Problem Set 11 Solutions

Problem 1: Robust Tracking Problem

(a)

1. *Feasibility.* The problem is feasible because no zero of the plant is a pole of the exosystem.
2. *Preparation.* We must find the state equations of the plant

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \qquad & x \in \mathbb{R}^n \\ y &= Cx\,. \end{aligned}$$

and of the exosystem

$$\begin{aligned} \dot{w} &= Sw\,, \qquad & w \in \mathbb{R}^q \\ y_d &= C_d w\,. \end{aligned}$$

In this case we have

$$\begin{aligned} \dot{x} &= x + u \\ y &= x\,. \end{aligned}$$

$$\begin{aligned} \dot{w} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} w \\ y_d &= \begin{bmatrix} 1 & 0 \end{bmatrix} w\,. \end{aligned}$$

3. *Pole placement for K.* We want to find K such that $\sigma(A + BK) = \{-1\}$. We obtain $K = -2$.
4. *Regulator equations.* We must solve

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma \\ C\Pi &= C_d \end{aligned}$$

for the unknowns $\Pi \in \mathbb{R}^{n\times q}$ and $\Gamma \in \mathbb{R}^{1\times q}$. This yields

$$\Pi = \begin{bmatrix} 1 & 0 \end{bmatrix}\,, \qquad \Gamma = \begin{bmatrix} -1 & 1 \end{bmatrix}\,.$$

Therefore the (non-robust) asymptotic tracking controller is:

$$u = \Gamma w + K(x - \Pi w) = 1 + t - 2x\,.$$

Note that this controller contains a feedforward and feedback piece.

5. *Pole placement for G.* We must find G such that

$$\sigma\left(\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\begin{bmatrix} -C & C_d \end{bmatrix}\right) = \{-1, -2, -3\}\,,$$

where $G_1 \in \mathbb{R}^n$ and $G_2 \in \mathbb{R}^q$. Using the method of matching coefficients this yields

$$G = \begin{bmatrix} -24 \\ -17 \\ -6 \end{bmatrix}\,.$$

6. *Robust regulator.* The robust regulator is

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} A + BK + G_1 C & B(\Gamma - K\Pi) - G_1 C_d \\ G_2 C & S - G_2 C_d \end{bmatrix} \xi + Ge \\
u &= \Gamma \xi_2 + K(\xi_1 - \Pi \xi_2)\,.
\end{aligned}$$

This yields

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} -25 & 25 & 1 \\ -17 & 17 & 1 \\ -6 & 6 & 0 \end{bmatrix} \xi + \begin{bmatrix} -24 \\ -17 \\ -6 \end{bmatrix} e \\
u &= \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \xi\,.
\end{aligned}$$

In transfer function form this is

$$C(s) = \frac{U(s)}{E(s)} = \frac{25s^2 + 17s + 6}{s^2(s+8)}\,.$$

Notice that the internal model principle is satisfied because $C(s)$ has a copy of the poles of the exosystem.

(b)

1. *Feasibility.* The problem is feasible because no zero of the plant is a pole of the exosystem.
2. *Preparation.* We must find the state equations of the plant

$$\begin{aligned}
\dot{x} &= Ax + Bu\,, \qquad && x \in \mathbb{R}^n \\
y &= Cx\,.
\end{aligned}$$

and of the exosystem

$$\begin{aligned}
\dot{w} &= Sw\,, \qquad && w \in \mathbb{R}^q \\
y_d &= C_d w\,.
\end{aligned}$$

In this case we have

$$\begin{aligned}
\dot{x} &= x + u \\
y &= x\,.
\end{aligned}$$

$$\begin{aligned}
\dot{w} &= \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} w \\
y_d &= [\,1 \;\; 0\,]\, w\,.
\end{aligned}$$

3. *Pole placement for K.* We want to find K such that $\sigma(A + BK) = \{-1\}$. We obtain $K = -2$.
4. *Regulator equations.* We must solve

$$\begin{aligned}
\Pi S &= A\Pi + B\Gamma \\
C\Pi &= C_d
\end{aligned}$$

for the unknowns $\Pi \in \mathbb{R}^{n \times q}$ and $\Gamma \in \mathbb{R}^{1 \times q}$. This yields

$$\Pi = \begin{bmatrix} 1 & 0 \end{bmatrix}\,, \qquad \Gamma = \begin{bmatrix} -1 & -2 \end{bmatrix}\,.$$

Therefore the (non-robust) asymptotic tracking controller is:

$$u = \Gamma w + K(x - \Pi w) = w_1 - 2w_2 - 2x\,.$$

Note that this controller contains a feedforward and feedback piece.

5. *Pole placement for G.* We must find G such that

$$\sigma\left(\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\begin{bmatrix} -C & C_d \end{bmatrix}\right) = \{-1, -2, -3\}\,,$$

where $G_1 \in \mathbb{R}^n$ and $G_2 \in \mathbb{R}^q$. Using the method of matching coefficients this yields

$$G = \begin{bmatrix} -4.8 \\ 2.2 \\ -4.6 \end{bmatrix}.$$

6. *Robust regulator.* The robust regulator is

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} A + BK + G_1 C & B(\Gamma - K\Pi) - G_1 C_d \\ G_2 C & S - G_2 C_d \end{bmatrix} \xi + Ge \\ u &= \Gamma\xi_2 + K(\xi_1 - \Pi\xi_2)\,. \end{aligned}$$

This yields

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} -5.8 & 5.8 & -2 \\ 2.2 & -2.2 & -2 \\ -4.6 & 6.6 & 0 \end{bmatrix} \xi + \begin{bmatrix} -4.8 \\ 2.2 \\ -4.6 \end{bmatrix} e \\ u &= \begin{bmatrix} -2 & 1 & -2 \end{bmatrix} \xi\,. \end{aligned}$$

In transfer function form this is

$$C(s) = \frac{U(s)}{E(s)} = \frac{21s^2 - 11s + 38}{s^3 + 8s^2 + 4s + 32}\,.$$

Because the poles of this transfer function are $\{-8, \pm 2j\}$, the internal model principle is satisfied.