fig. 4. Fig. 4(a) shows that the retinal error is regulated to zero on a short timescale. Fig. 4(b) shows that when the reflex gains are not adapted ($\varepsilon = 0$), then the output of the cerebellum u_{im} is not reduced over a long timescale. Fig. 4(c)–(d) show the effect of including adaptation on the reflex gains. The retinal error is regulated to zero on a short timescale, as seen in Fig. 4(c), whereas in Fig. 4(d) we see that on a long timescale the reflexes are able to fully offload the work of the cerebellum. Figs. 4(e)–(f) show the adaptation of the reflex gains to their ideal values.



(a) Retinal error e

(b) Cerebellar output u_{im}

(c) Retinal error e

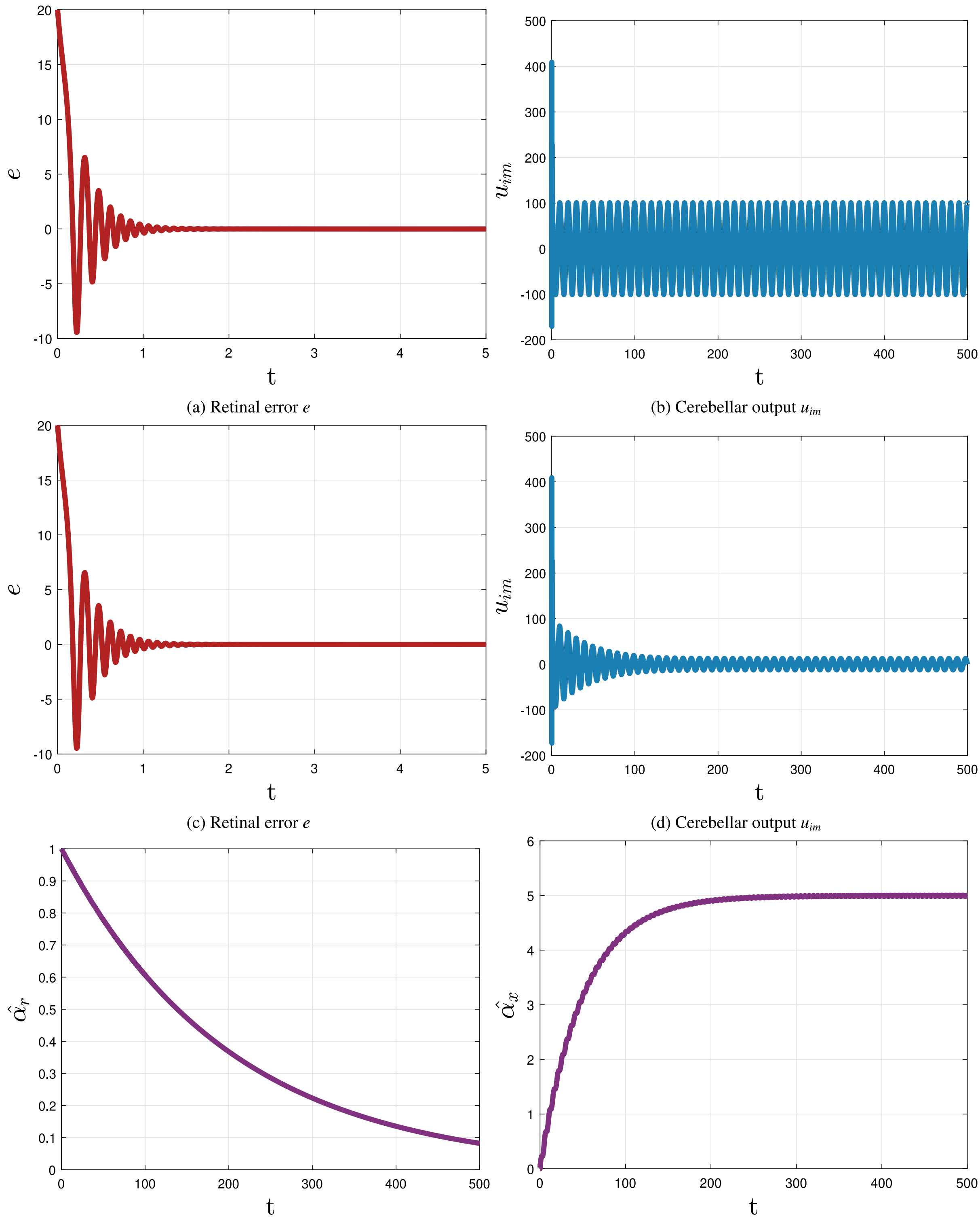
(d) Cerebellar output u_{im}

(e) Adaptation of $\hat{\alpha}_r$

(f) Adaptation of $\hat{\alpha}_x$

Fig. 4. Oculomotor system without reflex adaptation (a,b) and with reflex adaptation (c,d,e,f).

In the second scenario, we consider a case when List (R8) does not hold, yet the model still produces the correct behavior. The reference signal is $r = 10\cos(0.2\pi t)$, and the head is fixed $\dot{x}_h = 0$. In this case the cerebellum and the reflexes must share the work of disturbance rejection at steady state. The results are illustrated in Fig. 5. Figs. 5(a), (c) again show that the retinal error is regulated to zero on a short timescale, whether or not reflex gains are adapted. Fig. 5(b) shows that when the reflex gains are not adapted ($\varepsilon = 0$), then the output of the cerebellum u_{im} is not reduced. Fig. 4(d) shows the reduction in work by the cerebellum when the reflex gains are adapted, although we do not



(a) Retinal error e

(b) Cerebellar output u_{im}

(c) Retinal error e

(d) Cerebellar output u_{im}

(e) Adaptation of $\hat{\alpha}_r$

(f) Adaptation of $\hat{\alpha}_x$

Fig. 5. Oculomotor system without reflex adaptation (a,b) and with reflex adaptation (c,d,e,f).

have $u_{im} \to 0$ in this case. Figs. 5(e)–(f) show the adaptation of the reflex gains. The VOR gain $\hat{\alpha}_r$ tends to zero since it receives no excitation from its regressor $\dot{x}_h$, so it is forgotten due to μ-modification.

8. Open problems

There are several open theoretical problems encountered in the paper. First, we made an assumption in (R8) and Assumption 7 that the reflexes can perfectly cancel disturbances acting on the plant. While this assumption is valid for certain reflexes such as the stretch reflexes, it does not hold more generally, the prime example being the reflexes of the oculomotor system. We will develop the more general result without the assumption in a forthcoming paper using the same methods shown here. Second, we assumed in Assumption 12 that $\tilde{w} \to 0$ exponentially, independently of other states in the model. Of course such an assumption does not hold in practice, though we have noted in Section 7 that the proposed model behaves correctly in simulation. This theoretical gap must be addressed in our future work. Third, there are technicalities associated with constructing converse Lyapunov functions and establishing existence of solutions due to the non differentiability of the subspace estimator. As we proceed with continuous and discrete time designs, such questions will resurface.

A number of open problems in the area of adaptive regulator theory are independent of the reflex architecture presented here. Currently there is a thrust to remove the standard assumptions of adaptive control: the assumption of an upper bound on the relative degree of the plant (Marino & Tomei, 2021; Tomei & Marino, 2023); the assumption of a minimum phase plant (Bin, Marconi, & Teel, 2019); and the assumption of an upper bound on the order of the exosystem (Marino & Tomei, 2007, 2011). We have already mentioned the interesting results in Tomei and Marino (2023) on stabilization of minimum phase systems without the use of a high-gain observer.

