Reach Control Problem on Simplices

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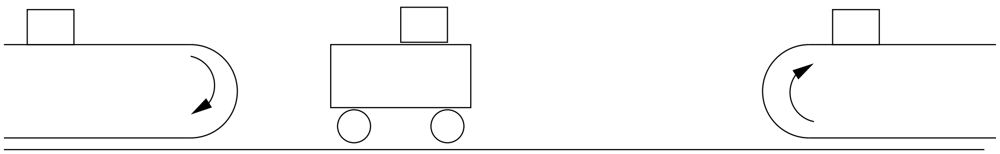
- [HvS04] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica* 40, 2004.

Motivating Example

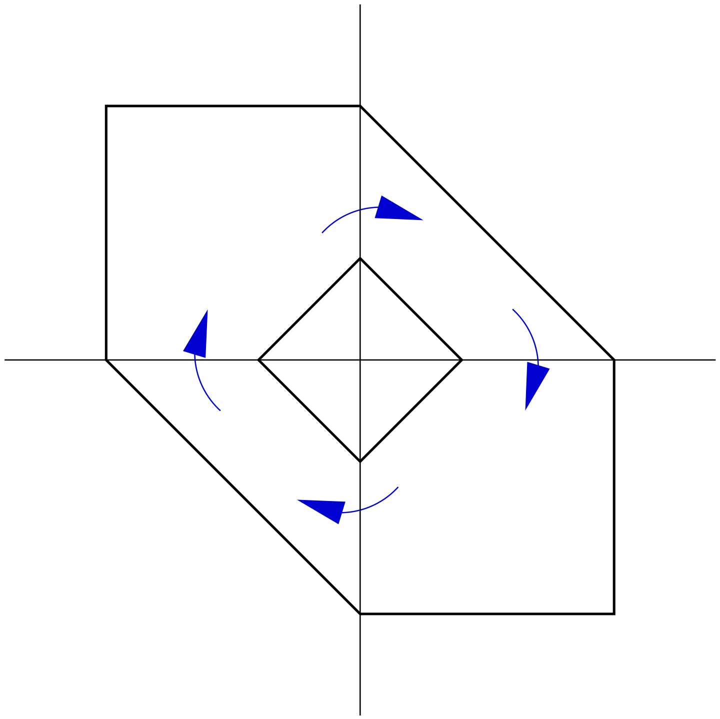
Cart and Conveyor Belts



Control Specifications:

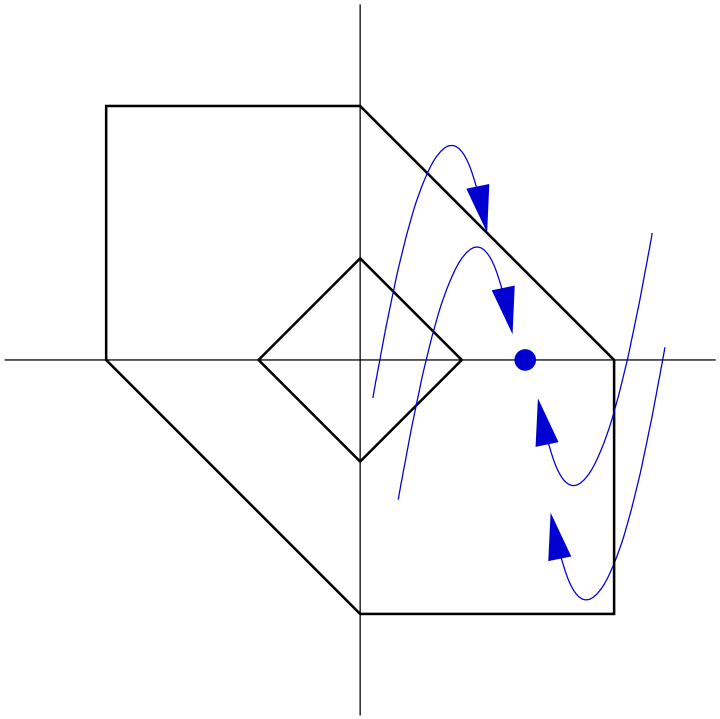
1. Safety: $|x| \leq 3$, $|\dot{x}| \leq 3$, $|x + \dot{x}| \leq 3$.
2. Liveness: $|x| + |\dot{x}| \geq 1$.
3. Temporal behavior: Every box arriving on conveyor 1 is picked up and deposited on conveyor 2.

State Space View



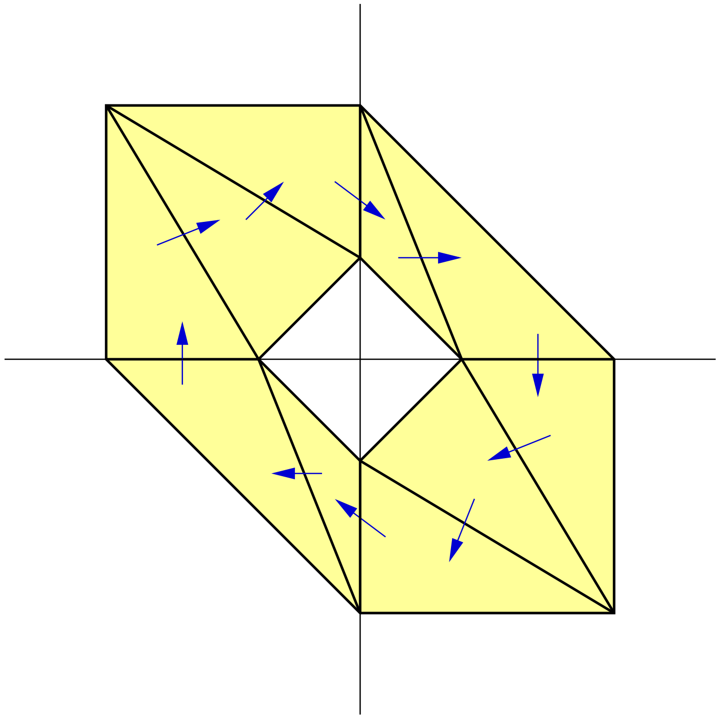
- The state space is a polytope with a hole, not $\mathbb{R}^n$.
- The safety specification determines the outer boundary.
- The liveness specification creates the hole.
- The arrows capture the temporal behavior.

