ECE311 - DYNAMIC SYSTEMS AND CONTROL
MIDTERM SOLUTIONS

1. Consider a system

$$\ddot{y} + 2\dot{y} + 2y = u\,.$$

(a) Find the transfer function $G(s) = \frac{Y(s)}{U(s)}$.

(b) Find a state space model of the system.

(c) Suppose $u(t) = 3e^{-t}$ for $t \geq 0$, and $y(0) = \dot{y}(0) = 0$. Find the solution $y(t)$.

(d) Now suppose $u(t) = 0$ and $y(0) = \dot{y}(0) = 2$. Find the solution $y(t)$.

(a)

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2s + 2}\,.$$

(b) Let $x = (y, \dot{y})$. Then

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\,, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x\,.$$

(c) We have

$$U(s) = \frac{3}{s+1}\,.$$

Using partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{3}{(s+1)(s^2+2s+2)} = \frac{3}{(s+1)(s+1-j)(s+1+j)} \\ &= \frac{A}{s+1} + \frac{B}{s+1-j} + \frac{C}{s+1+j}\,. \end{aligned}$$

We get

$$A = 3\,, \qquad B = -\frac{3}{2}\,, \qquad C = B^* = -\frac{3}{2}\,.$$

Taking the inverse Laplace transform

$$\begin{aligned} y(t) &= 3e^{-t} - \frac{3}{2}\mathrm{e}^{-t}\left(\mathrm{e}^{jt} + \mathrm{e}^{-jt}\right) \\ &= 3e^{-t} - 3\mathrm{e}^{-t}\cos t\,, \qquad t \geq 0\,. \end{aligned}$$

(d) Suppose $\dot{y}(0) = 1$ and $y(0) = 1$. Taking Laplace transforms

$$s^2 Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + 2Y(s) = 0\,.$$

After simplifying

$$Y(s) = \frac{s+3}{(s+1+j)(s+1-j)} = \frac{A}{s+1+j} + \frac{B}{s+1-j}\,.$$

We get

$$A = \frac{1}{2} + j\,, \qquad B = A^* = \frac{1}{2} - j\,.$$

Thus

$$\begin{aligned}
y(t) &= \left(\frac{1}{2} + j\right) \mathrm{e}^{-t+jt} + \left(\frac{1}{2} - j\right) \mathrm{e}^{-t+jt} \\
&= \mathrm{e}^{-t} \left[\frac{1}{2}(\mathrm{e}^{jt} + \mathrm{e}^{-jt}) + \frac{2}{2j}(\mathrm{e}^{jt} - \mathrm{e}^{-jt})\right] \\
&= \mathrm{e}^{-t} \cos t + 2\mathrm{e}^{-t} \sin t\,.
\end{aligned}$$

2. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 (\sin x_1 + 2) \\ \dot{x}_2 &= x_1 \left(1 + e^{-x_3}\right) \\ \dot{x}_3 &= \sin x_3 + u \\ y &= x_1 + x_2^2 + x_3^3 \,.\end{aligned}$$

(a) Suppose the input is a constant $u = \overline{u}$. Find all equilibrium points $\overline{x}$.

(b) Assume $\overline{u} = 0$. Verify that $\overline{x} = (0, 0, 0)$ is an equilibrium point of the system.

(c) Linearize the system about the equilibrium point in (b).

(a) We have

$$\begin{aligned}\overline{x}_2 (\sin \overline{x}_1 + 2) &= 0 \\ \overline{x}_1 \left(1 + e^{-\overline{x}_3}\right) &= 0 \\ \sin \overline{x}_3 + \overline{u} &= 0\end{aligned}$$

This yields

$$\begin{aligned}\overline{x}_3 &= \sin^{-1} \overline{u} \\ \overline{x}_1 &= 0 \qquad\qquad (\text{since } \ \mathrm{e}^{-\overline{x}_3} > 0) \\ \overline{x}_2 &= 0 \,.\end{aligned}$$

Therefore equilibria are of the form $(0, 0, \sin^{-1}(\overline{u}))$.