On the systems neuroscience side, the most important open problem in the area of reflex adaptation is to understand how the cerebellum trains reflexes (both in the short-term and long-term) that apparently operate in discrete-time, such as the eyeblink reflex.

9. Supporting proofs

9.1. Proof of Proposition 1

The implication that ES over every ball implies GUAS and LES is immediate from the definitions, so we focus on the reverse implication. Fix $\delta > 0$. By LES, there exist $\delta_0, c_0, \lambda_0 > 0$ such that

$$\|x(t)\| \le c_0\|x(t_0)\|\mathrm{e}^{-\lambda_0(t-t_0)}, \quad \forall t \ge t_0,\ \forall \|x(t_0)\| < \delta_0.$$

By GUAS, there exists $T(\delta) > 0$ such that if $\|x(t_0)\| < \delta$, then $\|x(t)\| < \delta_0$ for all $t \ge t_0 + T(\delta)$. Therefore,

$$\begin{aligned}\|x(t)\| &\le c_0\|x(t_0+T(\delta))\|\mathrm{e}^{-\lambda_0(t-t_0-T(\delta))}\\ &\le c_1(\delta)\|x(t_0)\|\mathrm{e}^{-\lambda_0(t-t_0-T(\delta))}, \quad \forall t \ge t_0+T(\delta)\end{aligned}$$

and for some $c_1(\delta) > 0$. For the time period $[t_0, t_0 + T(\delta)]$, by GUAS (see Khalil (2002, Lemma 4.5)) there exists $c_2(\delta) > 0$ such that

$$\|x(t)\| \le c_2(\delta)\|x(t_0)\|, \quad \forall t \in [t_0, t_0+T(\delta)].$$

Combining our two bounds, we have

$$\begin{aligned}\|x(t)\| &\le \max\left\{c_1(\delta), c_2(\delta)\right\}\|x(t_0)\|\mathrm{e}^{-\lambda_0(t-t_0-T(\delta))}\\ &= \max\left\{c_1(\delta), c_2(\delta)\right\}\mathrm{e}^{T(\delta)}\|x(t_0)\|\mathrm{e}^{-\lambda_0(t-t_0)}, \quad \forall t \ge t_0.\end{aligned}$$

This proves the equilibrium $x = 0$ of (1) is ES over a ball of radius δ. Since $\delta > 0$ is arbitrary, the result follows. Finally, we note that the exponential rate λ_0 does not depend on δ.

9.2. Proof of Lemma 4

The proof here is similar to that of Serrani et al. (2001, Section VI.B), with the LES proof being slightly more concise by writing the linearization with respect to $w(t)$ rather than $\hat{w}(t)$.

The system (15) can be cast as an autonomous system because $\hat{w}_0 = w_0 + M_x x_\circ$ where w is generated by the LTI exosystem

$$\dot{\zeta}_w = S_w\zeta_w \tag{63a}$$

$$w_0 = \Gamma_w\zeta_w \tag{63b}$$

for an appropriate (Γ_w, S_w) and $\zeta_w(t) \in \mathbb{R}^{q_w}$. In particular, since S_w can be selected to have simple eigenvalues on the $j\omega$-axis, there exist $P_w > 0$ such that $S_w^{\mathsf{T}}P_w + P_wS_w = 0$.

Now consider the Lyapunov function

$$V(x_\circ, \tilde{\psi}_0, \zeta_w) := x_\circ^{\mathsf{T}}Px_\circ + |b|\gamma^{-1}\|\tilde{\psi}_0\|^2 + \zeta_w^{\mathsf{T}}P_w\zeta_w,$$

where $P > 0$ is provided by Lemma 3. One can verify that its Lie derivative with respect to (15), (63) is

$$\dot{V}(x_\circ, \tilde{\psi}_0, \zeta_w) \le -\rho\|x_\circ\|^2.$$

We deduce the equilibrium $(x_\circ, \tilde{\psi}_0, \zeta_w) = (0,0,0)$ is stable. Next we apply a LaSalle argument. Let

$$\mathcal{E} := \{\,(x_\circ, \tilde{\psi}_0, \zeta_w) \ : \ \rho\|x_\circ\|^2 = 0\,\}.$$

We claim the largest positively invariant set in $\mathcal{E}$ is $\{\,(0,0)\,\} \times \mathbb{R}^{q_w}$. To see this, note that $x_\circ(t) = 0$ for all $t \ge t_0 \ge 0$ implies

$$0 = A(K)0 + bB_\circ\hat{w}_0^{\mathsf{T}}\tilde{\psi} \implies 0 = \hat{w}_0^{\mathsf{T}}\tilde{\psi}_0 = (w_0(t) + M_w0)^{\mathsf{T}}\tilde{\psi}_0.$$

Because w_0 is PE, we can apply Lemma 2 to find

$$\dot{\tilde{\psi}}_0 = 0 \text{ and } w_0^{\mathsf{T}}(t)\tilde{\psi}_0 = 0 \implies \tilde{\psi}_0 = 0.$$

Since $V(\cdot)$ is radially unbounded, we may apply LaSalle's Theorem (Khalil, 2002, Corollary 4.2) to conclude $(x_\circ, \tilde{\psi}_0) \to 0$ for every initial condition. Thus, the equilibrium $(x_\circ, \tilde{\psi}_0) = (0,0)$ is GAS. Moreover, we can follow up with Serrani et al. (2001, Lemma III.1) to conclude GUAS.

To show LES, consider the linearization of (15) resulting in the LTV system

$$\begin{bmatrix}\dot{x}_\circ\\ \dot{\tilde{\psi}}_0\end{bmatrix} = \begin{bmatrix}A(K) & bB_\circ w_0^{\mathsf{T}}(t)\\ -\operatorname{sgn}(b)\gamma w_0(t)B_\circ^{\mathsf{T}} & 0\end{bmatrix}\begin{bmatrix}x_\circ\\ \tilde{\psi}_0\end{bmatrix}.$$

Since w_0 is PE and $w_0, \dot{w}_0$ are bounded, by Theorem 2 the equilibrium $(x_\circ, \tilde{\psi}_0) = (0,0)$ of the linearization is GES. Finally, since the dynamics (15) are locally Lipschitz in $(x_\circ, \tilde{\psi}_0)$ uniformly in $t \ge 0$, we can apply (Khalil, 2002, Theorem 4.13) to deduce LES.