Naive Approach



- Equilibrium stabilization gives no guarantee of safety or liveness.
- Safety, if achieved, is not robust.
- Difficult to design for trade-off between safety and liveness.
- Operationally inefficient - system must be run unnaturally slowly.

Preferred Approach



- Safety is built in up front and is provably robust.
- Liveness can be traded off with safety by adjusting the size of the hole.
- Triangulation is used in algebraic topology, physics, numerical solution of PDE's, computer graphics, etc. Why not in control theory?

Reach Control Problem

What is Given

1. An affine system

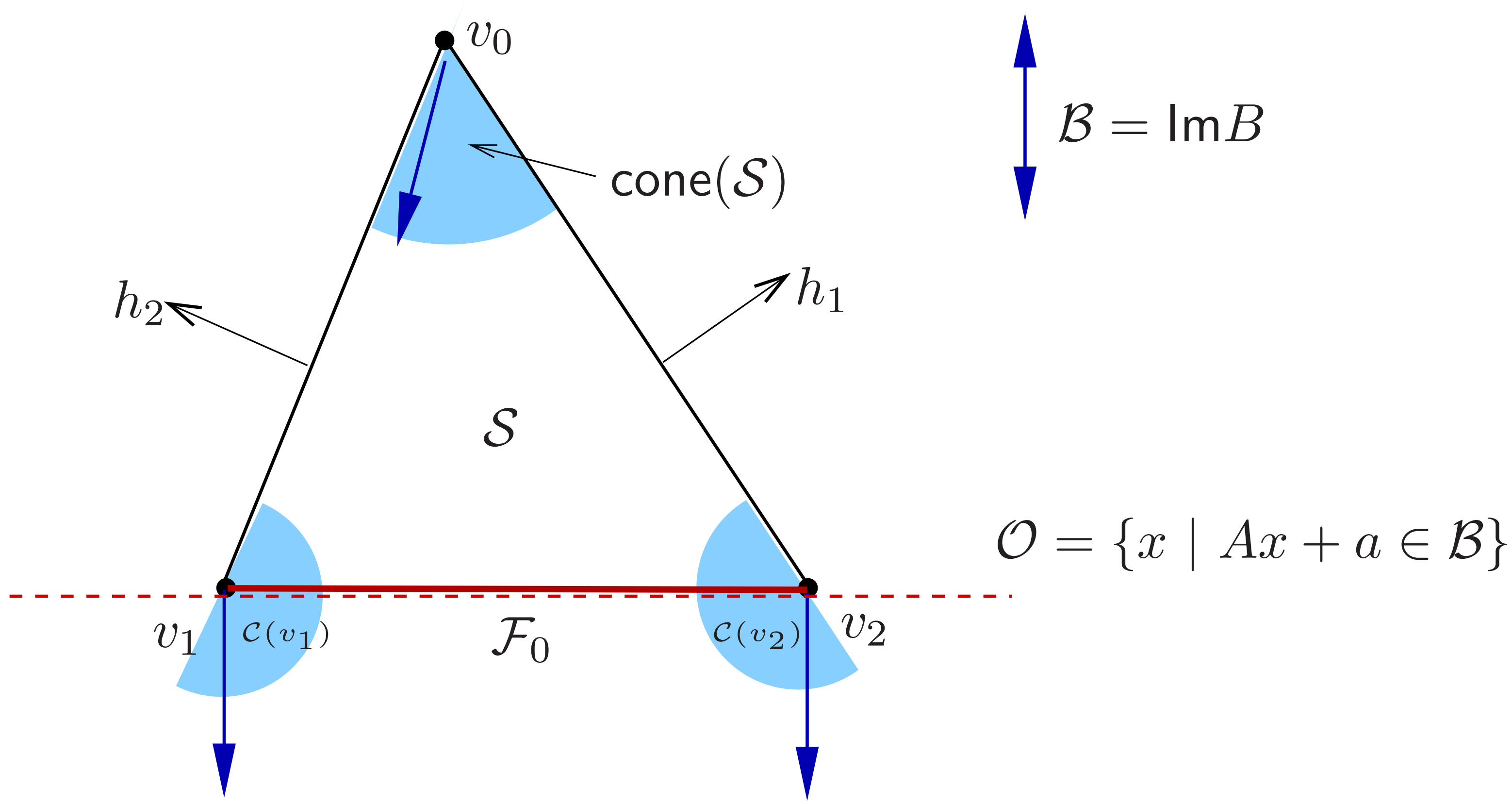
$$\dot{x} = Ax + Bu + a\,, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m\,. \tag{1}$$

2. An n-dimensional simplex $\mathcal{S} = \text{conv}\{v_0, \ldots, v_n\}$.

3. A set of restricted facets $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$.

4. One exit facet $\mathcal{F}_0$.

The Setup



Notation: $\mathcal{C}(v_i) := \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0 \,, j \neq i\}$

Problem Statement

Problem. (RCP) *Given simplex $\mathcal{S}$ and system* (1), *find $u(x)$ such that: for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\gamma > 0$ such that*

- $x(t) \in \mathcal{S}$ *for all* $t \in [0, T]$,
- $x(T) \in \mathcal{F}_0$, *and*
- $x(t) \notin \mathcal{S}$ *for all* $t \in (T, T + \gamma)$.

Notation: $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_{\mathbf{0}}$ by feedback of class $\mathbb{U}$.

