ECE311 - Dynamic Systems and Control

Linearization of Nonlinear Systems

Objective

This handout explains the procedure to linearize a nonlinear system around an equilibrium point. An example illustrates the technique.

1 State-Variable Form and Equilibrium Points

A system is said to be in *state-variable form* if its mathematircal model is described by a system of n first-order differential equations and an algebraic output equation:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \dots, x_n, u) \\
\dot{x}_2 &= f_2(x_1, \dots, x_n, u) \\
&\cdots \\
\dot{x}_n &= f_n(x_1, \dots, x_n, u) \\
y &= h(x_1, \dots, x_n, u).
\end{aligned} \tag{1}$$

The column vector $\mathbf{x} = [x_1, \dots, x_n]^\top$ is called the *state* of the system. The scalars u and y are called the *control input* and the *system output*, respectively. Denoting

$$\mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} f_1(x_1, \dots, x_n, u) \\ f_2(x_1, \dots, x_n, u) \\ \vdots \\ f_n(x_1, \dots, x_n, u) \end{bmatrix},$$

we concisely rewrite (1) as

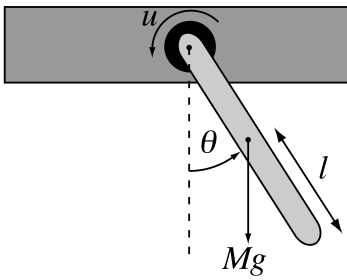
$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, u) \\
y &= h(\mathbf{x}, u).
\end{aligned} \tag{2}$$

When $\mathbf{f}$ and h are nonlinear functions of $\mathbf{x}$ and u, then we say that the system is nonlinear. In this course we will work exclusively with linear systems, i.e., systems for which (2) becomes

$$\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + Bu \\
y &= C\mathbf{x} + Du,
\end{aligned} \tag{3}$$

where A is $n \times n$, B is $n \times 1$, C is $1 \times n$, and D is a scalar. Sometimes, physical systems are described by nonlinear models such as (2), and the tools we will learn in this course can not be employed to design controllers. However, if a nonlinear system operates around an *equilibrium point*, i.e., around a configuration where the system is at rest, then it is possible to study the behavior of the system in a neighborhood of such point.

Example 1 (A simple pendulum). Consider the dynamics of the pendulum depicted below, where u denotes an input torque provided by a DC motor.



The equation of motion for this system is

$$\begin{aligned} I\frac{d^2\theta}{dt^2} + Mgl\sin\theta &= u \\ y &= \theta, \end{aligned} \tag{4}$$

where I is the moment of inertia of the pendulum around the pivot point, and y is the output of the system, i.e., the variable one wants to control. Consider now the equivalent state-variable representation of (4), obtained by choosing $x_1 = \theta$ and $x_2 = \dot{\theta}$,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \\ y &= x_1 \end{aligned} \tag{5}$$

The model (5) has precisely the form (2), where in this case $\mathbf{x} = [x_1, x_2]^\top$ and

$$\mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} x_2 \\ -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \end{bmatrix}, \qquad h(\mathbf{x}, u) = x_1.$$

Since $\mathbf{f}$ contains the term $\sin x_1$, the system (5) is nonlinear. Observe that when $(\mathbf{x}, u) = (0, 0)$, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) = 0$ which implies that $\mathbf{x}(t) = [x_1(t)\ x_2(t)]^\top$ is constant for all t. In other words, if the pendulum is in the vertical downward position with no angular velocity (i.e., $x_1 = x_2 = 0$), and with no input torque (i.e., $u = 0$), then the pendulum stays in the vertical downward position for all time (i.e., $x(t)$ is constant for all t). For this reason, the configuration $\mathbf{x} = [0\ 0]^\top$ is referred to as an *equilibrium point*.

△

We now generalize the intuition developed in the previous example by defining the notion of an equilibrium point.

Definition 1 *(Equilibrium Point)* Consider a system in state-variable form (2). Suppose that u is set to be a constant value u^*. Then, $\mathbf{x}^*$ is said to be an *equilibrium point* for (2) if $\mathbf{f}(\mathbf{x}^*, u^*) = [0\ 0\ \ldots 0]^\top$.

Example 2 (Pendulum - continued) Back to the pendulum example, suppose we turn off the DC motor, that is, we set $u = u^* = 0$. Let's use the definition above to find all corresponding equilibria. We set $\mathbf{f}(\mathbf{x}, u)\Big|_{u=u^*=0} = [0\ 0]^\top$, in other words,

$$\begin{aligned} x_2 &= 0 \\ -\frac{Mgl}{I}\sin x_1 &= 0 \end{aligned}$$

Thus, the equilibrium points of the pendulum with $u = 0$ are given by

$$\mathbf{x}^* = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \quad k \text{ integer.}$$

Physically, this means that the pendulum is at equilibrium whenever the angle θ is either 0 (pendulum pointing downward) or π (pendulum pointing upward), and the angular velocity $\dot{\theta}$ is zero. Qualitatively, the equilibrium $\mathbf{x}^* = [0\ 0]^\top$ is stable, while the equilibrium $\mathbf{x}^* = [\pi\ 0]^\top$ is unstable.

Now suppose we turn on the DC motor in such a way that it produces a desired constant torque $u = u^* \neq 0$. The corresponding equilibria must satisfy the equation $f(\mathbf{x}, u^*) = [0\ 0]^\top$, i.e.,

$$\begin{aligned} x_2 &= 0 \\ -\frac{Mgl}{I}\sin x_1 + \frac{u^*}{I} &= 0. \end{aligned}$$

Note that, setting $u = u^* = Mgl \sin x_1^*$, the state

$$\mathbf{x}^* = \begin{bmatrix} x_1^* \\ 0 \end{bmatrix}$$

is an equilibrium point of the pendulum. Physically, that means that by imparting a suitable constant torque to the pendulum one can make the pendulum be at rest at any desired angle x_1^*. For instance, by imparting a torque $u = u^* = Mgl$, the configuration $x_1 = \pi/2$, $x_2 = 0$ is an equilibrium of the pendulum. Is such configuration stable or unstable?

