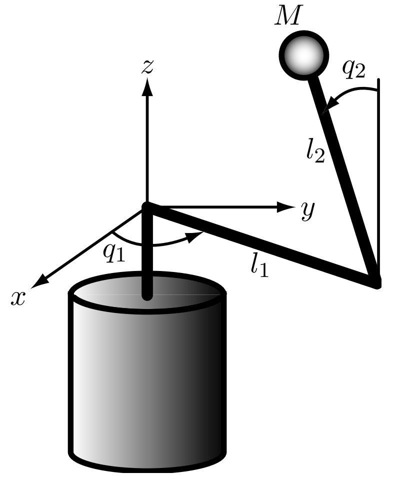
ECE1647F Introduction to Nonlinear Control Systems
Energy-Based Control of a Rotary Pendulum

Due: **Wednesday, December 3**

This problem introduces you to the experimental setup you will be working on in the Systems Control Group lab. Consider the rotary pendulum system depicted below.



The system consists of an arm which is driven by a DC motor and rotates in the horizontal plane, and a pendulum rod mounted on the arm which is free to rotate in the plane orthogonal to the arm. This mechanical system has therefore two degrees of freedom, of which only one is actuated (the arm).

The control input for this system is the DC motor current, I_m. The control objective we consider here is to stabilize the pendulum in the inverted position, corresponding to $q_2 = 0$.

The following is a list of physical parameters of the system and their description.

$J = 1.647 \cdot 10^{-3}$ Kg $\cdot$ m^2	Moment of inertia of the motor plus arm
$M = 7.3 \cdot 10^{-2}$ Kg	Mass of the pendulum rod
$l_1 = 0.215$ m	Length of the arm
$l_2 = 8.18 \cdot 10^{-2}$ m	Distance of the pendulum pivot to the pendulum c.o.g.
$b_1 = 1.1 \cdot 10^{-3}$ N $\cdot$ m $\cdot$ s/rad	Viscous damping coefficient at the motor shaft
$b_2 = 5 \cdot 10^{-4}$ N $\cdot$ m $\cdot$ s/rad	Viscous damping coefficient at the pendulum pivot
$\eta = 0.69$	Electro-mechanical efficiency of the motor
$K_t = 7.42 \cdot 10^{-2}$ N $\cdot$ m/A	Motor torque constant

The torque delivered by the motor is

$$\tau = \eta K_t I_m.$$

We use the Euler-Lagrange method to derive the system model. Referring to the figure above, let (x_p, y_p, z_p) denote the coordinates of the center of gravity of the pendulum rod with respect to the inertial reference frame x, y, z. These are given by

$$\begin{aligned} x_p &= l_1 \cos q_1 + l_2 \sin q_1 \sin q_2 \\ y_p &= l_1 \sin q_1 - l_2 \cos q_1 \sin q_2 \\ z_p &= l_2 \cos q_2. \end{aligned}$$

The velocity of the center of mass is obtained by differentiation

$$\begin{aligned} \dot{x}_p &= l_2 \cos q_1 \sin q_2 \, \dot{q}_1 + l_2 \sin q_1 \cos q_2 \, \dot{q}_2 - l_1 \sin q_1 \, \dot{q}_1 \\ \dot{y}_p &= l_2 \sin q_1 \sin q_2 \, \dot{q}_1 - l_2 \cos q_1 \cos q_2 \, \dot{q}_2 + l_1 \cos q_1 \, \dot{q}_1 \\ \dot{z}_p &= -l_2 \sin q_2 \, \dot{q}_2. \end{aligned}$$

The potential energy of the system is

$$V = Mgl_2(\cos q_2 - V_0).$$

The kinetic energy of the motor plus arm is

$$T_1 = \frac{1}{2} J \dot{q}_1^2.$$

The kinetic energy of the pendulum rod is

$$T_2 = \frac{1}{2} M(\dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2).$$

The total kinetic energy is

$$T = T_1 + T_2$$

The Lagrangian of the system is $L = T - V$,

$$L = \frac{1}{2} J \, \dot{q}_1^2 + \frac{1}{2} M \left[(l_1^2 + l_2^2 \sin^2 q_2) \, \dot{q}_1^2 + l_2^2 \, \dot{q}_2^2 - 2 l_1 l_2 \cos q_2 \, \dot{q}_1 \, \dot{q}_2 \right] - M l_2 g (\cos q_2 - V_0).$$