Basic Principles

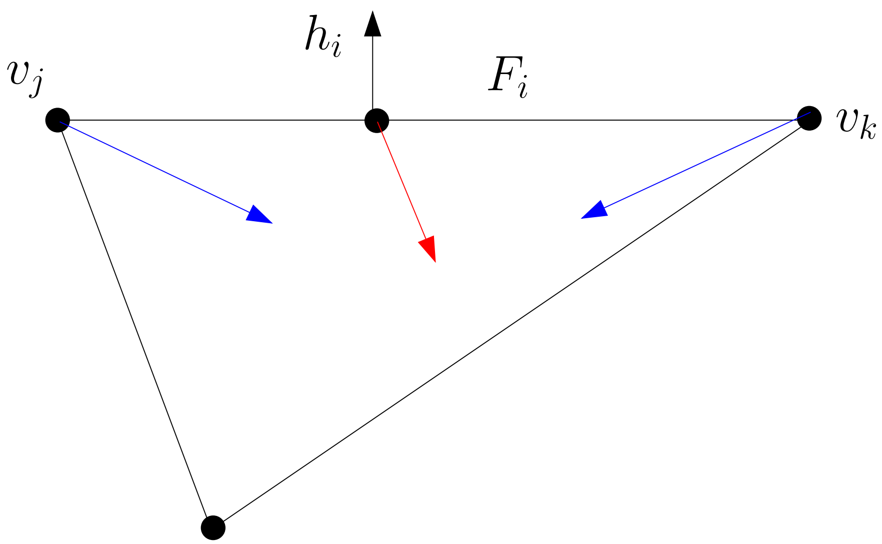
Convexity and Affine Systems

Consider an affine system $\dot{x} = Ax + a$. If for all vertices v in $\mathcal{F}_i$,

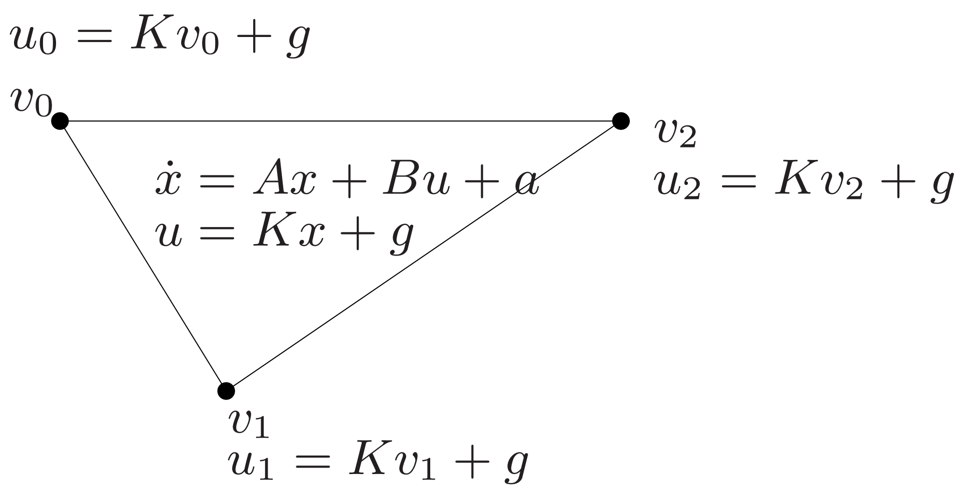
$$h_i \cdot (Av + a) \leq 0,$$

then

- $h_i \cdot (Ax + a) \leq 0, \qquad \forall x \in F_i.$
- Trajectories that leave $\mathcal{S}$ do so through a facet $\mathcal{F}_j$, $j \neq i$.



Affine Feedback



$$\underbrace{\begin{bmatrix} v_0^T & 1 \\ \vdots & \\ v_n^T & 1 \end{bmatrix}}_{\text{invertible}} \begin{bmatrix} K^T \\ g^T \end{bmatrix} = \begin{bmatrix} u_0^T \\ \vdots \\ u_n^T \end{bmatrix}, \qquad \begin{aligned} \dot{x} &= Ax + B(Kx + g) + a \\ &= \tilde{A}x + \tilde{a}. \end{aligned}$$

[HvS04] L.C.G.J.M. Habets and J.H. van Schuppen. Automatica 2004.

Escaping Compact, Convex Sets

Theorem. *Consider an affine system* $\dot{x} = Ax + a$ *on* $\mathcal{S}$*. If*

$$Ax + a \neq 0\,, \qquad\qquad \forall x \in \mathcal{S}\,,$$

then trajectories starting in $\mathcal{S}$ *leave* $\mathcal{S}$ *in finite time.*

Affine Feedback

A First Result

Theorem. $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by affine feedback iff there exists* $u(x) = Kx + g$ *such that*

(a) The **invariance conditions** *hold:*

$$Av_i + a + Bu(v_i) \in \mathcal{C}(v_i), \qquad i \in \{0, \ldots, n\}.$$

(b) The closed-loop system has **no equilibrium** *in* $\mathcal{S}$*.*

[HvS06] L.C.G.J.M. Habets and J.H. van Schuppen. IEEE TAC 2006.
[RosBro06] B. Roszak and M.E. Broucke. Automatica 2006.