(b) Suppose $\overline{u} = 0$. Then $\overline{x} = 0$ and we get

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 &= 0 \\ \dot{x}_3 &= \sin 0 = 0 \,.\end{aligned}$$

(c)

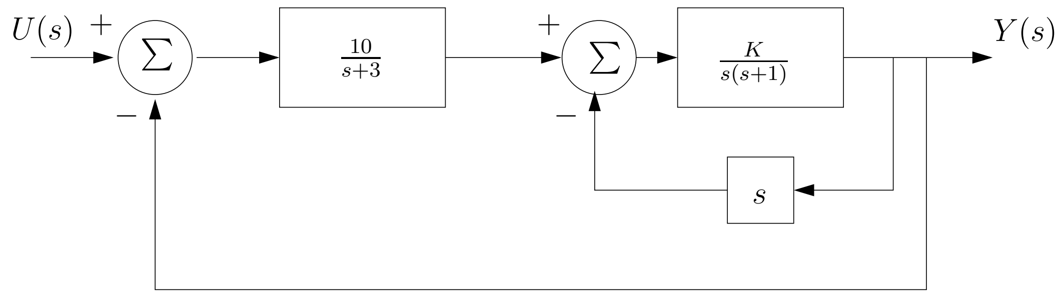
$$A = \left.\frac{\partial f}{\partial x}\right|_{\overline{x}} = \left[\begin{array}{ccc} x_2 \cos x_1 & \sin x_1 + 2 & 0 \\ (1 + \mathrm{e}^{-x_3}) & 0 & -x_1 \mathrm{e}^{-x_3} \\ 0 & 0 & \cos x_3 \end{array}\right]_{\overline{x}=0} = \left[\begin{array}{ccc} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\overline{x}} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right].$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{\overline{x}} = \left[\begin{array}{ccc} 1 & 2x_2 & 3x_3^2 \end{array}\right]_{\overline{x}} = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right].$$

$$D = \left.\frac{\partial h}{\partial u}\right|_{\overline{x}} = 0 \,.$$

3. Consider the system



(a) Find a state-space model of the system.

(b) For what values of K is the system asymptotically stable?

(a) The simplified transfer function is

$$G(s) = \frac{10K}{s^3 + (4+K)s^2 + (3+3K)s + 10K}$$

Therefore, letting $x = (y, \dot{y}, \ddot{y})$, a state model is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10K & -(3+3K) & -(4+K) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u\,.$$

$$y = \begin{bmatrix} 10K & 0 & 0 \end{bmatrix} x\,.$$

(b) For asymptotic stability we must check that the eigenvalues of A are in the OLHP. This can be done using a Routh table.

s^3	1	$3K+3$
s^2	$K+4$	10K
s^1	$\frac{3(K+4)(K+1)-10K}{K+4}$	0
s^0	10K	0

For no sign changes in the first column we require

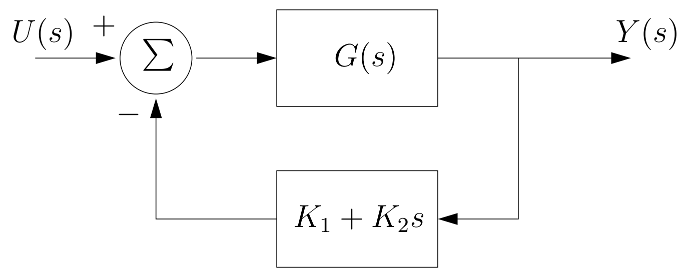
$$K > 0$$
$$K + 4 > 0$$
$$3K^2 + 5K + 12 > 0\,.$$

In sum, the system is asymptotically stable if $K > 0$.

4. Consider a vehicle traveling on a rough road, with y the height of the vehicle seats and u the height of the tires from a given reference. Assume the vehicle can be described by the system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad 0 < \zeta < 1, \qquad \omega_n > 0 .$$

It is found experimentally that the percent overshoot to a unit step input for this system is excessive, i.e. the damping ratio (or damping factor) ζ is too small. To improve the damping ratio an active suspension system is designed in the form of the following feedback:



(a) What choice of K_1 and K_2 should be used to obtain a damping ratio of 2ζ for the closed-loop system?

(b) What is the % overshoot to a unit step in the closed-loop system, when the correct choice of (K_1, K_2) is made?

(a) The closed-loop system is

$$Y(s) = \frac{\omega_n^2}{s^2 + (2\zeta\omega_n + \omega_n^2 K_2)s + \omega_n^2(1 + K_1)} U(s) .$$

To obtain a new damping ratio which is 2ζ we can select $K_1 = 0$ so that the natural frequency is unchanged and set

$$K_2 = \frac{2\zeta}{\omega_n} .$$

(b) The new percent overshoot is:

$$100 e^{-\frac{2\pi\zeta}{\sqrt{1-4\zeta^2}}} .$$