Let $q = (q_1, q_2)$, and define the vector of generalized forces $Q = (\tau - b_1 \dot{q}_1, -b_2 \dot{q}_2)$. The model of the system is given by the Euler-Lagrange equations,

$$Q = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.$$

By straightforward computation and rearrangement of terms one gets

$$\begin{aligned} &\begin{bmatrix} J + M(l_1^2 + l_2^2 \sin^2 q_2) & -M l_1 l_2 \cos q_2 \\ -M l_1 l_2 \cos q_2 & M l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} \\ &+ \begin{bmatrix} M l_2^2 \sin q_2 \cos q_2 \dot{q}_2 & M l_2^2 \sin q_2 \cos q_2 \dot{q}_1 + M l_1 l_2 \sin q_2 \dot{q}_2 \\ -M l_2^2 \sin q_2 \cos q_2 \dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} b_1 \dot{q}_1 \\ b_2 \dot{q}_2 - M g l_2 \sin q_2 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}, \end{aligned} \tag{1}$$

where $\tau = \eta K_t I_m$. The system has the standard form

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q, \dot{q})\dot{q} + \mathbf{N}(q, \dot{q}) = \mathbf{T}.$$

The state of the system is $(q, \dot{q}) = (q_1, q_2, \dot{q}_1, \dot{q}_2) \in \mathbb{R}^4$. The control input is I_m.
Control objective: To design a C^1 feedback $I_m(q, \dot{q})$ stabilizing the unstable equilibrium $(q, \dot{q}) = (0, 0)$.

Energy-Based control design

The energy of the system is

$$\mathbb{E}(q,\dot{q}) = T + V = \frac{1}{2}\dot{q}^\top \mathbf{M}(q)\dot{q} + Ml_2 g(\cos(q_2) - V_0).$$

Let $V_0 = 1$, and suppose the system is frictionless, i.e., $b_1 = b_2 = 0$. Consider the set

$$W = \{(q,\dot{q}) : \mathbb{E}(q,\dot{q}) = 0,\ q_1 = 0,\ \dot{q}_1 = 0\}.$$

Note that W contains the unstable equilibrium $(q,\dot{q}) = (0,0)$. Our control design strategy is this. We first design a controller that makes the phase curves of the closed-loop system approach the set W. Additionally, the set W is the union of the stable and unstable manifolds of the equilibrium $(q,\dot{q}) = (0,0)$ for the closed-loop system. While approaching the set W, phase curves of the closed-loop system time and again enter an arbitrarily small neighborhood of $(q,\dot{q}) = (0,0)$. We then design a linear controller based on the linearization of the system at the unstable equilibrium. Once the phase curves enter a sufficiently small neighborhood of the unstable equilibrium, we switch to the linear controller which stabilizes the equilibrium.

Stabilization of W

After the lab, you'll be asked to justify a number of statements in this section.

Let

$$V(q,\dot{q}) = \frac{1}{2}k_e\mathbb{E}^2 + \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2\dot{q}_1^2, \quad k_e, k_1, k_2 > 0.$$

It can be shown that $\dot{\mathbb{E}} = \dot{q}_1\tau$, and hence

$$\dot{V} = k_e\mathbb{E}\dot{q}_1\tau + k_1 q_1\dot{q}_1 + k_2\dot{q}_1\ddot{q}_1.$$

Since

$$\ddot{q} = -\mathbf{M}(q)^{-1}\mathbf{C}(q,\dot{q})\dot{q} - \mathbf{M}(q)^{-1}\mathbf{N}(q,\dot{q}) + \mathbf{M}(q)^{-1}\mathbf{T},$$

using Cramer's rule we obtain

$$\ddot{q}_1 = \varphi(q,\dot{q}) + \frac{Ml_2^2}{\det \mathbf{M}(q)}\tau,$$

where

$$\varphi(q,\dot{q}) = -\frac{1}{\det \mathbf{M}(q)}\begin{bmatrix} Ml_2^2 & Ml_1l_2\cos q_2 \end{bmatrix}[\mathbf{C}(q,\dot{q})\dot{q} + N(q,\dot{q})].$$

In conclusion,

$$\begin{aligned}\dot{V} &= \dot{q}_1\left[k_e\mathbb{E}\tau + k_1q_1 + k_2\left(\varphi(q,\dot{q}) + \frac{Ml_2^2}{\det \mathbf{M}(q)}\tau\right)\right] \\ &= \dot{q}_1\left[\left(k_e\mathbb{E} + \frac{k_2Ml_2^2}{\det \mathbf{M}(q)}\right)\tau + k_1q_1 + k_2\varphi(q,\dot{q})\right].\end{aligned}$$

Choose the feedback

$$\tau = \frac{1}{k_e\mathbb{E}(q,\dot{q}) + \dfrac{k_2Ml_2^2}{\det \mathbf{M}(q)}}\left[-\mu\dot{q}_1 - k_1q_1 - k_2\varphi(q,\dot{q})\right], \tag{2}$$

where $\mu > 0$, so that

$$\dot{V} = -\mu\dot{q}_1^2.$$

Is the feedback (2) well-defined? It can be shown that if

$$\frac{k_2}{k_e} > 2Ml_2g[J + M(l_1^2 + l_2^2)],$$

then

$$(\forall (q, \dot{q}) \in \mathbb{R}^4) \;\; k_e\mathbb{E}(q, \dot{q}) + \frac{k_2 M l_2^2}{\det \mathbf{M}(q)} \neq 0,$$

and so the feedback is well-defined. It can be shown that this feedback makes the set W asymptotically stable. In particular, for all initial conditions $(q_0, \dot{q}_0)$ such that $V(q_0, \dot{q}_0) < 2k_e M^2 l_2^2 g^2$, the solution $(q(t), \dot{q}(t))$ asymptotically approaches W.