Numerical Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

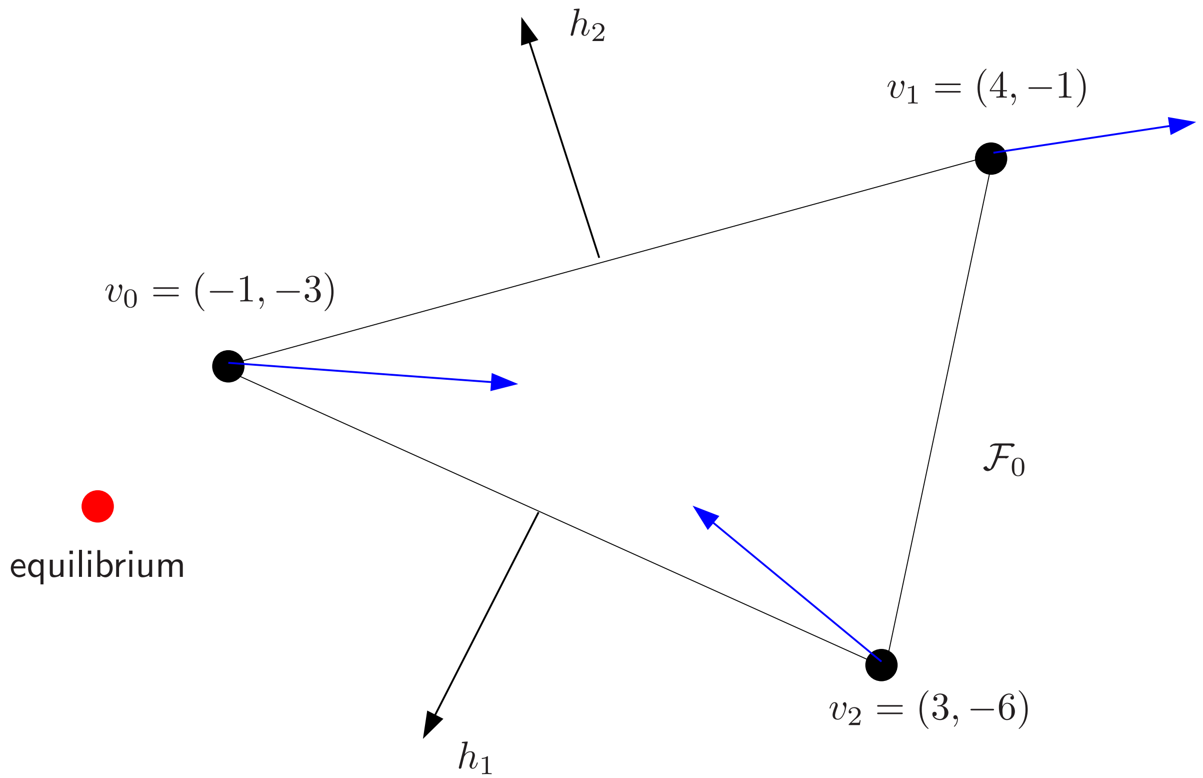
$\mathcal{S}$ determined by: $v_0 = (-1, -3)$, $v_1 = (4, -1)$, and $v_2 = (3, -6)$.

Invariance conditions give:

- $h_1^T(Av_0 + Bu_0 + a) \leq 0 \quad \Rightarrow \quad u_0 \geq -1.75$
- $h_2^T(Av_0 + Bu_0 + a) \leq 0 \quad \Rightarrow \quad u_0 \leq -0.6$
- $h_2^T(Av_1 + Bu_1 + a) \leq 0 \quad \Rightarrow \quad u_1 \leq 0.2$
- $h_1^T(Av_2 + Bu_2 + a) \leq 0 \quad \Rightarrow \quad u_2 \geq 0.5.$

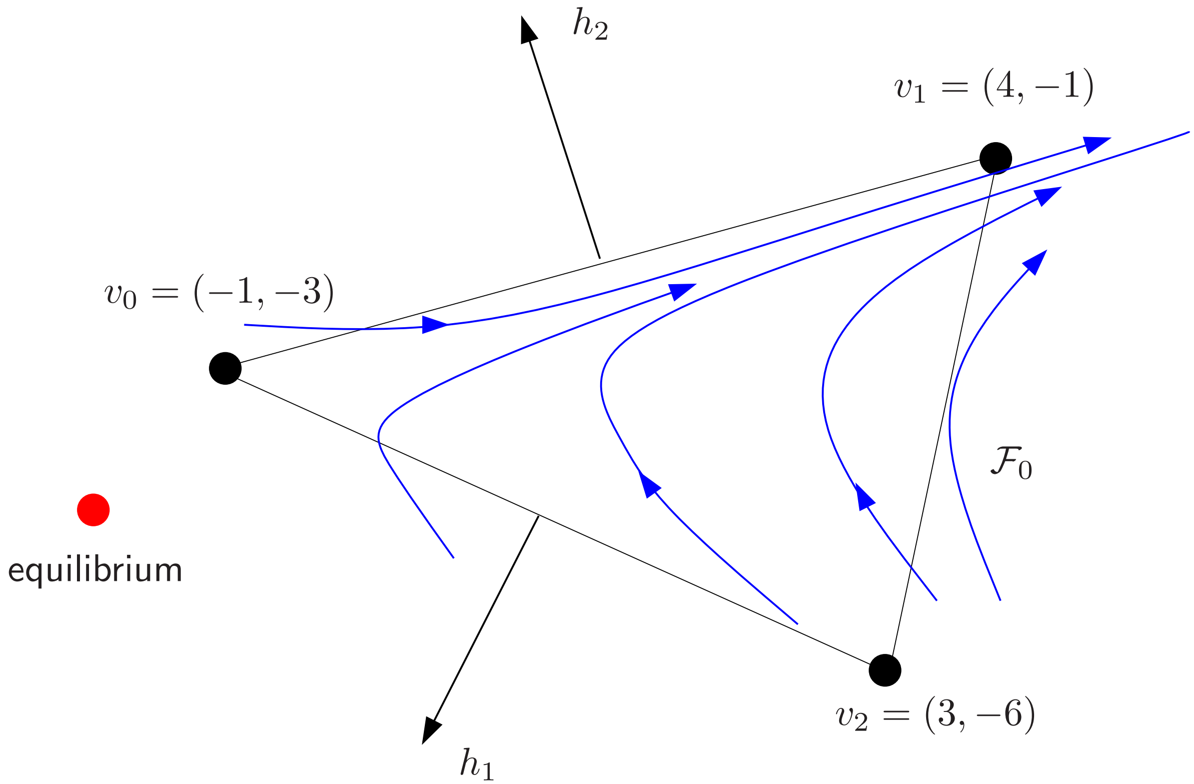
Numerical Example

Choose $u_0 = -1.175$, $u_1 = 0.2$, $u_2 = 0.5$.



Numerical Example

The affine feedback is: $u = [0.325 \;\; -0.125]x - 1.225$



Equilibrium Set and Triangulations

Let $\mathcal{B} = \mathrm{Im}B$. The **equilibrium set** is

$$\mathcal{O} = \{x \mid Ax + a \in \mathcal{B}\} .$$

Define

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O} .$$

Assumption. *If $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \leq \kappa < n$. Reorder indices so that*

$$\mathcal{G} = \mathrm{conv}\{v_1, \ldots, v_{\kappa+1}\} .$$

Note: $v_0 \notin \mathcal{G}$, otherwise there's a trivial solution to RCP.

Can be achieved using the **placing triangulation**.