9.3. Proof of Theorem 3

The proof is adapted from Serrani et al. (2001) with minor variations. Let $K_\star$ be defined as in Lemma 3 and fix any $K \ge K_\star$ and $\delta_1 > 0$. We begin by showing uniform boundedness of $(x_\circ, \tilde{\psi}, \tilde{\xi})$ using Teel and Praly (1995, Lemma 2.4), with appropriate modifications. To this end we verify:

- the system (14) is clearly continuously differentiable;
- there exists a compact set $\mathcal{W}$ such that $w_0(t) \in \mathcal{W}$ for all $t \ge 0$ by boundedness of $w_0(t)$;
- a time-varying form of the uniform Lyapunov property (Teel & Praly, 1995, Assumption ULP) is satisfied by the continuously differentiable (uniformly in $t \ge t_0 \ge 0$) converse Lyapunov function $V_1(t, x_\circ, \tilde{\psi})$ provided by Massera (1956, Theorem 23) because the equilibrium $(x_\circ, \tilde{\psi}_0) = (0,0)$ is GUAS by Lemma 4 and since the dynamics (14) are locally Lipschitz in $(x_\circ, \tilde{\psi}_0)$ uniformly in $w_0 \in \mathcal{W}$ (see Proposition 2 to conclude the Lipschitz property over compact sets in $(x_\circ, \tilde{\psi}_0)$). That is, there exists class-$\mathcal{K}_\infty$ functions $\alpha_i(\cdot)$ such that

$$\begin{aligned}\alpha_1(\|(x_\circ, \tilde{\psi}_0)\|) \le V_1(t, x_\circ, \tilde{\psi}_0) &\le \alpha_2(\|(x_\circ, \tilde{\psi}_0)\|)\\ \dot{V}_1(t, x_\circ, \tilde{\psi}_0) &\le -\alpha_3(\|(x_\circ, \tilde{\psi}_0)\|)\end{aligned}$$

  for all $(x_\circ, \tilde{\psi}_0)$;
- every sublevel set $\mathcal{S}_{c+1}^t := \{\,(x_\circ, \tilde{\psi}_0) \ : \ V_1(t, x_\circ, \tilde{\psi}_0) \le c+1\,\}$ is compact because $V_1(t,\cdot)$ is continuous and radially unbounded. Moreover,

$$\begin{aligned}\mathcal{S}_{c+1} &:= \{\,(x_\circ, \tilde{\psi}_0) \ : \ \alpha_2(\|(x_\circ, \tilde{\psi}_0)\|) \le c+1\,\}\\ \mathcal{S}_{c+1}^\circ &:= \{\,(x_\circ, \tilde{\psi}_0) \ : \ \alpha_1(\|(x_\circ, \tilde{\psi}_0)\|) \le c+1\,\}\end{aligned}$$

  are compact and $\mathcal{S}_{c+1} \subseteq \mathcal{S}_{c+1}^t \subseteq \mathcal{S}_{c+1}^\circ$ for all $t \ge t_0 \ge 0$;
- $A_\circ - LC_\circ$ is Hurwitz by design;

- for any fixed $c > 1$ and all $\kappa \geq 1$ we have
$$b_1 \|\tilde{\xi}\| \geq \|B_1(K, \kappa, \hat{w}_0)\tilde{\xi}\|$$
$$b_2 + b_3\|\tilde{\xi}\| \geq \|B_2(K, \hat{w}_0) \begin{bmatrix} x_\circ \\ \tilde{\psi}_0 \end{bmatrix} \| + \|B_3(K, \kappa)\tilde{\xi}\|$$
for some $b_i(K, c) > 0$ provided $(x_\circ, \tilde{\psi}_0) \in \mathcal{S}^\circ_{c+1}$ and $\tilde{\xi} \in \mathbb{R}^r$ because $w_0 \in \mathcal{W}$.

Given that $A_\circ - LC_\circ$ is Hurwitz, let $P_L > 0$ solve the Lyapunov equation $(A_\circ - LC_\circ)^\intercal P_L + P_L(A_\circ - LC_\circ) = -I$ and define

$$\mu(\kappa) := \ln\left(1 + \lambda_{\max}(P_L)\delta_1^2 \kappa^{2(r-1)}\right) \tag{64}$$

so that Teel and Praly (1995, Eq. (70)) is satisfied. Since $\alpha_2(\cdot)$ is radially unbounded, we can always find a $c(\delta_1) > 1$ such that $\mathcal{B}(\delta_1) \subseteq \mathcal{S}_c$. Define the set

$$\mathcal{D}_t := \mathrm{int}(\mathcal{S}^t_{c+1}) \times \{\, \tilde{\xi} \;:\; U(\tilde{\xi}) < \mu(\kappa) + 1 \,\}$$
$$U(\tilde{\xi}) = \ln(1 + \tilde{\xi}^\intercal P_L \tilde{\xi}).$$

Following the proof of Teel and Praly (1995, Lemma 2.4), there exists a continuously differentiable function

$$W(t, x_\circ, \tilde{\psi}_0, \tilde{\xi}) = \frac{cV_1(t, x_\circ, \tilde{\psi}_0)}{c + 1 - V_1(t, x_\circ, \tilde{\psi}_0)} + \frac{\mu(\kappa)U(\tilde{\xi})}{\mu(\kappa) + 1 - U(\tilde{\xi})} \tag{65}$$

defined over $(x_\circ, \tilde{\psi}_0, \tilde{\xi}) \in \mathcal{D}_t$ for each $t \geq t_0 \geq 0$, and a constant $\kappa_\star(K, \delta_1) \geq 1$ such that for any $\kappa \geq \kappa_\star$ the Lie derivative of (the time-varying) $W(\cdot)$ with respect to (14) satisfies

$$\dot{W}(t, x_\circ, \tilde{\psi}_0, \tilde{\xi}) \leq 0$$

over $(x_\circ, \tilde{\psi}_0, \tilde{\xi}) \in \mathcal{D}'_t$ for each $t \geq t_0 \geq 0$ where

$$\mathcal{D}'_t := \{\, (x_\circ, \tilde{\psi}_0, \tilde{\xi}) \in \mathcal{D}_t \;:\; 1 \leq W(t, x_\circ, \tilde{\psi}_0, \tilde{\xi}) \leq c^2 + \mu^2(\kappa) + 1 \,\}.$$

Similarly, define the (strict) sublevel sets

$$\mathcal{E}_t := \{\, (x_\circ, \tilde{\psi}_0, \tilde{\xi}) \in \mathcal{D}_t \;:\; W(t, x_\circ, \tilde{\psi}_0, \tilde{\xi}) < c^2 + \mu^2(\kappa) + 1 \,\}$$

and let $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t)$ denote solutions of the system (14). In order to show uniform boundedness of states, we need to establish positive invariance of a collection of time-varying sets.

Claim. If $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_0) \in \mathcal{E}_{t_0}$, then $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t$ for all $t \geq t_0 \geq 0$.

Proof of Claim. Suppose not and define

$$\mathcal{T} := \{\, t \in [t_0, \infty) \;:\; (x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \notin \mathcal{E}_t \,\}.$$

By assumption $\mathcal{T}$ is non-empty and bounded below, so its infimum $t_\star := \inf \mathcal{T}$ exists. First, we show that $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \notin \mathcal{E}_{t_\star}$. Define the continuous functions

$$W(t) := W(t, x_\circ(t), \tilde{\psi}_0(t), \tilde{\xi}(t))$$
$$V_1(t) := V_1(t, x_\circ(t), \tilde{\psi}_0(t))$$
$$U(t) := U(\tilde{\xi}(t)).$$

For the sake of contradiction, suppose that $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \mathcal{E}_{t_\star}$ so that

$$W(t_\star) < c^2 + \mu^2(\kappa) + 1, \quad V_1(t_\star) < c + 1, \quad U(t_\star) < \mu(\kappa) + 1.$$

By continuity, there exists a $\Delta t > 0$ such that the above inequalities hold for $t \in [t_\star, t_\star + \Delta t)$. Equivalently, $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t$ for all $t \in [t_\star, t_\star + \Delta t)$ and so $t_\star < \inf \mathcal{T}$, which is a contradiction. An immediate consequence is that $t_\star > t_0$, otherwise $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_0) \in \mathcal{E}_{t_0}$ and $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_0) \notin \mathcal{E}_{t_0}$.

