ECE1647F
Introduction to Nonlinear Control Systems
Assignment 4

Due: **Thursday, November 20**

Problem 1

(Khalil, Problem 4.30) Consider the nonlinear system

$$\dot{x}_1 = -\frac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2$$
$$\dot{x}_2 = x_1 - \frac{1}{2}\tan\left(\frac{\pi x_2}{2}\right).$$

1. Find all equilibria.
2. Using linearization, study stability of each equilibrium point.
3. Using quadratic Lyapunov functions suggested by the linearization, estimate the domain of attraction of one of the asymptotically stable equilibria. To facilitate your analysis, you can use a computer program to plot various sets. If you use Matlab, you may use the command `contour`.
4. Draw the phase portrait (again, you can use a computer program) and show on it the exact domain of attraction, as well as your estimate.

Problem 2

Consider the following class of nonlinear systems

$$\begin{aligned} \dot{x} &= Ax + f(x) \\ y &= Cx \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *globally Lipschitz* vector field, and $C \in \mathbb{R}^{1\times n}$. Assume that the state of the system is unknown, while $y(t)$ can be measured. An *observer* for (1) is a dynamical system

$$\dot{\hat{x}} = \hat{f}(\hat{x}, y)$$

where $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, such that, $\forall(x_0, \hat{x}_0) \in \mathbb{R}^n \times \mathbb{R}^n$, $\lim_{t\to\infty}(\phi(t, x_0) - \hat{\phi}(t, \hat{x}_0)) = 0$. Hence, an observer is a device which *estimates* the state of (1) by only using the information given by y.

Assume that f in (1) is globally Lipschitz and that there exists $L \in \mathbb{R}^{n\times 1}$ such that $A - LC$ is Hurwitz. Find conditions on the Lipschitz constant of f such that

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x})$$

is an observer for (1).

[Hint: consider the estimation error $\tilde{x} = x - \hat{x}$ and write the error dynamics. Find a quadratic Lyapunov function for the linear part and use it for the nonlinear error dynamics. Then use the fact that f is globally Lipschitz to find an upper bound for the Lie derivative of V.]

Problem 3

Consider the following dynamical system

$$\dot{x} = Ax + g(x),$$

where $A \in \mathbb{R}^{n \times n}$ is a Hurwitz matrix and $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz on $\mathbb{R}^n$ and such that, for some $\gamma > 0$,

$$(\forall x \in \mathbb{R}^n) \ \|g(x)\| \leq \gamma \|x\|^2.$$

(i) Prove that the origin is an exponentially stable equilibrium;

(ii) Find an estimate of its domain of attraction;

(iii) Let now $n = 2$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} x_1^2 + x_2^2 \\ 0 \end{bmatrix}.$$

Find the value of γ for this example and simulate the system for a few different initial conditions which are increasingly distant from the origin. Compare your estimate of the domain of attraction to the simulation results: how conservative is your estimate?

[Hint: in parts (i) and (ii) use Lyapunov's equation.]

Problem 4

Consider the model of a controlled pendulum with unknown mass (we assume that the state space is $\mathbb{R}^2$)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\theta \sin x_1 + \frac{1}{I} u. \end{aligned}$$

The parameter θ is unknown. You'll now design an adaptive controller that stabilizes the inverted configuration $x = (\pi, 0)$. Note that, if θ were known, the control law

$$u = I\left(-c_1(x_1 - \pi) - c_2 x_2 + \theta \sin x_1\right), \quad c_1, c_2 > 0$$

would globally exponentially stabilize the equilibrium $(\pi, 0)$. Consider the adaptive control law

$$\begin{aligned} u(x_1, x_2, \hat{\theta}) &= I\left(-c_1(x_1 - \pi) - c_2 x_2 + \hat{\theta} \sin x_1\right), \quad c_1, c_2 > 0 \\ \dot{\hat{\theta}} &= g(x_1, x_2), \end{aligned}$$

where $g(x_1, x_2)$ is to be designed.

1. Using a function of the form

$$V = \tilde{x}^\top P \tilde{x} + \frac{1}{2}(\hat{\theta} - \theta)^2, \quad \text{where } \tilde{x} = (\tilde{x}_1, \tilde{x}_2) = (x_1 - \pi, x_2),$$

and LaSalle's invariance principle, find $g(x_1, x_2)$ such that

(i) All trajectories $(x_1(t), x_2(t), \theta(t))$ of the closed-loop system are bounded.

(ii) The straight line

$$L = \{(x_1, x_2, \hat{\theta}) \in \mathbb{R}^3 : x_1 = \pi,\ x_2 = 0\}$$

is a set of equilibria for the closed-loop system.

(iii) All trajectories of the closed-loop system approach L as $t \to \infty$.

2. Choose a numerical value for θ, and simulate the closed-loop system with your adaptive controller. Plot $(x_1(t), x_2(t))$ on the x_1-x_2 plane for several initial conditions to verify that $(x_1(t), x_2(t)) \to (\pi, 0)$. Does $\hat{\theta}(t)$ converge to θ?

Problem 5

Consider again the controlled pendulum model but now suppose that both its mass and its inertia are unknown. We write the model as follows

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\theta_1 \sin x_1 + \theta_2 u,\end{aligned}$$

where θ_1 and θ_2 are unknown. Note that, although θ_2 is unknown, it is known that its sign is positive. If θ_1 and θ_2 were known, the controller

$$u(x_1, x_2) = \frac{\theta_1}{\theta_2}\sin x_1 - \frac{1}{\theta_2}(c_1(x_1 - \pi) + c_2 x_2),\ \ c_1, c_2 > 0$$

would make the equilibrium $(\pi, 0)$ globally exponentially stable. Let $a_1 = \frac{\theta_1}{\theta_2}$ and $a_2 = \frac{1}{\theta_2}$ and consider the adaptive control law

$$\begin{aligned}u(x_1, x_2, \hat{a}_1, \hat{a}_2) &= \hat{a}_1 \sin x_1 - \hat{a}_2(c_1\tilde{x}_1 + c_2\tilde{x}_2)\\ \dot{\hat{a}}_1 &= g_1(x_1, x_2)\\ \dot{\hat{a}}_2 &= g_2(x_1, x_2).\end{aligned}$$

Using a function of the form

$$V = \tilde{x}^\top P \tilde{x} + \frac{\theta_2}{2}(\hat{a}_1 - a_1)^2 + \frac{\theta_2}{2}(\hat{a}_2 - a_2)^2,$$

find $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ yielding properties analogous to (i)-(iii) in Problem 4. Produce simulations analogous to those in Problem 4 that confirm your theoretical results.

Problem 6

Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_1(1 - x_1^2 - x_2^2)\\ \dot{x}_2 &= x_1 + x_2(1 - x_1^2 - x_2^2).\end{aligned}$$

Using LaSalle's invariance principle, prove that unit circle is a stable limit cycle of the system. Additionally, prove that any phase curve through a nonzero initial condition asymptotically approaches the unit circle.