ECE1647F Introduction to Nonlinear Control Systems
Final Project

Due: **Monday, December 22**

Work independently without consulting anyone else, **including the instructor.**

Problems

In this project you'll work on some theoretical issues concerning the nonlinear controller you designed in the lab. The notation in this document refers to the lab handout. Throughout this document we assume that $b_1 = b_2 = 0$, $V_0 = 1$, and that $\tau(q, \dot{q})$ has the expression in (2) in the lab handout. If you don't manage to solve one of the problems, just continue along and try solving the other ones.

1. Show that the linearization of the plant at $(q, \dot{q}) = (0, 0)$ has matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{l_1 g M}{J} & 0 & 0 \\ 0 & \dfrac{g(J + M l_1^2)}{l_2 J} & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1/J \\ \dfrac{l_1}{l_2 J} \end{bmatrix}.$$

2. Prove that the set $W = \{(q, \dot{q}) : \mathbb{E}(q, \dot{q}) = 0,\ q_1 = 0,\ \dot{q}_1 = 0\}$ is the union of the stable and unstable manifolds of the equilibrium $(q, \dot{q}) = (0, 0)$ of the closed-loop system (i.e., the system obtained when using the feedback (2)).

3. Prove that if

$$\frac{k_2}{k_e} > 2 M l_2 g [J + M(l_1^2 + l_2^2)],$$

then

$$(\forall (q, \dot{q}) \in \mathbb{R}^4) \quad k_e \mathbb{E}(q, \dot{q}) + \frac{k_2 M l_2^2}{\det \mathbf{M}(q)} > 0,$$

and therefore the feedback τ is well-defined. [Hint: find a lower bound to $\mathbb{E}(q, \dot{q})$ and an upper bound to $\det(\mathbf{M}(q))$.]

4. Let ϵ be an arbitrary scalar in $(0, 2 k_e M^2 l_2^2 g^2)$. Prove that any initial condition in the set

$$\Omega_\epsilon := \{(q, \dot{q}) : V(q, \dot{q}) \le 2 k_e M^2 l_2^2 g^2 - \epsilon\},$$

the corresponding solution $(q(t), \dot{q}(t))$ cannot approach any of the equilibria $(q, \dot{q}) = (0, \pi + 2k\pi, 0, 0)$, k integer. This means that the pendulum cannot get "stuck" in the vertical downward configuration.

5. Recall that the nonlinear controller yields $\dot{V} = -\mu \dot{q}_1^2$. Consider the set $E_\epsilon = \{(q, \dot{q}) \in \Omega_\epsilon : \dot{q}_1 = 0\}$. Note that, if M is *any* invariant subset of E_ϵ, then necessarily

$$(\forall (q_0, \dot{q}_0) \in M)\ (V(q(t), \dot{q}(t)) = \text{constant}, \quad q_1(t) = \text{constant}, \quad \mathbb{E}(q(t), \dot{q}(t)) = \text{constant}).$$

(where $(q(t), \dot{q}(t))$ denotes the solution with IC $(q_0, \dot{q}_0)$). Use this fact to prove that

$$(\forall (q_0, \dot{q}_0) \in M)\ \mathbb{E}(q(t), \dot{q}(t)) \cdot \tau(q(t), \dot{q}(t)) = \text{constant}.$$

6. You'll now use the previous result to prove that, for all $(q_0, \dot{q}_0) \in M$, $\mathbb{E}(q(t), \dot{q}(t)) \equiv 0$. Suppose, by way of contradiction, that for some initial condition in M, $\mathbb{E}(q(t), \dot{q}(t)) = \text{constant} \neq 0$. Use part 5 to prove that $\tau(q(t), \dot{q}(t))$ must be constant.

7. [Difficult] Use parts 5 and 6 to prove that

   (a) $\sin(q_2(t)) \equiv 0$.

   (b) $\tau(q(t), \dot{q}(t)) \equiv 0$.

   [Hint: start from equation (1) in the lab handout. Among other things, use the fact that the energy is constant.]

8. Use part 7 to prove that $q_1(t) \equiv 0$.

   [Hint: use equation (2) in the lab handout.]

9. Use the results in part 4, 7, and 8 to show that for all initial conditions in the set M, $\mathbb{E}(q(t), \dot{q}(t)) \equiv 0$, which contradicts the assumption in part 6.

10. Prove that the largest invariant subset M of E_ϵ is contained in $W \cap \Omega_\epsilon$.