Local stabilization of $(q, \dot{q}) = (0, 0)$

It can be shown that the linearization of the system at $(q, \dot{q}) = (0, 0)$ has matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{l_1 g M}{J} & -\dfrac{b_1}{J} & -\dfrac{l_1 b_2}{l_2 J} \\ 0 & \dfrac{g(J + Ml_1^2)}{l_2 J} & -\dfrac{l_1 b_1}{l_2 J} & -\dfrac{b_2(J + Ml_1^2)}{Ml_2^2 J} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1/J \\ \dfrac{l_1}{l_2 J} \end{bmatrix}.$$

The pair (A, B) is controllable, therefore a feedback that locally stabilizes the unstable equilibrium is given by

$$\tau = K \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix},$$

where $K \in \mathbb{R}^{1\times 4}$ is chosen so that $A + BK$ is Hurwitz.

Lab preparation

Please produce the material described below. Do not include extra-material in your preparation.

1. Let $b_1 = b_2 = 0$ and $V_0 = 1$. Pick $k_1 = 1$ and $\mu = 10$, while k_2 and k_e will have to be tuned. Start with $k_2 = 1$, $k_e = 200$. Simulate the pendulum with controller (2) in Matlab (use the parameters provided in the table on page 1) and verify that the controller stabilizes the set W. Tune the parameters k_2, k_e to improve the performance of the controller. Monitor the current $I_m = \tau/(\eta K_t)$. You should make sure that I_m remains most of the times below 5A with peaks of at most 10A with duration of at most 0.1s.

   Produce:

   - The values of k_2 and k_e you have found.
   - One plot of $(q_1(t), \dot{q}_1(t))$ on the $(q_1, \dot{q}_1)$ plane and $(q_2(t), \dot{q}_2(t))$ on the $(q_2, \dot{q}_2)$ plane when the initial condition is $(q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)) = (0, 3/4\pi, 0, 0)$.
   - One plot of the current $I_m(t)$ for the same initial condition.

   Comment: What is the physical behavior of the pendulum as the set W is stabilized?

2. Now you'll design a linear feedback. Pick a stabilizing feedback gain vector K to satisfy the following criteria: the domain of attraction of $(q, \dot{q}) = (0, 0)$ should include points such that $|q_1| < \pi/10$ rad, $|q_2| < \pi/10$ rad, $|\dot{q}_1| < 1$ rad/s, and $|\dot{q}_2| < 1$ rad/s. The current $I_m(t)$ should satisfy the constraints outlined in part 1. Note: your choice should be guided by appropriate simulations of the nonlinear system, no theoretical analysis is needed.

   Produce: The feedback gain vector K and the location of the poles of the closed-loop system.

3. Now you'll simulate a controller that merges the two controllers in parts 1 and 2 by means of a rudimentary switching logic: it begins by using the energy controller in part 1. The first time that the phase curve enters the set

   $$\{(q, \dot{q}) : |q_1| < \pi/10, |q_2| < \pi/10, |\dot{q}_1| < 1, |\dot{q}_2| < 1\}$$

   it switches to the linear controller and disables the energy controller thereafter.

   Produce: One plot of $(q_1(t), \dot{q}_1(t))$ on the $(q_1, \dot{q}_1)$ plane and $(q_2(t), \dot{q}_2(t))$ on the $(q_2, \dot{q}_2)$ plane when the initial condition is $(q_1(0), q_2(0), \dot{q}_1(0), \dot{q}_2(0)) = (0, 3/4\pi, 0, 0)$.

4. Now you'll investigate the effect of friction. Let b_1 and b_2 be as in the table on the first page. Using the switching controller, simulate the closed-loop system using the parameters that came out of your tuning in parts 1 and 2.

   Comment: What difference do you observe with respect to part 3? What is the reason for the difference? Explore the effect of varying the parameter V_0 (and possibly k_2 and k_e) on what you observe.

   Produce: One plot illustrating the difference you observe with respect to part 3. An analogous plot for the value of V_0 you have chosen.

5. **Prepare:** A Simulink block diagram implementing the switching controller. You'll use this diagram in the lab. The block takes in input the system state $(q_1, q_2, \dot{q}_1, \dot{q}_2)$ and returns the control input $I_m = \tau/(\eta K_t)$. The diagram should have parameters k_e, k_1, k_2, and μ that should be easily accessible for the user to tune.

   Note: In creating a Simulink block for your pendulum controller, you can use "user-defined function" blocks. Open the library browser, look at User-Defined functions, and use the "Fcn" block. In order for the Real Time Workshop to compile your code, you can't embed Matlab `.m` files.