A Second Result

Theorem. *Suppose the triangulation assumption holds. TFAE:*

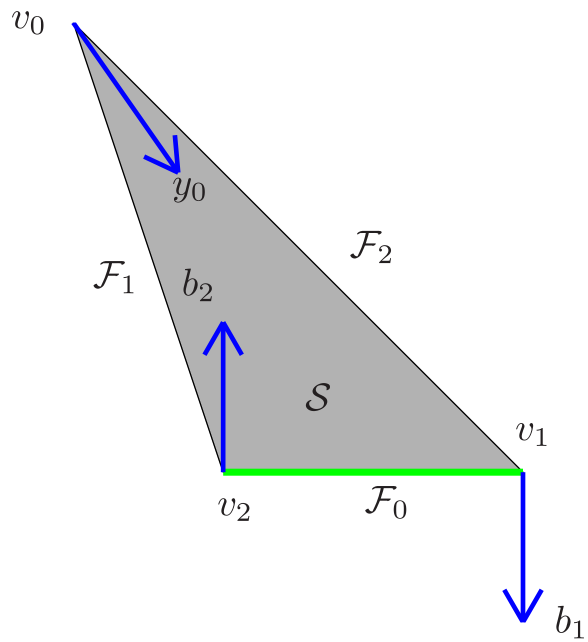
(a) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by affine feedback.*

(b) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by continuous state feedback.*

Proof. *Fixed point argument using Sperner's lemma,* $\mathscr{M}$*-matrices.*

[B10] M.E. Broucke. SIAM J. Control and Opt. 2010.

Limits of Continuous State Feedback



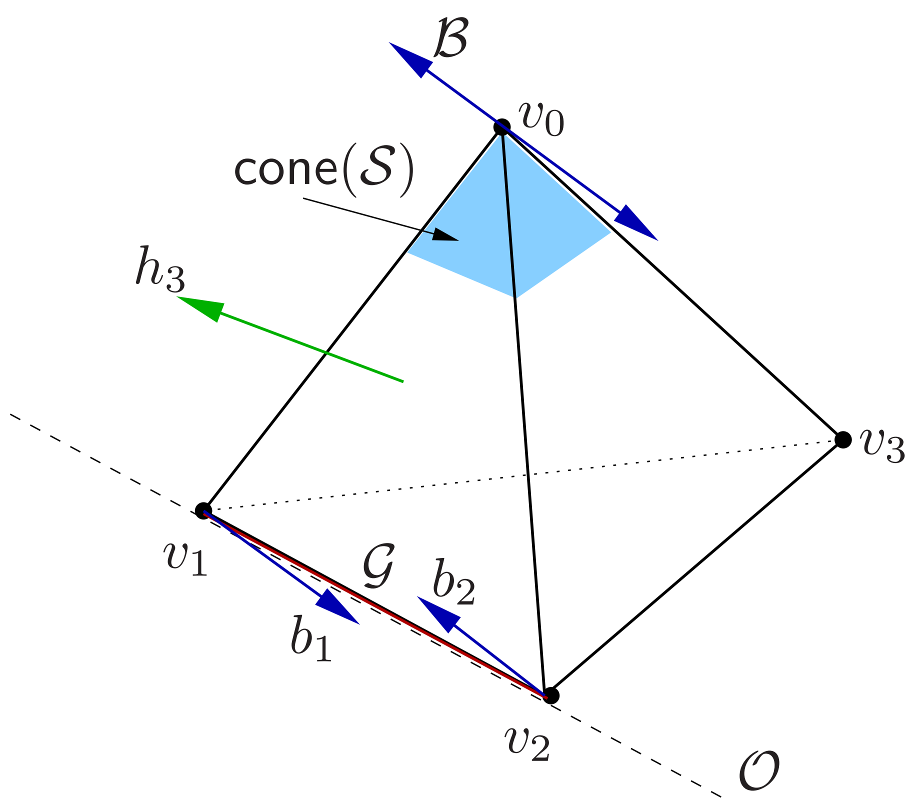
Let $u(x)$ be a continuous state feedback satisyfing the invariance conditions. If $\mathcal{B} = \text{sp}\{b\}$ and $\mathcal{G} = \overline{v_1 v_2}$, then

$$y(x) := c(x)b\,, \qquad x \in \overline{v_1 v_2} \qquad c : \mathbb{R}^n \to \mathbb{R} \;\text{ continuous}\,,$$

where $c(v_1) \geq 0$ and $c(v_2) \leq 0$. By Intermediate Value Theorem there exists $\overline{x}$ s.t. $c(\overline{x}) = 0$.

Conditions for a Topological Obstruction

1. $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$ nontriviality condition
2. $\nexists$ lin. indep. set $\{b_1, \ldots, b_{\kappa+1} \mid b_i \in \mathcal{B} \cap \mathcal{C}(v_i)\}$ system is "underactuated"



Reach Control Indices

$\mathscr{M}$-Matrices

Let $1 \le p \le q \le \kappa + 1$ and $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$. Define

$$M_{p,q} := \begin{bmatrix} (h_p \cdot b_p) & (h_p \cdot b_{p+1}) & \cdots & (h_p \cdot b_q) \\ \vdots & \vdots & & \vdots \\ (h_q \cdot b_p) & (h_q \cdot b_{p+1}) & \cdots & (h_q \cdot b_q) \end{bmatrix} \in \mathbb{R}^{(q-p+1)\times(q-p+1)}.$$

- A matrix is a **$\mathscr{Z}$-matrix** if the off-diagonal elements are non-positive.
- Because $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix.
- A $\mathscr{Z}$-matrix is a **non-singular $\mathscr{M}$-matrix** if every real eigenvalue is positive.
- Because $\mathcal{B} \cap \text{cone}(\mathcal{S}) = \mathbf{0}$, certain $M_{p,q}$ are non-singular $\mathscr{M}$-matrices.