Next, by continuity of $W(t)$ and since $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t$ for all $t \in [t_0, t_\star)$, we have that $W(t_\star) \leq c^2 + \mu^2(\kappa) + 1$. Additionally, there exists a $t_1 < t_\star$ such that $W(t) \geq 1$ for all $t \in [t_1, t_\star)$. Equivalently $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{D}'_t$ and so its time derivative satisfies $\dot{W}(t) \leq 0$ for $t \in [t_1, t_\star)$. By the Fundamental Theorem of Calculus and the fact $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_1) \in \mathcal{E}_{t_1}$ since $t_1 < t_\star$, we compute

$$W(t_\star) = W(t_1) + \int_{t_1}^{t_\star} \dot{W}(\tau)\, d\tau \leq W(t_1) < c^2 + \mu^2(\kappa) + 1.$$

Lastly, we claim that $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \mathcal{D}_{t_\star}$. Again by continuity of $V_1(t)$ and $U(t)$, it must be that

$$(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \mathcal{S}^{t_\star}_{c+1} \times \{\, \tilde{\xi} \;:\; U(\tilde{\xi}) \leq \mu(\kappa) + 1 \,\} =: \bar{\mathcal{D}}_{t_\star}.$$

For the sake of contradiction, suppose $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \partial\bar{\mathcal{D}}_{t_\star}$. Then it must be that at least one of

$$V_1(t_\star, x_\circ(t_\star), \tilde{\psi}_0(t_\star)) = c + 1, \quad U(\tilde{\xi}(t_\star)) = \mu(\kappa) + 1$$

hold. By the form of (65) and since $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t$ for $t \in [t_0, t_\star)$, it is clear that

$$\lim_{t \to t_\star^-} W(t) = +\infty,$$

implying there exists $t_2 < t_\star$ such that $W(t) \geq c^2 + \mu^2(\kappa) + 1$ for $t \in [t_2, t_\star)$. This contradicts $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t$ for $t \in [t_2, t_\star)$ and so it must be that $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \mathrm{int}(\bar{\mathcal{D}}_{t_\star}) = \mathcal{D}_{t_\star}$.

Altogether, we have shown $W(t_\star) < c^2 + \mu^2(\kappa) + 1$ and $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \mathcal{D}_{t_\star}$. This is equivalent to $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_\star) \in \mathcal{E}_{t_\star}$. Given that we have already showed this is a contradiction, it must be that such a $t_\star$ does not exist and so $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t$ for all $t \geq t_0 \geq 0$. □

From the definition of $\mathcal{S}_c$ and $\mu(\kappa)$ in (64), we can see our set of initial conditions satisfy

$$\mathcal{B}(\delta_1) \times \mathcal{B}(\delta_1 \kappa^{r-1}) \subseteq \mathcal{S}_c \times \mathcal{B}(\delta_1 \kappa^{r-1}) \subseteq \mathcal{E}_t$$

for all $t \geq t_0 \geq 0$, then by our claim we have that

$$(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{E}_t \subseteq \mathcal{D}_t \subseteq \mathcal{S}^\circ_{c+1} \times \{\, \tilde{\xi} \;:\; U(\tilde{\xi}) \leq \mu(\kappa) + 1 \,\}$$

for all $t \geq t_0 \geq 0$. By our choice of $V_1(\cdot)$ and $U(\cdot)$, it is clear that both $\mathcal{S}^\circ_{c+1}$ and $\{\, \tilde{\xi} \;:\; U(\tilde{\xi}) \leq \mu(\kappa)+1 \,\}$ are compact, meaning that all our states $(x_\circ, \tilde{\psi}_0, \tilde{\xi})$ are uniformly bounded provided our initial conditions satisfy $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t_0) \in \mathcal{B}(\delta_1) \times \mathcal{B}(\delta_1 \kappa^{r-1})$. Remark that the trajectories $(x_\circ, \tilde{\psi}_0)(t)$ in fact remain in $\mathcal{S}^\circ_{c+1}$ (which is independent of κ) regardless of how large κ is selected.

Finally we show ES over $(x_\circ, \tilde{\psi}_0)(t_0) \in \mathcal{B}(\delta_1)$ and $\tilde{\xi}(t_0) \in \mathcal{B}(\delta_1 \kappa^{r-1})$. Since every sublevel set $\mathcal{S}^\circ_{c+1}$ is compact, there exists some $\delta_2(c) > 0$ so that $\mathcal{S}^\circ_{c+1} \subseteq \mathcal{B}(\delta_2)$. By Proposition 1 and the fact that the equilibrium $(x_\circ, \tilde{\psi}_0) = (0, 0)$ of (15) is GUAS and LES from Lemma 4, there exists a converse Lyapunov function

$$V_2(t, x_\circ, \tilde{\psi}_0) \;:\; \mathbb{R}_+ \times \mathcal{B}(\delta_2) \to \mathbb{R}_+$$

for ES satisfying the conclusions of Khalil (2002, Theorem 4.14). Note that such a $V_2(\cdot)$ depends on the choice of δ_2. For the system (14), consider the Lyapunov function

$$V_3(t, x_\circ, \tilde{\psi}_0) := V_2(t, x_\circ, \tilde{\psi}_0) + \tilde{\xi}^\intercal P_L \tilde{\xi}.$$

Recalling that we have previously shown $(x_\circ, \tilde{\psi}_0, \tilde{\xi})(t) \in \mathcal{S}^\circ_{c+1} \times \{\, \tilde{\xi} \;:\; U(\tilde{\xi}) \leq \mu(\kappa) + 1 \,\}$, this implies that $(x_\circ, \tilde{\psi}_0)(t) \in \mathcal{S}^\circ_{c+1} \subseteq \mathcal{B}(\delta_2)$. Therefore, our Lyapunov function $V_3(\cdot)$ is valid for all the considered trajectories and one can compute its time derivative (omitting the argument t) as

$$\begin{aligned} \dot{V}_3(t, x_\circ, \tilde{\psi}_0, \tilde{\xi}) &\leq -c_3\|(x_\circ, \tilde{\psi}_0)\|^2 - \left(\kappa - 2\lambda_{\max}(P_L)b_3\right)\|\tilde{\xi}\|^2 \\ &\quad + \left(c_4 b_1 + 2\lambda_{\max}(P_L)b_2\right)\|(x_\circ, \tilde{\psi}_0)\|\,\|\tilde{\xi}\| \\ &\leq -\gamma_3 V_3(t, x_\circ, \tilde{\psi}_0, \tilde{\xi}) \end{aligned}$$

for an appropriately redefined $b_2 > 0$ and where the second inequality holds for some $\gamma_3 > 0$ provided κ is selected sufficiently large. It is wlog that we redefine $\kappa_\star(K, \delta_1) \geq 1$ to be possibly larger so that a unique $\gamma_3 > 0$ exists for all $\kappa \geq \kappa_\star$. At last, the Comparison Lemma then proves the result.

9.4. Results for averaging analysis

This section provides supporting results that have been used to obtain the averaging conditions in Assumption 2.