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2 Linearization

Although almost every physical system contains nonlinearities, oftentimes its behavior *within a certain operating range of an equilibrium point* can be reasonably approximated by that of a linear model. One reason for approximating the nonlinear system (2) by a linear model of the form (3) is that, by so doing, one can apply rather simple and systematic linear control design techniques such as those introduced in this course. Keep in mind, however, that a linearized model is valid only when the system operates in a sufficiently small range around an equilibrium point. To take into account the presence of nonlinearities, more sophisticated tools are needed which are beyond the scope of this course.

Given the nonlinear system (2) and an equilibrium point $\mathbf{x}^* = [x_1^* \ \cdots \ x_n^*]^\top$ obtained when $u = u^*$, we define a *coordinate transformation* as follows. Denote $\mathbf{\Delta x} = \mathbf{x} - \mathbf{x}^*$, i.e.,

$$\mathbf{\Delta x} = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}.$$

Further, denote $\Delta u = u - u^*$, and $\Delta y = y - h(\mathbf{x}^*, u^*)$. The new coordinates $\mathbf{\Delta x}$, Δu, and Δy represent the variations of $\mathbf{x}$, u, and y from their equilibrium values. You have to think of these as a new state, new control input, and new output respectively.

The linearization of (2) at $\mathbf{x}^*$ is given by

$$\begin{aligned} \dot{\mathbf{\Delta x}} &= A\mathbf{\Delta x} + B\Delta u \\ \Delta y &= C\mathbf{\Delta x} + D\Delta u, \end{aligned} \tag{6}$$

where

$$A = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}^*, u^*} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, \dots, x_n^*, u^*) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1^*, \dots, x_n^*, u^*) \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1}(x_1^*, \dots, x_n^*, u^*) & \cdots & \frac{\partial f_n}{\partial x_n}(x_1^*, \dots, x_n^*, u^*) \end{bmatrix},$$

$$B = \left[\frac{\partial \mathbf{f}}{\partial u}\right]_{\mathbf{x}^*, u^*} = \begin{bmatrix} \frac{\partial f_1}{\partial u}(x_1^*, \dots, x_n^*, u^*) \\ \vdots \\ \frac{\partial f_n}{\partial u}(x_1^*, \dots, x_n^*, u^*) \end{bmatrix},$$

$$C = \left[\frac{\partial h}{\partial \mathbf{x}}\right]_{\mathbf{x}^*, u^*} = \begin{bmatrix} \frac{\partial h}{\partial x_1}(x_1^*, \dots, x_n^*, u^*) & \cdots & \frac{\partial h}{\partial x_n}(x_1^*, \dots, x_n^*, u^*) \end{bmatrix}, \quad D = \left[\frac{\partial h}{\partial u}\right]_{\mathbf{x}^*, u^*}.$$

Remark 1: The linearization (6), also referred to as a *small-signal model*, is valid only in a sufficiently small neighborhood of the equilibrium point $\mathbf{x}^*$. Notice that, as expected, (6) has the linear structure (3).

Remark 2: Note that the matrices A, B, C, D have *constant* coefficients in that all partial derivatives are evaluated at the numerical values $(x_1^*, \dots, x_n^*, u^*)$. Please avoid the common mistake of producing matrices A, B, C, D containing expressions that are functions of the symbolic variables $x_1, \dots, x_n$!

Example 3 (Linearization of the pendulum system). Return to the pendulum example. Recall that the state-variable model is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \\
y &= x_1
\end{aligned}$$

Consider the equilibrium point, obtained by setting $u = u^* = 0$, corresponding to the vertical upward position and no control input, i.e., $\mathbf{x}^* = [\pi\ 0]^\top$. Following the procedure outlined above, we define

$$\mathbf{\Delta x} = \mathbf{x} - \mathbf{x}^* = \begin{bmatrix} x_1 - \pi \\ x_2 - 0 \end{bmatrix}, \qquad \Delta u = u - 0 = u, \qquad \Delta y = y - \pi,$$

and we form the matrices containing partial derivatives

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{Mgl}{I}\cos x_1 & 0 \end{bmatrix}, \frac{\partial \mathbf{f}}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix}, \frac{\partial h}{\partial \mathbf{x}} = [1, 0], \frac{\partial h}{\partial u} = 0.$$

Next, we evaluate the matrices above at $(x_1^*, x_2^*, u^*) = (\pi, 0, 0)$ and we write the linearized model

$$\begin{aligned}
\begin{bmatrix} \dot{\Delta x_1} \\ \dot{\Delta x_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{I} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \Delta u \\
\Delta y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \Delta x_1.
\end{aligned}$$

Notice how the linearized model is expressed in terms of a new state $\mathbf{\Delta x}$, new control input δu, and new output δy. Recall that these represent variations of $\mathbf{x}$, u, and y from their equilibrium values. The linearized model above is only valid in a small neighborhood of the equilibrium $\mathbf{x}^* = [\pi\ 0]^\top$, that is, it is only valid when the components Δx_1 and Δx_2 of the vector $\mathbf{\Delta x}$, are small. Physically this can be rephrased as follows: the linearized model of the pendulum at the vertical upward position is only valid when the angle θ is in a small neighborhood of π and the angular velocity $\dot{\theta}$ is small.

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