Reach Control Indices

Theorem. *There exist integers* $r_1, \ldots, r_p \geq 2$ *such that w.l.o.g.*

$$\begin{aligned}
\mathcal{B} \cap \mathcal{C}(v_i) \;&\subset\; \mathsf{sp}\{b_{m_1}, \ldots, b_{m_1+r_1-1}\}\,, \; i = m_1, \ldots, m_1 + r_1 - 1\,, \\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots \\
\mathcal{B} \cap \mathcal{C}(v_i) \;&\subset\; \mathsf{sp}\{b_{m_p}, \ldots, b_{m_p+r_p-1}\}\,, \; i = m_p, \ldots, m_p + r_p - 1\,,
\end{aligned}$$

where $b_i \in \mathcal{B} \cap \mathcal{C}(v_i)$ *and*
$m_k := r_1 + \cdots + r_{k-1} + 1\,, \qquad k = 1, \ldots, p$. *Moreover, for each* $k = 1, \ldots, p$, $\{b_{m_k}, \ldots, b_{m_k+r_k-2}\}$ *are linearly independent and*

$$b_{m_k+r_k-1} = c_{m_k} b_{m_k} + \cdots + c_{m_k+r_k-2} b_{m_k+r_k-2}\,, \quad c_i < 0\,.$$

$\{r_1, \ldots, r_p\}$ are called the **reach control indices** of system (1).

[BG12] M.E. Broucke and M. Ganness. IEEE TAC, in revision 2012.

Reach Control Indices

For $k = 1, \ldots, p$ define

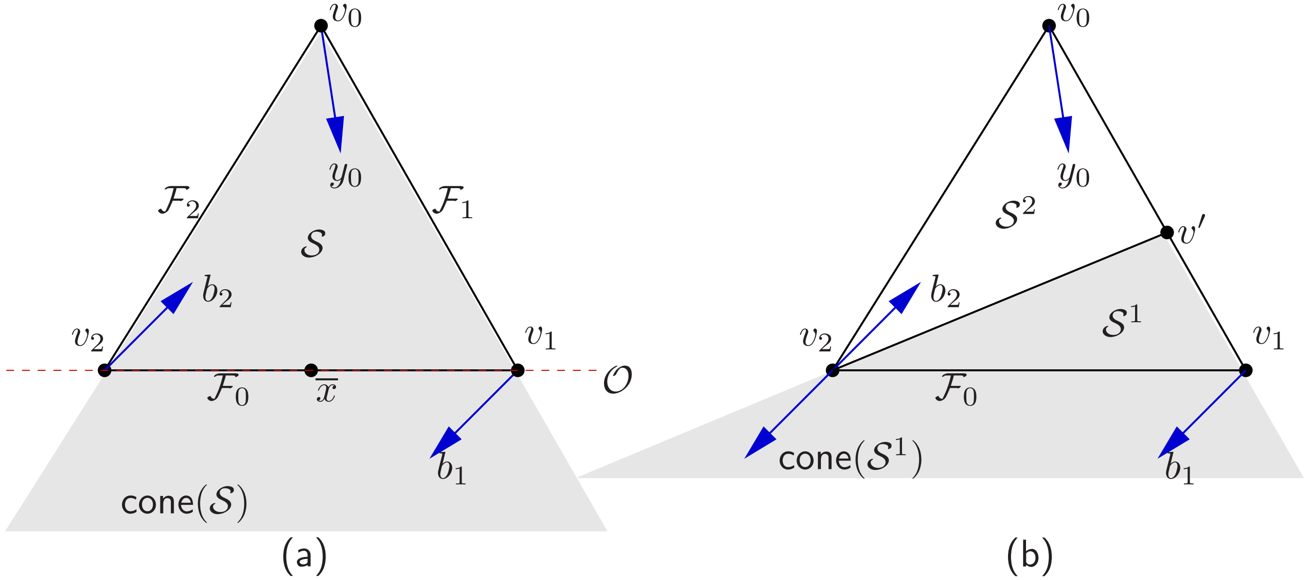
$$\mathcal{G}_k := \text{conv}\left\{v_{m_k}, \ldots, v_{m_k + r_k - 1}\right\}.$$

Theorem. *Let $u(x)$ be a continuous state feedback satisfying the invariance conditions. Then each $\mathcal{G}_k$ contains an equilibrium of the closed-loop system.*

[B10] M.E. Broucke. SIAM J. Control and Opt. 2010.

Piecewise Affine Feedback

Piecewise Affine Feedback

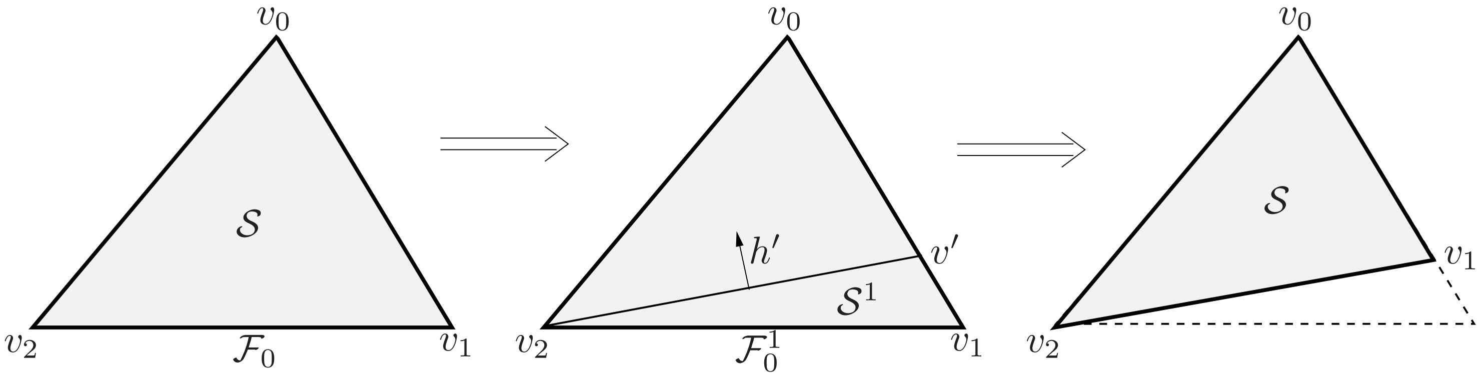


- As we slide v' from v_0 to v_1, cone($\mathcal{S}^1$) widens at v_2 until b_1 points into cone($\mathcal{S}^1$). For such v', $\mathcal{S}^1 \xrightarrow{\mathcal{S}^1} \mathcal{F}_0$ by affine feedback.
- $\mathcal{S}^2$ is not "underactuated" since $\mathcal{G}^2 = \{v_2\}$. Thus, $\mathcal{S}^2 \xrightarrow{\mathcal{S}^2} \mathcal{S}^1 \cap \mathcal{S}^2$ by affine feedback.