Proposition 6. *If $f(t,\tilde{\alpha})$ is continuous or locally Lipschitz in $\tilde{\alpha}$, both uniformly in $t \geq 0$, then so is $f_{av}(\tilde{\alpha})$.*

Proposition 7. *Suppose $\partial_{\tilde{\alpha}}\rho(t,\tilde{\alpha})$ is continuous in $\tilde{\alpha}$ uniformly in $t \geq 0$ and $(\partial_{\tilde{\alpha}}\rho)_{av}(\cdot)$ exists with convergence uniform in $\tilde{\alpha}$ over compact sets. If $\rho_{av}(\cdot)$ exists, then $(\partial_{\tilde{\alpha}}\rho)_{av}(\cdot) = \partial_{\tilde{\alpha}}\rho_{av}(\cdot)$.*

Proof. Consider $\rho_{av}(\tilde{\alpha} + \lambda v)$ for all $\lambda \in [0,1]$ and $\|v\| = 1$. Taking its total derivative with respect to λ we have

$$
\begin{aligned}
d_\lambda \rho_{av}(\tilde{\alpha} + \lambda v) &= d_\lambda \left[\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \rho(\tau, \tilde{\alpha} + \lambda v)\, d\tau \right] \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} d_\lambda \rho(\tau, \tilde{\alpha} + \lambda v)\, d\tau \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \partial_{\tilde{\alpha}} \rho(\tau, \tilde{\alpha} + \lambda v) v\, d\tau \\
&= (\partial_{\tilde{\alpha}}\rho)_{av}(\tilde{\alpha} + \lambda v) v,
\end{aligned}
$$

where we note that the average (which is a limit) and the derivative with respect to λ can be swapped in the second equality by uniform convergence in $\tilde{\alpha}$ over compact sets of $\partial_{\tilde{\alpha}}\rho(\cdot)$ to $(\partial_{\tilde{\alpha}}\rho)_{av}(\cdot)$ and by the existence of $\rho_{av}(\cdot)$ (Rudin, 1976, Theorem 7.17). Applying the chain rule to $d_\lambda \rho_{av}(\cdot)$ and then setting $\lambda = 0$, we conclude that

$$(\partial_{\tilde{\alpha}}\rho)_{av}(\tilde{\alpha}) v = \partial_{\tilde{\alpha}}\rho_{av}(\tilde{\alpha}) v$$

for all $\|v\| = 1$, as desired. □

Proposition 8. *Suppose $\rho(t,\tilde{\alpha})$ and $\partial_{\tilde{\alpha}}\rho(t,\tilde{\alpha})$ satisfy Proposition 7, $\rho(t,0) = 0$ for all $t \geq 0$, and for a compact set $\mathcal{S}$ there exists a function $\Delta(T,\mathcal{S})$ satisfying*

$$\Big\|\frac{1}{T}\int_{t_0}^{t_0+T} \partial_{\tilde{\alpha}}\rho(\tau,\tilde{\alpha})\, d\tau\Big\| \leq \Delta(T,\mathcal{S})$$

for all $\tilde{\alpha} \in \mathcal{S}$ and $t_0 \geq 0$. Then one has that

$$\Big\|\frac{1}{T}\int_{t_0}^{t_0+T} \rho(\tau,\tilde{\alpha})\, d\tau\Big\| \leq \Delta(T,\mathcal{S})\|\tilde{\alpha}\|$$

for all $\tilde{\alpha} \in \mathcal{S}$ and $t_0 \geq 0$.

Proof. Recalling that $d_\lambda \rho(t, \lambda\tilde{\alpha}) = \partial_{\tilde{\alpha}}\rho(t, \lambda\tilde{\alpha})\tilde{\alpha}$ by the chain rule, we have that

$$
\begin{aligned}
\Big\|\frac{1}{T}\int_{t_0}^{t_0+T} \rho(\tau,\tilde{\alpha})\, d\tau\Big\| &= \Big\|\frac{1}{T}\int_{t_0}^{t_0+T}\int_0^1 \partial_{\tilde{\alpha}}\rho(\tau,\lambda\tilde{\alpha})\tilde{\alpha}\, d\lambda\, d\tau\Big\| \\
&= \Big\|\int_0^1 \frac{1}{T}\int_{t_0}^{t_0+T} \partial_{\tilde{\alpha}}\rho(\tau,\lambda\tilde{\alpha})\tilde{\alpha}\, d\tau\, d\lambda\Big\| \\
&\leq \int_0^1 \Big\|\frac{1}{T}\int_{t_0}^{t_0+T} \partial_{\tilde{\alpha}}\rho(\tau,\lambda\tilde{\alpha})\, d\tau\Big\|\, d\lambda \|\tilde{\alpha}\| \\
&\leq \int_0^1 \Delta(T,\mathcal{S})\, d\lambda \|\tilde{\alpha}\| = \Delta(T,\mathcal{S})\|\tilde{\alpha}\|,
\end{aligned}
$$

where the first equality follows from $\rho(\tau,0) = 0$, the second equality follows from the Fubini–Tonelli Theorem, and finally the inequality on the third line follows from the triangle inequality and the definition of the induced matrix 2-norm. □

Proposition 9. *Consider the system $\dot{\chi}_f = F(t,\chi_f,\tilde{\alpha})$ where $F(\cdot)$ is piecewise continuous in $t \geq 0$ and continuously differentiable in $(\chi_f,\tilde{\alpha})$ uniformly in $t \geq 0$. If the equilibrium $\chi_f = 0$ is ES over $\chi_f(t_0) \in \mathcal{B}(\delta_f + 1)$ uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$, then the exists a converse Lyapunov function $V : \mathbb{R}_+ \times \mathcal{B}(\delta_f + 1) \times \mathcal{B}(\delta_\alpha) \to \mathbb{R}_+$ such that*

$$
\begin{gathered}
c_1\|\chi_f\|^2 \leq V(t,\chi_f,\tilde{\alpha}) \leq c_2\|\chi_f\|^2 \\
\partial_t V(t,\chi_f,\tilde{\alpha}) + \partial_{\chi_f} V(t,\chi_f,\tilde{\alpha}) F(t,\chi_f,\tilde{\alpha}) \leq -c_3\|\chi_f\|^2 \\
\|\partial_{\chi_f} V(t,\chi_f,\tilde{\alpha})\| \leq c_4\|\chi_f\| \\
\|\partial_{\tilde{\alpha}} V(t,\chi_f,\tilde{\alpha})\| \leq c_5\|\chi_f\|
\end{gathered}
$$

for some constants $c_i(\delta_f,\delta_\alpha) > 0$.

Proof. The first three inequalities are obtained by applying (Khalil, 2002, Theorem 4.14), so we only need to show the last one. Let $\phi(\tau; t, \chi_f, \tilde{\alpha})$ denote a trajectory starting at (t,χ_f) with fixed parameter $\tilde{\alpha}$. Then by Khalil (2002, Theorem 4.14) we know

$$V(t,\chi_f,\tilde{\alpha}) = \int_t^{t+\delta} \phi^\intercal(\tau;t,\chi_f,\tilde{\alpha})\phi(\tau;t,\chi_f,\tilde{\alpha})\, d\tau$$

for some appropriate $\delta > 0$. By the Leibniz integral rule followed by the chain rule we have

$$\partial_{\tilde{\alpha}} V(t,\chi_f,\tilde{\alpha}) = 2\int_t^{t+\delta} \phi^\intercal(\tau;t,\chi_f,\tilde{\alpha})\partial_{\tilde{\alpha}}\phi(\tau;t,\chi_f,\tilde{\alpha})\, d\tau.$$

By ES uniformly in $\tilde{\alpha}$ we have there exists constants $c_0(\delta_f,\delta_\alpha)$, $\lambda(\delta_f,\delta_\alpha) > 0$ such that

$$\|\phi(\tau;t,\chi_f,\tilde{\alpha})\| \leq c_0\|\chi_f\| e^{-\lambda(\tau-t)}$$

for all $\tau \geq t \geq 0$. Letting $s(\tau;t,\chi_f,\tilde{\alpha}) := \partial_{\tilde{\alpha}}\phi(\tau;t,\chi_f,\tilde{\alpha})$, one can show it satisfies the variational equation

$$
\begin{aligned}
\partial_\tau s(\tau;t,\chi_f,\tilde{\alpha}) &= \partial_\tau \partial_{\tilde{\alpha}}\phi(\tau;t,\chi_f,\tilde{\alpha}) = \partial_{\tilde{\alpha}}\partial_\tau\phi(\tau;t,\chi_f,\tilde{\alpha}) \\
&= \partial_\tau f(\tau,\phi(\tau;t,\chi_f,\tilde{\alpha}),\tilde{\alpha}) s(\tau;t,\chi_f,\tilde{\alpha}) \\
&\quad + \partial_{\tilde{\alpha}} f(\tau,\phi(\tau;t,\chi_f,\tilde{\alpha}),\tilde{\alpha}) \\
&=: A(\tau;t,\chi_f,\tilde{\alpha}) s(\tau;t,\chi_f,\tilde{\alpha}) + B(\tau;t,\chi_f,\tilde{\alpha})
\end{aligned}
$$

for all $\tau \geq t \geq 0$. Again by ES uniformly in $\tilde{\alpha}$ as well as continuous differentiability of $f(\cdot)$ uniformly in $\tau \geq 0$, there exists $s_1(\delta_f,\delta_\alpha)$, $s_2(\delta_f,\delta_{\tilde{\alpha}}) > 0$ such that $\|A(\cdot)\| \leq s_1$ and $\|B(\cdot)\| \leq s_2$. As such, one can show

$$\partial_\tau \left[\|s(\tau;t,\chi_f,\tilde{\alpha})\|^2\right] \leq 2s_1\|s(\tau;t,\chi_f,\tilde{\alpha})\|^2 + 2s_2$$

for all $\tau \geq t \geq 0$. Applying the Comparison Lemma, we have

$$\|s(\tau;t,\chi_f,\tilde{\alpha})\| \leq \sqrt{2s_2} e^{s_1(\tau-t)}$$

for all $\tau \geq t \geq 0$. Putting it all together, it is straightforward to show there exists $c_5(\delta_f,\delta_{\tilde{\alpha}}) > 0$ such that

$$\|\partial_{\tilde{\alpha}} V(t,\chi_f,\tilde{\alpha})\| \leq c_5\|\chi_f\|$$

as desired. □

9.5. *Proof of Theorem 4*

By (C5) there exists c_s, $\lambda_s > 0$ dependent solely on $f_{av}(\cdot)$ such that

$$\|\tilde{\alpha}_{av}(t)\| \leq c_s\|\tilde{\alpha}_{av}(t_0)\| e^{-\varepsilon\lambda_s(t-t_0)} \tag{66}$$

for all $t \geq t_0 \geq 0$ and any $\tilde{\alpha}_{av}(t_0) \in \mathbb{R}^{n_s}$. Define

$$\delta_s(\delta_\alpha) := \frac{\delta_\alpha}{c_s + 1}.$$

Clearly if $f_{av}(\cdot)$ is unchanged (even if $F(\cdot)$ is different), then the choice of $\delta_s(\cdot)$ remains the same and $\delta_s \to \infty$ as $\delta_\alpha \to \infty$. At this point, the result follows immediately from Teel et al. (2003, Theorem 1), so we verify the relevant assumptions hold. In particular, we will state all assumptions explicitly in a time-varying framework.

- Assumptions 1–2: Trivial as there are no "slow" or "fast" disturbances inducing a steady-state error on the states, and because $\varepsilon f(\cdot) = 0$ if $\varepsilon = 0$. In other words, we may take $d_s = 0$ and $d_f = 0$ in Teel et al. (2003).
- Assumption 3: By (C4) there exists $c_f(\delta_f,\delta_\alpha)$, $\lambda_f(\delta_f,\delta_\alpha) > 0$ such that for the system (18a) we have

  $$\|\chi_f(t)\| \leq c_f\|\chi_f(t_0)\| e^{-\lambda_f(t-t_0)}$$

  for all $t \geq t_0 \geq 0$ over $\chi_f(t_0) \in \mathcal{B}(\delta_f + 1)$ uniformly in $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$.

- Assumption 4: As shown at the start of the proof, (C5) implies (66) holds over any compact set of initial conditions. To show we indeed have an admissible average (Teel et al., 2003, Definition 1), let $\mathcal{R}_s \subseteq \mathbb{R}^{n_s}$ be any compact set. By (C1)–(C3), we can apply Proposition 8 to obtain $\Delta(T, \mathcal{R}_s)$ such that

$$\|\frac{1}{T}\int_{t_0}^{t_0+T} f(\tau,0,\tilde{\alpha}) - f_{av}(\tilde{\alpha})\, d\tau\| \le \Delta(T,\mathcal{R}_s)\|\tilde{\alpha}\|$$

for all $\tilde{\alpha} \in \mathcal{R}_s$ and $t_0 \ge 0$. In particular, by (C3) we may take $\Delta(T,\mathcal{R}_s)$ to be a *convergence function* for $(\partial_{\tilde{\alpha}} f)_{av}(\cdot)$; that is, $\Delta(\cdot)$ is continuous and decreasing in $T \ge 0$, and $\lim_{T\to\infty} \Delta(T,\mathcal{R}_s) = 0$.

Instead of directly checking Assumptions 5 and 6, we will use (Teel et al., 2003, Proposition 2). To this end, we verify the following.

- Assumption 7: In the notation of Teel et al. (2003) we let $\mathcal{K}_s$ and $\mathcal{K}_f$ be sets of initial conditions we would like to consider, and we let $\mathcal{H}_s$ and $\mathcal{H}_f$ be sets of initial conditions we want our nominal stability bounds to hold. Since $d_s = 0$, we can simplify and take the elements of $\mathcal{K}_f$ and $\mathcal{H}_f$ to be tuples $(\chi_f, \tilde{\alpha})$ rather than $(\chi_f, \tilde{\alpha}, d_s)$. Below we verify:
  1. $\mathcal{K}_s := \mathcal{B}(\delta_s) \subseteq \mathcal{B}((c_s+1)\delta_s) = \mathcal{B}(\delta_\alpha) =: \mathcal{H}_s$;
  2. $\mathcal{K}_f := \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_\alpha) \subseteq \mathcal{B}(\delta_f+1) \times \mathcal{B}(\delta_\alpha) =: \mathcal{H}_f$;
  3. $c_{f,o} := \sup_{(\chi_f,\tilde{\alpha})\in\mathcal{K}_f} \|\chi_f\| = \delta_f < \infty$;
  4. $c_{f,i} = 0 < \infty$ because $d_f = 0$;
  5. $c_{s,o} := \sup_{\tilde{\alpha}_{av}\in\mathcal{K}_s} \|\tilde{\alpha}_{av}\| = \delta_s < \infty$;
  6. $c_{s,i} = 0 < \infty$ because $(d_s, d_f) = (0,0)$;
  7. for any $\delta_\circ \in (0, \min\{\delta_f, \delta_s\}] \neq \emptyset$, we have

$$\mathcal{K}_s \supseteq \{\, \tilde{\alpha}_{av} \;:\; \|\tilde{\alpha}_{av}\| \le c_{s,i} + \delta_\circ \,\} = \mathcal{B}(\delta_\circ)$$
$$\mathcal{K}_f \supseteq \{\, (\chi_f,\tilde{\alpha}) \;:\; \|\chi_f\| \le c_{f,i} + \delta_\circ,\ \|\tilde{\alpha}\| \le \max\{\, c_s c_{s,o}, c_{s,i} \,\} + \delta_\circ \,\} = \mathcal{B}(\delta_\circ) \times \mathcal{B}(c_s\delta_s + \delta_\circ)\,.$$