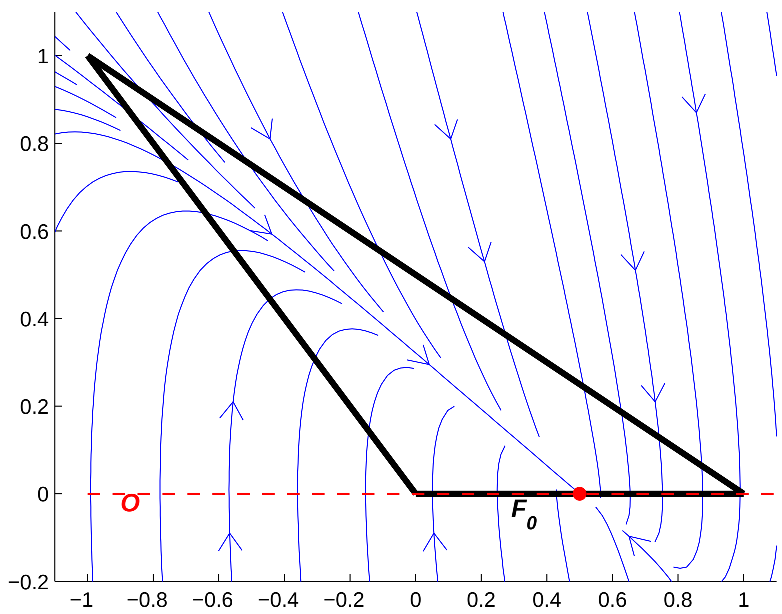
Recursive Subdivision Algorithm

Subdivision Algorithm:

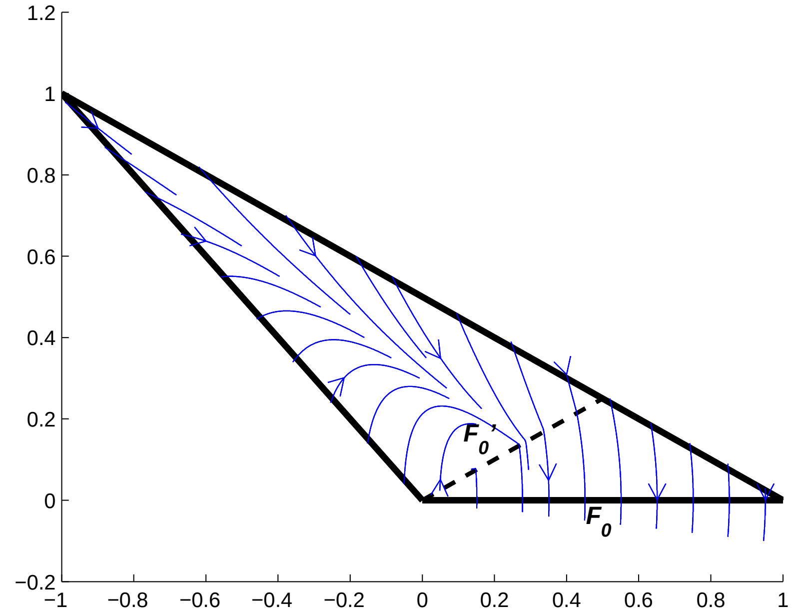
1. Set $k = 1$.
2. Select $v' \in (v_0, v_{m_k})$ such that $\mathcal{B} \cap \text{cone}(\mathcal{S}^k) \neq \mathbf{0}$, where $\mathcal{S}^k := \text{conv}\{v', v_1, \ldots, v_n\}$.
3. Set $\mathcal{S} := \text{conv}\{v_0, v_1, \ldots, v_{m_k - 1}, v', v_{m_k+1}, \ldots, v_n\}$.
4. If $k < p$, set $k := k + 1$ and go to step 2.
5. Set $\mathcal{S}^{p+1} := \mathcal{S}$.



Piecewise Affine Feedback



(a) Affine feedback



(b) PWA feedback

A Third Result

Theorem. *Suppose the triangulation assumption holds. For a somewhat stronger version of RCP, TFAE:*

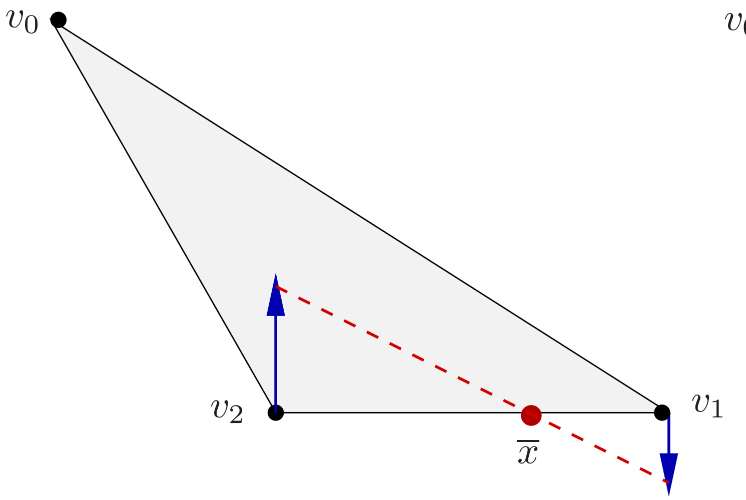
1. $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by piecewise affine feedback.*
2. $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ *by open-loop controls.*

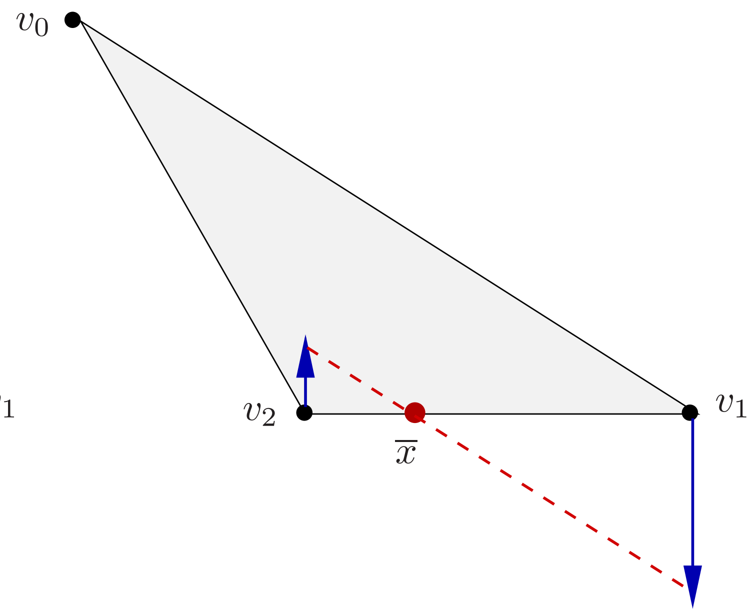
[BG12] M.E. Broucke and M. Ganness. IEEE TAC, in revision 2012.

Time-varying Affine Feedback

Time-varying Affine Feedback

Equilibria are such a drag...





A Fourth Result

Theorem. *Suppose the triangulation assumption holds and the invariance conditions are solvable. There exists $c > 0$ sufficiently small such that $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ using the time-varying affine feedback*

$$u(x,t) = \mathrm{e}^{-ct} u^0(x) + (1 - \mathrm{e}^{-ct}) u^\infty(x)$$

where $u^0(x) = K^0 x + g^0$ places closed-loop equilibria at

$$v_{m_1}, \ldots, v_{m_p}$$

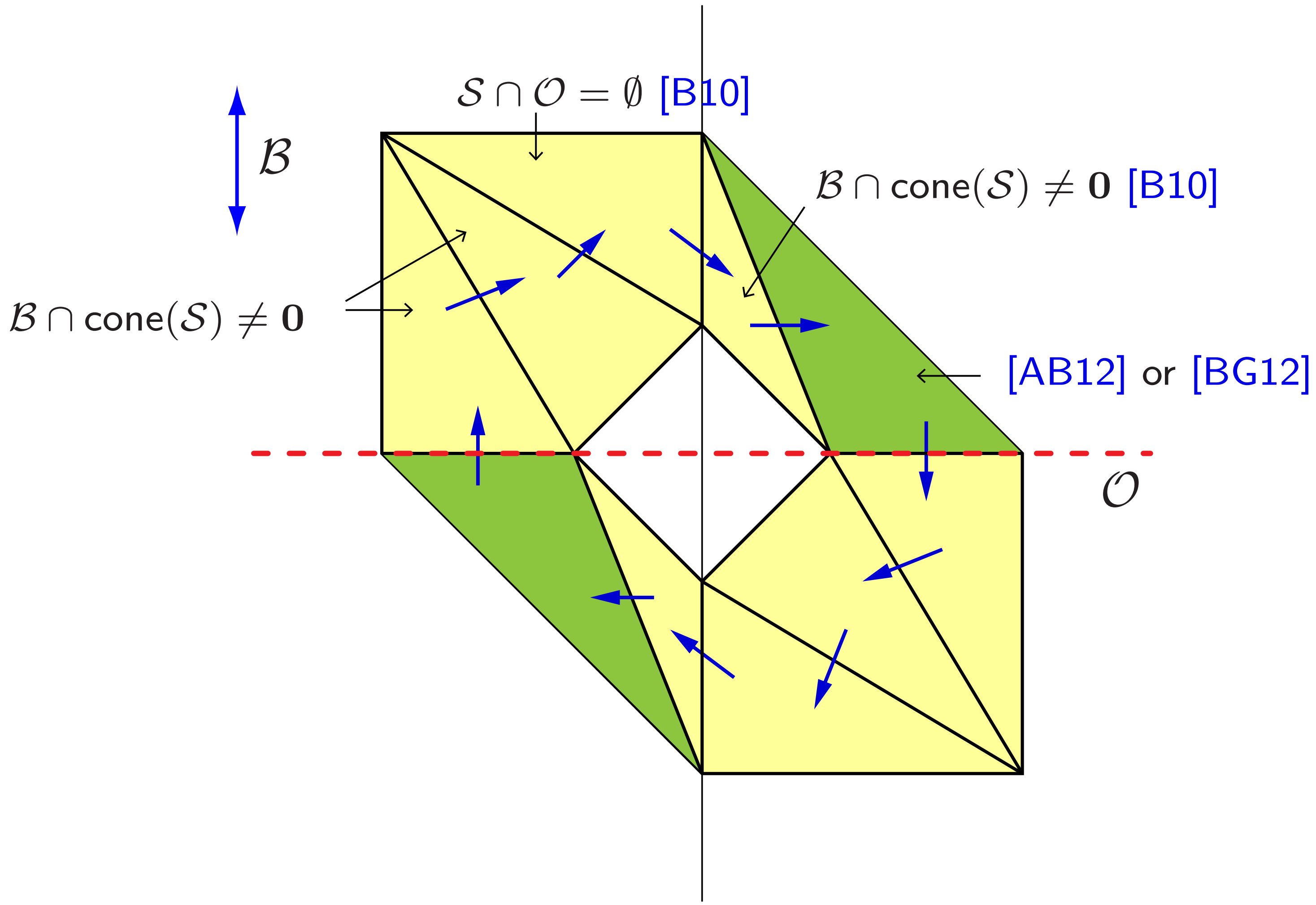
and $u^\infty(x) = K^\infty x + g^\infty$ places closed-loop equilibria at

$$v_{m_1+r_1-1}, \ldots, v_{m_p+r_p-1} \, .$$

[AB12] G. Ashford and M. Broucke. Automatica, accepted 2012.

Motivating Example

Preferred Approach



Final Design