- Assumption 8: In the notation of Teel et al. (2003), define $\mathcal{X}_s := \mathcal{B}(c_s\delta_s)$ and $\mathcal{Z}_f := \mathcal{B}(c_f\delta_f) \times \mathbb{R}^{n_s}$, where $c_s, c_f > 0$ come from Assumptions 3 and 4. Let $\sigma > 0$, then we can verify:
  1. continuity of $\|\tilde{\alpha}_{av}\|$ and compactness of $\mathcal{X}_s + \mathcal{B}(\sigma) = \mathcal{B}(c_s\delta_s + \sigma)$ implies uniform continuity;
  2. for any compact set $\mathcal{R}_s \supseteq \mathcal{X}_s + \mathcal{B}(\sigma)$, which exists by compactness of $\mathcal{X}_s + \sigma\mathcal{B}(1)$, we have an admissible average over $\tilde{\alpha}_{av} \in \mathcal{R}_s$ by Assumption 4;
  3. continuity of $\|\chi_f\|$ and compactness of $\mathcal{U}(\sigma) := \mathcal{B}(c_f\delta_f + \sigma) \times \mathcal{B}(c_s\delta_s + \sigma)$ implies uniform continuity;
  4. Fix any $\rho > 0$, then
     (a) for any $\varepsilon^* \in (0,\rho]$ we have that for each $c \in [0, c_{f,i}] = \{\,0\,\}$, if $\|\chi_f\| \le 0 + \varepsilon^*$ then 0 is the only element such that $\|0\| \le 0$ and $\|\chi_f - 0\| \le \rho$;
     (b) there exists an $\varepsilon^* > 0$ sufficiently small such that Teel et al. (2003, Eq. (65)-(69)) are satisfied by (C1) and compactness of $\mathcal{U}(\sigma)$.

With all the assumptions verified and given that our initial conditions $(\chi,\tilde{\alpha})(t_0) \in \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_s)$ are contained in $\mathcal{K}_s$ and $\mathcal{K}_f$, the result then follows.

9.6. Proof of Theorem 5

Fix some $\epsilon \in (0, \min\{\,\delta_\alpha, 1\,\})$. Our point of departure is Theorem 4, where we redefine $\varepsilon_\star(\delta_f, \delta_\alpha) := \varepsilon_\star(\delta_f, \delta_\alpha, \epsilon)$. As such, we have uniform boundedness of trajectories of (16)

$$\chi(t) \in \mathcal{B}(c_f\delta_f + \epsilon), \qquad \tilde{\alpha}(t) \in \mathcal{B}(c_s\delta_s + \epsilon)$$

for all $t \ge t_0 \ge 0$ and all $\varepsilon \in (0, \varepsilon_\star)$ over $(\chi,\tilde{\alpha})(t_0) \in \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_s)$. Moreover, by our choice of ϵ there exists a (uniform) time $\Delta t \ge 0$ such that for all $t \ge t_0 + \Delta t \ge 0$ we have

$$\chi(t) \in \mathcal{B}(\delta_f + 1), \qquad \tilde{\alpha}(t) \in \mathcal{B}(\delta_\alpha).$$

By uniform boundedness, it suffices to show exponential stability of trajectories for $t \ge t_0 + \Delta t \ge 0$ rather than $t \ge t_0 \ge 0$; see the proof of Proposition 1. Exponential stability can be established by a similar two timescale Lyapunov argument as in Sastry and Bodson (1989, Theorem 4.4.3) (see also Hafez et al. (2023, Theorem 10)). To this end, we verify assumptions (B1)–(B5) of Sastry and Bodson (1989, p. 184) hold for $\chi \in \mathcal{B}(\delta_f + 1)$ and $\tilde{\alpha} \in \mathcal{B}(\delta_\alpha)$.

(B1): (C1) states the relevant continuity and differentiability properties of $f(\cdot)$ and $g(\cdot)$, whereas (C2) states that all functions vanish at the origin;

(B2): (C3) establishes the existence of the averages, whereas Propositions 6 and 7 propagate the continuity properties from (C1) to the averages;

(B3): (C1)–(C3) form the assumptions to apply both Propositions 7 and 8. They establish the zero average value and the convergence function properties of $\rho(t,\tilde{\alpha}) := f(t,0,\tilde{\alpha}) - f_{av}(\tilde{\alpha})$, respectively;

(B4): (C1)–(C2), (C4) form the assumptions to apply Proposition 9, which characterizes the ES of the fast system (18a) through an appropriate Lyapunov function defined over $(t,\chi,\tilde{\alpha}) \in \mathbb{R}_+ \times \mathcal{B}(\delta_f + 1) \times \mathcal{B}(\delta_\alpha)$;

(B5): This technical assumption need not be verified since we know a priori from Theorem 4 that the trajectories of the closed-loop system (16) converge uniformly and in finite time to $\mathcal{B}(\delta_f+1) \times \mathcal{B}(\delta_\alpha)$.

As a result, we conclude the equilibrium $(\chi,\tilde{\alpha}) = (0,0)$ of (16) is ES over $(\chi,\tilde{\alpha})(t_0) \in \mathcal{B}(\delta_f) \times \mathcal{B}(\delta_s)$ for every $\varepsilon \in (0,\varepsilon_\star)$, where we have possibly redefined $\varepsilon_\star > 0$ to be smaller.

9.7. Proof of Lemma 8

The case when $q_{pe} = 0$ is trivial since $\Sigma = 0$, and thus omitted. Let the block partition

$$\hat{\Sigma} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} V_1^\intercal \\ V_2^\intercal \end{bmatrix}$$

denote the SVD of $\hat{\Sigma}$ where $\sigma_{\min}(D_1) \ge \sigma_{tol}$ and $\sigma_{\max}(D_2) < \sigma_{tol}$. Then the $(\cdot)_{tol}$ operator is defined as

$$\hat{\Sigma}_{tol} := \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\intercal \\ V_2^\intercal \end{bmatrix} = U_1 D_1 V_1^\intercal,$$

which is the matrix $\hat{\Sigma}$ but with its singular values less than σ_{tol} treated as zero. Note that when D_1 is a matrix of dimension zero, we may take $U_1 D_1 V_1^\intercal = 0 \in \mathbb{R}^{q\times q}$. An analogous statement holds for $U_2 D_2 V_2^\intercal$. Consequently, observe that

$$\operatorname{pinv}(\hat{\Sigma}; \sigma_{tol}) := \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^\intercal \\ U_2^\intercal \end{bmatrix} = \hat{\Sigma}_{tol}^\dagger.$$

Now let $\hat{\Sigma} = \Sigma + \tilde{\Sigma}$. Recalling that $\Sigma(t)\Sigma^\dagger(t) = WW^\intercal$ for all $t \ge t_0 \ge 0$ by Lemma 6, one has

$$\begin{aligned}
&\|\hat{\Sigma}\operatorname{pinv}(\hat{\Sigma};\sigma_{tol}) - WW^\intercal\| \\
&\quad \le \|\Sigma\|\,\|(\Sigma+\tilde{\Sigma})_{tol}^\dagger - \Sigma^\dagger\| + \|\tilde{\Sigma}\|\,\|(\Sigma+\tilde{\Sigma})_{tol}^\dagger\| \\
&\quad \le \|\Sigma(t)\|_{\mathcal{L}_\infty}\|(\Sigma+\tilde{\Sigma})_{tol}^\dagger - \Sigma^\dagger\| + \sigma_{tol}^{-1}\|\tilde{\Sigma}\|\,.
\end{aligned}$$

To deal with the first term in the above inequality, we use a perturbation bound on the pseudoinverse given by Stewart (1977, Theorem 3.3), stating that

$$\|\hat{\Sigma}_{tol}^\dagger - \Sigma^\dagger\| \le \frac{1+\sqrt{5}}{2}\max\left\{\,\|\hat{\Sigma}_{tol}^\dagger\|^2, \|\Sigma^\dagger\|^2\,\right\}\|\hat{\Sigma}_{tol} - \Sigma\|.$$

Given that the smallest non-zero singular value of $\hat{\Sigma}_{tol}$ is forced to be greater than or equal to σ_{tol}, it must be that $\|\hat{\Sigma}^{\dagger}_{tol}\| \le \sigma^{-1}_{tol}$. Also, since $\delta(\cdot)$ given by Lemma 6 is a class-$\mathcal{K}$ function, it must satisfy $\lim_{\varepsilon\to 0^+}\delta(\varepsilon) = 0$. Therefore, given that $\sigma_{tol} \in (0,\beta_0)$, there exists an $\varepsilon_\star(\sigma_{tol}) > 0$ such that $\beta_0 - \delta(\varepsilon) > \sigma_{tol} > 0$ for all $\varepsilon \in (0,\varepsilon_\star]$. Note that $\varepsilon_\star$ must be selected sufficiently small to also satisfy Lemma 6. Then $\sigma_{\min}(\Sigma) > \sigma_{tol}$ for $\varepsilon \in (0,\varepsilon_\star]$, implying that $\|\Sigma^{\dagger}\| \le \sigma^{-1}_{tol}$. Altogether

$$\begin{aligned}\|\hat{\Sigma}^{\dagger}_{tol} - \Sigma^{\dagger}\| &\le \frac{1+\sqrt{5}}{2\sigma^2_{tol}}\|(\Sigma+\tilde{\Sigma})_{tol} - \Sigma\| \\ &\le \frac{1+\sqrt{5}}{2\sigma^2_{tol}}\left(\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| + \|\tilde{\Sigma}\|\right).\end{aligned}$$

Once again, we need to develop an appropriate bound for the first term. Let $\hat{\Sigma} = u_1\sigma_1 v_1^{\intercal} + \cdots + u_q\sigma_q v_q^{\intercal}$ denote its SVD with singular values ordered as $\sigma_1 \ge \cdots \ge \sigma_q$. Then for some subset $\mathcal{I} \subseteq \{1,\ldots,q\}$ we have

$$\begin{aligned}\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| &= \|\sum_{i\in\mathcal{I}} u_i\sigma_i v_i^{\intercal}\| = \sigma_{(\min\mathcal{I})}(\Sigma+\tilde{\Sigma}) \\ &\le \|\Sigma+\tilde{\Sigma}\| \le \|\Sigma(t)\|_{\mathcal{L}_\infty} + \|\tilde{\Sigma}\|.\end{aligned}$$

Note that if $\mathcal{I} = \emptyset$, then $\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| = 0$. To obtain a tighter bound, recall Lemma 7 stating

$$\begin{aligned}\sigma_{q_{pe}}(\Sigma+\tilde{\Sigma}) &\ge \beta_0 - \delta(\varepsilon) - \|\tilde{\Sigma}\| \\ \sigma_{q_{pe}+1}(\Sigma+\tilde{\Sigma}) &\le \|\tilde{\Sigma}\|.\end{aligned}$$

Since we have selected ε such that $\beta_0 - \delta(\varepsilon) > \sigma_{tol}$ for $\varepsilon \in (0,\varepsilon_\star]$, there exists a $\delta_\circ(\sigma_{tol}) > 0$ such that

$$\sigma_{q_{pe}}(\Sigma+\tilde{\Sigma}) \ge \sigma_{tol}, \quad \sigma_{q_{pe}+1}(\Sigma+\tilde{\Sigma}) < \sigma_{tol}$$

for all $\|\tilde{\Sigma}\| \in [0,\delta_\circ)$. Therefore, for $\|\tilde{\Sigma}\| \in [0,\delta_\circ)$ one concludes that $\hat{\Sigma}_{tol} = (\Sigma+\tilde{\Sigma})_{tol} = u_1\sigma_1 v_1^{\intercal} + \cdots + u_{q_{pe}}\sigma_{q_{pe}} v_{q_{pe}}^{\intercal}$, implying that

$$\begin{aligned}\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| &= \|u_{q_{pe}+1}\sigma_{q_{pe}+1}v^{\intercal}_{q_{pe}+1} + \cdots + u_q\sigma_q v_q^{\intercal}\| \\ &= \sigma_{q_{pe}+1}(\Sigma+\tilde{\Sigma}) \le \|\tilde{\Sigma}\|.\end{aligned}$$

Notice that if $q_{pe} = q$, then $\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| = 0 \le \|\tilde{\Sigma}\|$. Altogether, we have established

$$\begin{aligned}\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| &\le \begin{cases}\|\tilde{\Sigma}\|, & \|\tilde{\Sigma}\| \in [0,\delta_\circ) \\ \|\Sigma(t)\|_{\mathcal{L}_\infty} + \|\tilde{\Sigma}\|, & \|\tilde{\Sigma}\| \in [\delta_\circ,\infty)\end{cases} \\ &\le a_\circ\|\tilde{\Sigma}\|\end{aligned}$$

for some $a_\circ(\delta_\circ) \ge 1$ sufficiently large. Putting all our inequalities together, we obtain the upper bound

$$\begin{aligned}\|\hat{\Sigma}^{\dagger}_{tol} - \Sigma^{\dagger}\| &\le \frac{1+\sqrt{5}}{2\sigma^2_{tol}}\left(\|(\Sigma+\tilde{\Sigma})_{tol} - (\Sigma+\tilde{\Sigma})\| + \|\tilde{\Sigma}\|\right) \\ &\le \frac{1+\sqrt{5}}{2\sigma^2_{tol}}\left(a_\circ + 1\right)\|\tilde{\Sigma}\| =: b_\circ\|\tilde{\Sigma}\|\end{aligned}$$

resulting in

$$\|\hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma};\sigma_{tol}) - WW^{\intercal}\| \le \left(\|\Sigma(t)\|_{\mathcal{L}_\infty} b_\circ + \sigma^{-1}_{tol}\right)\|\tilde{\Sigma}\| =: c_\circ\|\tilde{\Sigma}\|.$$

To conclude the proof, we note that for any two orthogonal projection matrices A and B, it must be $\|A - B\| \le 1$ (Stewart, 1977). The result then follows because a bit of matrix multiplication shows

$$\hat{\Sigma}\,\mathrm{pinv}(\hat{\Sigma};\sigma_{tol}) = \hat{\Sigma}_{tol}\hat{\Sigma}^{\dagger}_{tol}$$

is an orthogonal projection matrix.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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